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ON THE IMAGES OF ELLIPSES UNDER SIMILARITIES

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Abstract

We consider ellipses corresponding to any norm function on the complex plane and determine their images under the similarities which are special Möbius transformations.

1 Introduction

It is well-known that Möbius transformations map circles to circles where straight lines are considered to be circles through ∞. It is also well-known that all norms on $\mathbb{C}$ are equivalent. In [5], the present author considered circles corresponding to any norm function and determined their images under the Möbius transformations on the complex plane. Recently, in [2] and [3], Adam Coffman and Marc Frantz considered the images of non-circular ellipses (corresponding to the Euclidean norm function) under the Möbius transformations. In [6], the present author determined the images of non-circular ellipses under the harmonic Möbius transformations.

Motivated by the above studies, we consider the images of ellipses corresponding to any norm function on $\mathbb{C}$ under the Möbius transformations.

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Throughout the paper, we consider the real linear space structure of the complex plane $\mathbb{C}$ and investigate the answer of the following question:

If $w = T(z)$ is a Möbius transformation and $\|.\|$ is any norm function on $\mathbb{C}$, then does T take ellipses to ellipses in this norm?

Note that all Möbius transformations do not map ellipses to ellipses corresponding to the Euclidean norm function on $\mathbb{C}$. From [2] and [3], we know that the Möbius transformations which map ellipses to ellipses are similarity transformations. In our case, we see that the rotation map $z \to e^{i\phi}z$ do not map ellipses to ellipses for every value of the real number ϕ. Thus we restrict our investigations to similarity transformations.

2 Main results

We give a brief account of Möbius transformations (see [1] and [4] for more details).

A Möbius transformation T is a function of the form

$$T(z) = \frac{az+b}{cz+d}; a, b, c, d \in \mathbb{C} \text{ and } ad - bc \neq 0. \tag{2.1}$$

Such transformations form a group under composition. The Möbius transformations with $c = 0$ form the subgroup of *similarities*. Such transformations have the form

$$z \to \alpha z + \beta; \alpha, \beta \in \mathbb{C}, \alpha \neq 0. \tag{2.2}$$

The transformation $z \to \frac{1}{z}$ is called an *inversion*. Here we use the well-known fact that every Möbius transformation T of the form (2.1) is a composition of finitely many similarities and inversions.

Let $\|.\|$ be any norm function on $\mathbb{C}$. A circle whose center is at z_0 and of radius r is denoted by $S_r(z_0)$ and defined by $S_r(z_0) = \{z \in \mathbb{C} : \|z - z_0\| = r\}$. An ellipse is the locus of points z with the property that the sum of the distances from z to two given fixed points, say F_1 and F_2, is a constant. The two fixed points are called foci. Thus the set $\{z \in \mathbb{C} : \|z - F_1\| + \|z - F_2\| = r\}$ is the ellipse with foci F_1 and F_2. We denote this ellipse by $E_r(F_1, F_2)$. If the two foci coincide, then the ellipse is a circle.

Now we recall the following lemma which will be used later.

Lemma 2.1. *[5] Let $\|.\|$ be any norm function on the complex plane. Then for every $\phi \in \mathbb{R}$, the following function define a norm on the complex plane:*

$$\|z\|_{\phi} = \left\|e^{-i\phi} z\right\|. \tag{2.3}$$

We begin the following lemma.

Lemma 2.2. *Let $\|.\|$ be any norm on $\mathbb{C}$. Then the similarity transformations of the form*

$$f(z) = \alpha z + \beta;\ \alpha \neq 0,\ \alpha \in \mathbb{R}, \tag{2.4}$$

map ellipses to ellipses corresponding to this norm function.

Proof. Let $\|.\|$ be any norm and let $E_r(F_1, F_2)$ be any ellipse corresponding to this norm. If $f(z)$ is a similarity transformation of the form (2.4), then the image of $E_r(F_1, F_2)$ under f is the ellipse $E_{|\alpha| r}(f(F_1), f(F_2))$. Indeed, we have

$$\begin{aligned} & \|f(z) - f(F_1)\| + \|f(z) - f(F_2)\| \\ = \ & \|\alpha z + \beta - (\alpha F_1 + \beta)\| + \|\alpha z + \beta - (\alpha F_2 + \beta)\| \\ = \ & \|\alpha(z - F_1)\| + \|\alpha(z - F_2)\| \\ = \ & |\alpha| \left(\|z - F_1\| + \|z - F_2\|\right) = |\alpha|\, r. \end{aligned}$$

□

Now we consider the norm functions defined in (2.3). Notice that for the Euclidean norm, all of the norm functions $\|.\|_{\phi}$ are equal to the Euclidean norm. For any other norm function we have $\|.\|_{k\pi} = \|.\|$ where $k \in \mathbb{Z}$.

