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On Some Class of Hypersurfaces in $\mathbb{E}^{n+1}$ Satisfying Chen's Equality

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Abstract

In this paper we study pseudosymmetry type hypersurfaces in the Euclidean space $\mathbb{E}^{n+1}$ satisfying B. Y. Chen's equality.

Key Words: Chen's equality, semisymmetric, pseudosymmetric manifold, hypersurface.

1. Introduction

Let (M, g), $n \geq 3$, be a connected Riemannian manifold of class C^∞. We denote by ∇, R, C, S and κ the Levi-Civita connection, the Riemann-Christoffel curvature tensor, the Weyl conformal curvature tensor, the Ricci tensor and the scalar curvature of (M, g), respectively. The Ricci operator $\mathcal{S}$ is defined by $g(\mathcal{S}X, Y) = S(X, Y)$, where $X, Y \in \chi(M)$, $\chi(M)$ being Lie algebra of vector fields on M. We next define endomorphisms $X \wedge Y$, $\mathcal{R}(X, Y)$ and $\mathcal{C}(X, Y)Z$ of $\chi(M)$ by

$$(X \wedge Y)Z = g(Y, Z)X - g(X, Z)Y, \tag{1.1}$$

$$\mathcal{R}(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \tag{1.2}$$

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$$\mathcal{C}(X,Y)Z = \mathcal{R}(X,Y)Z - \frac{1}{n-2}(X \wedge \mathcal{S}Y + \mathcal{S}X \wedge Y - \frac{\kappa}{n-1}X \wedge Y)Z, \tag{1.3}$$

respectively, where $X, Y, Z \in \chi(M)$.

The Riemannian Christoffel curvature tensor R and the Weyl curvature tensor C of (M, g) are defined by

$$R(X,Y,Z,W) = g(\mathcal{R}(X,Y)Z, W), \tag{1.4}$$

$$C(X,Y,Z,W) = g(\mathcal{C}(X,Y)Z, W), \tag{1.5}$$

respectively, where $W \in \chi(M)$.

For a $(0,k)$-tensor field T, $k \geq 1$, on (M, g) we define the tensors $R \cdot T$ and $Q(g,T)$ by

$$\begin{aligned} (R(X,Y)\cdot T)(X_1,...,X_k) &= -T(\mathcal{R}(X,Y)X_1, X_2,...,X_k) \\ &\quad -...-T(X_1,...,X_{k-1},\mathcal{R}(X,Y)X_k), \end{aligned} \tag{1.6}$$

$$\begin{aligned} Q(g,T)(X_1,...,X_k;X,Y) &= (X \wedge Y)T(X_1,...,X_k)-T((X \wedge Y)X_1, X_2,...,X_k) \\ &\quad -...-T(X_1,...,X_{k-1},(X \wedge Y)X_k), \end{aligned} \tag{1.7}$$

respectively.

If the tensors $R \cdot R$ and $Q(g,R)$ are linearly dependent then M is called *pseudosymmetric*. This is equivalent to

$$R \cdot R = L_R Q(g,R) \tag{1.8}$$

holding on the set $U_R = \{x \mid Q(g,R) \neq 0 \text{ at } x\}$, where L_R is some function on U_R. If $R \cdot R = 0$ then M is called *semisymmetric*. (see [11], Section 3.1; [19]).

If the tensors $R \cdot S$ and $Q(g,S)$ are linearly dependent then M is called *Ricci-pseudosymmetric*. This is equivalent to

$$R \cdot S = L_S Q(g,S) \tag{1.9}$$

holding on the set $U_S = \{x \mid S \neq \frac{\kappa}{n}g \text{ at } x\}$, where L_S is some function on U_S. Every pseudosymmetric manifold is Ricci pseudosymmetric but the converse statement is not true. If $R \cdot S = 0$ then M is called *Ricci-semisymmetric*. (see [10], [14]).

If the tensors $R \cdot C$ and $Q(g, C)$ are linearly dependent then M is called *Weyl-pseudosymmetric*. This is equivalent to

$$R \cdot C = L_C Q(g, C) \tag{1.10}$$

holding on the set $U_C = \{x \mid C \neq 0 \text{ at } x\}$. Every pseudosymmetric manifold is Weyl pseudosymmetric but the converse statement is not true. If $R \cdot C = 0$ then M is called *Weyl-semisymmetric*. (see [13]).

The manifold M is a *manifold with pseudosymmetric Weyl tensor* if and only if

$$C \cdot C = L_C Q(g, C) \tag{1.11}$$

holds on the set U_C, where L_C is some function on U_C (see [12]). The tensor $C \cdot C$ is defined in the same way as the tensor $R \cdot R$.

2. Submanifolds Satisfying Chen's Equality

Let M^n be an $n \geq 3$ dimensional connected submanifold immersed isometrically in the Euclidean space $\mathbb{E}^m$. We denote by $\widetilde{\nabla}$ and ∇ the Levi-Civita connections corresponding to $\mathbb{E}^m$ and M, respectively. Let ξ be a local unit normal vector field on M in $\mathbb{E}^m$. We can present the Gauss formula and the Weingarten formula of M in $\mathbb{E}^m$ in the form $\widetilde{\nabla}_X Y = \nabla_X Y + h(X, Y)$ and $\widetilde{\nabla}_X \xi = -A_\xi(X) + D_X \xi$, respectively, where X, Y are vector fields tangent to M and D is the normal connection of M. (see [4]).

Let M^n be a submanifold of $\mathbb{E}^m$ and $\{e_1,...,e_n\}$ be an orthonormal tangent frame field on M^n. For the plane section $e_i \wedge e_j$ of the tangent bundle TM spanned by the vectors e_i and e_j $(i \neq j)$ the scalar curvature of M is defined by $\kappa = \sum\limits_{i,j=1}^{n} K(e_i \wedge e_j)$ where K denotes the sectional curvature of M. Consider the real function $\inf K$ on M^n defined for every $x \in M$ by

$$(\inf K)(x) := \inf\{K(\pi) \mid \pi \text{ is a plane in } T_x M^n\}.$$

Note that since the set of planes at a certain point is compact, this infimum is actually a minimum. Then in [6], B. Y. Chen proved the following basic inequality between the intrinsic scalar invariants $\inf K$ and κ of M^n, and the extrinsic scalar invariant $|H|$, being the length of the mean curvature vector field H of M^n in $\mathbb{E}^m$.

Lemma 2.1 *[6]. Let M^n, $n \geq 2$, be any submanifold of $\mathbb{E}^m$, $m = n + p$, $p \geq 1$. Then*

$$\inf K \geq \frac{1}{2}\left\{\kappa - \frac{n^2(n-2)}{n-1}|H|^2\right\}. \tag{2.12}$$

Equality holds in (2.12) at a point x if and only if with respect to suitable local orthonormal frames $e_1,...,e_n \in T_x M^n$, the Weingarten maps A_t with respect to the normal sections $\xi_t = e_{n+t}$, $t = 1,...,p$ are given by

$$A_1 = \begin{bmatrix} a & 0 & 0 & 0 & \cdots & 0 \\ 0 & b & 0 & 0 & \cdots & 0 \\ 0 & 0 & \mu & 0 & \cdots & 0 \\ 0 & 0 & 0 & \mu & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \mu \end{bmatrix}, \quad A_t = \begin{bmatrix} c_t & d_t & 0 & \cdots & 0 \\ d_t & -c_t & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \ (t > 1),$$

where $\mu = a + b$ for any such frame, $\inf K(x)$ is attained by the plane $e_1 \wedge e_2$.

The relation (2.12) is called Chen's inequality. Submanifolds satisfying Chen's inequality have been studied with many authors. For more details see ([18],[8],[15] and recently [2] and [3]).

Remark 2.2 *For dimension $n = 2$ (2.12) is trivially satisfied.*

From now on we assume that M^n is a hypersurface in $\mathbb{E}^{n+1}$. We denote shortly $K_{rs} = K(e_r \wedge e_s)$.