Then we can give the following theorem:

Theorem 2.1. *Let $w = f(z) = \alpha z + \beta;\ \alpha \neq 0,\ \alpha, \beta \in \mathbb{C}$. Then for every ellipse $E_r(F_1, F_2)$ corresponding to any norm function $\|.\|$ on $\mathbb{C}$, $f(E_r(F_1, F_2))$ is an ellipse corresponding to the same norm function or corresponding to the norm function $\|z\|_{\phi} = \left\|e^{-i\phi}.z\right\|$, where $\phi = \arg(\alpha)$.*

Proof. Let $w = f(z) = \alpha z + \beta$; $\alpha \neq 0$, $\alpha, \beta \in \mathbb{C}$. If $E_r(F_1, F_2)$ is an Euclidean ellipse, then from [3] we know that $f(E_r(F_1, F_2))$ is again an Euclidean ellipse. Suppose that $E_r(F_1, F_2)$ is not an Euclidean ellipse. Let us write $f(z) =$

$|\alpha| e^{i\phi} z + \beta$; $\alpha \neq 0$, $\phi = \arg(\alpha)$ and let $f_1(z) = e^{i\phi} z$, $f_2(z) = |\alpha| z + e^{-i\phi}\beta$. We have $f(z) = (f_1 \circ f_2)(z)$.

Then by Lemma 2.2, the transformation $f_2(z)$ maps ellipses to ellipses corresponding to this norm function. Let $w = f_1(z) = e^{i\phi} z$, $\phi \neq k\pi$, $k \in \mathbb{Z}$. Now we consider the norm function $\|.\|_\phi$ given in Lemma 2.1. We get

$$\begin{aligned}
\|w - f(F_1)\|_\phi + \|w - f(F_2)\|_\phi &= \left\|e^{i\phi}(z - F_1)\right\|_\phi + \left\|e^{i\phi}(z - F_2)\right\|_\phi \\
&= \left\|e^{-i\phi}\left[e^{i\phi}(z - F_1)\right]\right\| + \left\|e^{-i\phi}\left[e^{i\phi}(z - F_2)\right]\right\| \\
&= \|z - F_1\| + \|z - F_2\| = r.
\end{aligned}$$

This shows that the image of the ellipse $E_r(F_1, F_2)$ under the transformation $w = f_1(z) = e^{i\phi} z$, $(\phi \neq k\pi,\ k \in \mathbb{Z})$ is the ellipse $E_r(f(F_1), f(F_2))$ corresponding to the norm function $\|.\|_\phi$ given in (2.3). □

We note that we do not know the exact values of ϕ for which $\|.\|_\phi = \|.\|$. This is an open problem. If $\|.\|_\phi = \|.\|$, then the transformation $f_1(z) = e^{i\phi} z$ maps ellipses to ellipses corresponding to this norm function. In general $f_1(z) = e^{i\phi} z$ do not map ellipses to ellipses corresponding to the same norm function. For example, let $\|.\|$ be any norm with $\|1\| \neq \|i\|$ and $\phi = \frac{\pi}{2}$. Assume that $\|z\|_{\frac{\pi}{2}} = \|z\|$ for all $z \in \mathbb{C}$. For $z = 1$ we have $\|i\| = \|1\|$, which is a contradiction. Therefore the transformation $z \to e^{\frac{\pi}{2} i} z$ maps ellipses corresponding to the norm function $\|.\|$ to ellipses corresponding to the norm function $\|.\|_{\frac{\pi}{2}}$. We give the following conjecture for the norm functions with the properties $\|1\| = \|i\|$ and $\|z\| = \|\overline{z}\|$ for all $z \in \mathbb{C}$.

Conjecture 2.1. *Let $\|.\|$ be any norm on $\mathbb{C}$ with $\|1\| = \|i\|$. Assume that $\|z\| = \|\overline{z}\|$ for all $z \in \mathbb{C}$. Then we have $\|.\|_{\frac{\pi}{2}} = \|.\|$ and hence the transformation $z \to e^{\frac{\pi}{2} i} z$ maps ellipses to ellipses corresponding to this norm function.*

If this conjecture is true, then we have also the transformation $z \to e^{\frac{\pi}{2} i} z$ maps circles to circles corresponding to this norm function as a corollary.

Example 2.1. *Let us consider the norm function*

$$\|z\| = 2|x| + |y|$$

on $\mathbb{C}$. Let $F_1 = -1$ and $F_2 = 1$. The image of the ellipse $E_6(F_1, F_2)$ under the transformation $w = e^{\frac{\pi}{2} i} z$ is not an ellipse corresponding to the same norm but

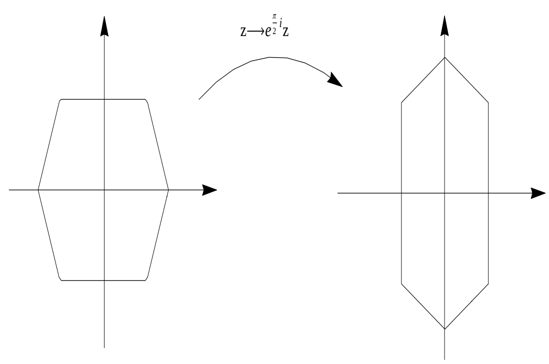

Figure 1:

it is the ellipse $E_6(-i,i)$ *corresponding to the norm function* $\|z\|_{\frac{\pi}{2}} = |x| + 2\,|y|$, (*see Figure* 1).

Finally we note that Lemma 2.2 and Theorem 2.1 hold also for hyperbolas corresponding to any norm function on the complex plane.

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