By the use of Lemma 2.1 we get the following corollaries;

Corollary 2.3 *Let M be a hypersurface of $\mathbb{E}^{n+1}$, $n \geq 3$, satisfying Chen's equality then*

$$K_{12} = ab, \quad K_{1j} = a\mu, \quad K_{2j} = b\mu, \quad K_{ij} = \mu^2, \tag{2.13}$$

where $i, j > 2$. Furthermore, $\mathcal{R}(e_i, e_j)e_k = 0$ if i, j and k are mutually different.

Corollary 2.4 *Let M be a hypersurface of $\mathbb{E}^{n+1}$, $n \geq 3$, satisfying Chen's equality then*

$$\begin{aligned} S(e_1,e_1) &= [(n-2)a\mu + ab], \\ S(e_2,e_2) &= [(n-2)b\mu + ab], \\ S(e_3,e_3) = ... = S(e_n,e_n) &= (n-2)\mu^2, \end{aligned} \tag{2.14}$$

and $S(e_i,e_j) = 0$ if $i \neq j$.

Remark 2.5 *Hypersurface M with three distinct principal curvatures, their multiplicities are 1, 1 and $n-2$, is said to be 2-quasi umbilical. Therefore hypersurfaces satisfying B. Y. Chen equality is a special type of 2-quasi umbilical hypersurfaces.*

Theorem 2.6 *[16]. Any 2-quasi-umbilical hypersurface M, $dimM \geq 4$, immersed isometrically in a semi-Riemannian conformally flat manifold N is a manifold with pseudosymmetric Weyl tensor.*

Corollary 2.7 *[15]. Every hypersurface M immersed isometrically in a Riemannian space of constant curvature $N^{n+1}(c)$, $n \geq 4$, realizing Chen's equality is a hypersurface with pseudosymmetric Weyl tensor.*

On the other hand, it is known that in a hypersurface M of a Riemannian space of constant curvature $N^{n+1}(c)$, $n \geq 4$, if M is a Ricci-pseudosymmetric manifold with pseudosymmetric Weyl tensor then it is a pseudosymmetric manifold (see [15]). Moreover from [1], we know that, in a hypersurface M of a Riemannian space of constant curvature $N^{n+1}(c)$, $n \geq 4$, the Weyl pseudosymmetry and the pseudosymmetry conditions are equivalent. So using the previous facts and Theorem 2.6 one can obtain the following corollary.

Corollary 2.8 *In the class of 2-quasiumbilical hypersurfaces of the Euclidean space $\mathbb{E}^{n+1}$, $n \geq 4$, the conditions of the pseudosymmetry, the Ricci-pseudosymmetry and the Weyl pseudosymmetry are equivalent.*

In [18] the authors gave the classification of semisymmetric submanifolds satisfying B. Y. Chen equality.

Theorem 2.9 *[18]. Let M^n, $n \geq 3$, be a submanifold of $\mathbb{E}^m$ satisfying Chen's equality. Then M^n is semisymmetric if and only if M^n is a minimal submanifold (in which case M^n is $(n-2)$-ruled), or M^n is a round hypercone in some totally geodesic subspace $\mathbb{E}^{n+1}$ of $\mathbb{E}^m$.*

Now our aim is to give an extension of Theorem 2.9 for the case M is a pseudosymmetric hypersurface in the Euclidean space $\mathbb{E}^{n+1}$. Since hypersurfaces satisfying Chen's equality is a special type of 2-quasiumbilical hypersurfaces, it is enough to investigate only the pseudosymmetry condition. By Corollary 2.8, this will include all types of the pseudosymmetry (1.8)-(1.10). Firstly we give the following lemmas;

Lemma 2.10 *Let M, $n \geq 3$, be a hypersurface of $\mathbb{E}^{n+1}$ satisfying Chen's equality. Then*

$$(\mathcal{R}(e_1, e_3) \cdot \mathcal{R})(e_2, e_3)e_1 = a\mu b^2 e_2, \tag{2.15}$$

$$(\mathcal{R}(e_2, e_3) \cdot \mathcal{R})(e_1, e_3)e_2 = b\mu a^2 e_1. \tag{2.16}$$

Proof. Using (1.6) we have

$$\begin{aligned}(\mathcal{R}(e_1, e_3) \cdot \mathcal{R})(e_2, e_3)e_1 &= \mathcal{R}(e_1, e_3)(\mathcal{R}(e_2, e_3)e_1) - \mathcal{R}(\mathcal{R}(e_1, e_3)e_2, e_3)e_1 \\ &\quad -\mathcal{R}(e_2, \mathcal{R}(e_1, e_3)e_3)e_1 - \mathcal{R}(e_2, e_3)(\mathcal{R}(e_1, e_3)e_1)\end{aligned} \tag{2.17}$$

and

$$\begin{aligned}(\mathcal{R}(e_2, e_3) \cdot \mathcal{R})(e_1, e_3)e_2 &= \mathcal{R}(e_2, e_3)(\mathcal{R}(e_1, e_3)e_2) - \mathcal{R}(\mathcal{R}(e_2, e_3)e_1, e_3)e_2 \\ &\quad -\mathcal{R}(e_1, \mathcal{R}(e_2, e_3)e_3)e_2 - \mathcal{R}(e_1, e_3)(\mathcal{R}(e_2, e_3)e_2).\end{aligned} \tag{2.18}$$

Since

$$\mathcal{R}(e_i, e_j)e_k = (A_\xi e_i \wedge A_\xi e_j)e_k$$

then using (2.13) one can get

$$\begin{aligned}\mathcal{R}(e_1, e_3)e_1 = -K_{13}e_1 \quad &, \quad \mathcal{R}(e_1, e_3)e_3 = K_{13}e_1 \\ \mathcal{R}(e_2, e_1)e_1 = K_{12}e_2 \quad &, \quad \mathcal{R}(e_2, e_1)e_2 = -K_{12}e_1 \\ \mathcal{R}(e_2, e_3)e_2 = -K_{23}e_2 \quad &, \quad \mathcal{R}(e_2, e_3)e_3 = K_{23}e_2.\end{aligned} \tag{2.19}$$

Therefore substituting (2.19), (2.13) into (2.17) and (2.18) respectively we get the result. □

Lemma 2.11 *Let M, $n \geq 3$, be a hypersurface of $\mathbb{E}^{n+1}$ satisfying Chen's equality. Then*

$$Q(g,\mathcal{R})(e_2,e_3,e_1;e_1,e_3) = b^2 e_2, \tag{2.20}$$

$$Q(g,\mathcal{R})(e_1,e_3,e_2;e_2,e_3) = a^2 e_1. \tag{2.21}$$

Proof. Using the relation (1.7) we obtain

$$\begin{aligned} Q(g,\mathcal{R})(e_2,e_3,e_1;e_1,e_3) &= (e_1\wedge e_3)\mathcal{R}(e_2,e_3)e_1 - \mathcal{R}((e_1\wedge e_3)e_2,e_3)e_1 \\ &\quad -\mathcal{R}(e_2,(e_1\wedge e_3)e_3)e_1 - \mathcal{R}(e_2,e_3)((e_1\wedge e_3)e_1) \end{aligned} \tag{2.22}$$

and

$$\begin{aligned} Q(g,\mathcal{R})(e_2,e_3,e_2;e_2,e_3) &= (e_2\wedge e_3)\mathcal{R}(e_1,e_3)e_2 - \mathcal{R}((e_2\wedge e_3)e_1,e_3)e_2 \\ &\quad -\mathcal{R}(e_1,(e_2\wedge e_3)e_3)e_2 - \mathcal{R}(e_1,e_3)((e_2\wedge e_3)e_2). \end{aligned} \tag{2.23}$$

So substituting respectively (2.19) and (2.13) into (2.22) and (2.23) we obtain (2.20)-(2.21). □

Theorem 2.12 *Let M, $n \geq 3$, be a hypersurface of $\mathbb{E}^{n+1}$ satisfying Chen's equality. Then M is pseudosymmetric if and only if*

(i) $M = \mathbb{E}^n$, or

(ii) M is a round hypercone in $\mathbb{E}^{n+1}$, or

(iii) M is a minimal hypersurface in $\mathbb{E}^{n+1}$ (in which case M is $(n-2)$-ruled), or

(iv) The shape operator of M in $\mathbb{E}^{n+1}$ is of the form

$$A_\xi = \begin{bmatrix} a & 0 & 0 & 0 & \cdots & 0 \\ 0 & a & 0 & 0 & \cdots & 0 \\ 0 & 0 & 2a & 0 & \cdots & 0 \\ 0 & 0 & 0 & 2a & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 2a \end{bmatrix}. \tag{2.24}$$

Proof. Let M be a pseudosymmetric hypersurface in $\mathbb{E}^{n+1}$. Then by definition one can write

$$(\mathcal{R}(e_1, e_3) \cdot \mathcal{R})(e_2, e_3)e_1 = L_R Q(g, \mathcal{R})(e_2, e_3, e_1; e_1, e_3) \tag{2.25}$$

and

$$(\mathcal{R}(e_2, e_3) \cdot \mathcal{R})(e_1, e_3)e_2 = L_R Q(g, \mathcal{R})(e_1, e_3, e_2; e_2, e_3). \tag{2.26}$$

Since M satisfies B. Y. Chen equality then by Lemma 2.10 and Lemma 2.11 the equations (2.25) and (2.26) turns, respectively, into

$$(a\mu - L_R)b^2 = 0 \tag{2.27}$$

and

$$(b\mu - L_R)a^2 = 0. \tag{2.28}$$

i) Firstly, suppose that M is semisymmetric, i.e., M is trivially pseudosymmetric then $L_R = 0$. So the equations (2.27) and (2.28) can be written as the following:

$$ab\mu = 0.$$

Now suppose $a = 0$, $b \neq 0$ then $\mu = b$ and by [9] M is a round hypercone in $\mathbb{E}^{n+1}$. If $a \neq 0$, $b = 0$ then $\mu = a$ and similarly M is a round hypercone in $\mathbb{E}^{n+1}$. If $\mu = 0$ then M is minimal. If $a = 0$, $b = 0$ then $\mu = 0$ so $M = \mathbb{E}^n$.

ii) Secondly, suppose M is not semisymmetric, i.e., $R \cdot R \neq 0$. For the subcases $a = b = 0$, $a = 0$, $b \neq 0$ or $a \neq 0$, $b = 0$ we get $R \cdot R = 0$ which contradicts the fact that

$R \cdot R \neq 0$. Therefore the only remaining possible subcase is $a \neq 0$, $b \neq 0$. So by the use of (2.27) and (2.28) we have $(a-b)\mu = 0$. Since $\mu = a + b \neq 0$ then $a = b$ and by Lemma 2.1 the shape operator of M is of the form (2.24).

This completes the proof of the theorem. □

Theorem 2.13 *Let M, $n \geq 3$, be a hypersurface of $\mathbb{E}^{n+1}$satisfying Chen's equality. If M is pseudosymmetric then $rankS = 0$ or 2 or $n-1$ or n.*

Proof. Suppose that M is a hypersurface of $\mathbb{E}^{n+1}$, $n \geq 3$, satisfying Chen equality. If M is semisymmetric then $M = \mathbb{E}^n$ or M is a round hypercone or M is minimal. It is easy to check that if $M = \mathbb{E}^n$ then $rankS = 0$, if M is a round hypercone then $rankS = n-1$, if M is minimal then $rankS = 2$. Now suppose M is not semisymmetric but it is pseudosymmetric. Then by Theorem 2.12 the principal curvatures of M are a, a, $2a$,...,$2a$. So by Corollary 2.4, $S(e_1, e_1) = S(e_2, e_2) = (2n-3)a^2$ and $S(e_3, e_3) = ... = S(e_n, e_n) = 2(n-2)a^2$, which gives $rankS = n$.

Hence we get the result, as required. □

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