On a class of generalized quasi-Einstein manifolds

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Abstract. In this study, we find the necessary conditions in order that a special class of generalized quasi-Einstein manifolds to be pseudo Ricci-symmetric and R-harmonic. We also consider these type manifolds with cyclic parallel Ricci tensor.

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1 Introduction

A non-flat Riemannian manifold (M^n, g), $n = \dim M \geq 3$, is said to be an Einstein if the condition $S = \frac{\kappa}{n} g$ is fulfilled on M^n, where S and κ denote the Ricci tensor and the scalar curvature of (M^n, g) respectively.

A non-flat Riemannian manifold (M^n, g), $n \geq 3$, is defined to be quasi-Einstein if its Ricci tensor S is not identically zero and satisfies the condition

$$S(X, Y) = ag(X, Y) + bA(X)A(Y),$$

where a, b are scalars of which $b \neq 0$ and A is non-zero 1-form such that $g(X, U) = A(X)$ for every vector field X and U is a unit vector field.

In [2], [3], [4] and [8], the authors studied quasi-Einstein manifolds and gave some examples of quasi-Einstein manifolds. In [6] and [7], quasi-Einstein hypersurfaces in semi-Euclidean spaces and semi-Riemannian space forms were considered, respectively.

A non-flat Riemannian manifold (M^n, g), $n \geq 3$, is called generalized quasi-Einstein if its Ricci tensor S is non-zero and satisfies the condition

$$S(X, Y) = ag(X, Y) + bA(X)A(Y) + cB(X)B(Y), \tag{1.1}$$

where a, b, c are certain non-zero scalars and A, B are two non-zero 1-forms defined by

$$g(X, U) = A(X) \quad , \quad g(X, V) = B(X) \tag{1.2}$$

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and the unit vector fields U and V are orthogonal, i.e., $g(U,V)=0$. The vector fields U and V are called the generators of the manifold. If $c=0$ then the manifold reduces to a quasi-Einstein manifold (see [5]).

In [5], U. C. De and G. C. Ghosh studied generalized quasi-Einstein manifolds and as an example they showed that a 2-quasi-umbilical hypersurface of the Euclidean space is generalized quasi-Einstein.

In this study, we consider a special class of generalized quasi-Einstein manifolds such that the generators U and V are parallel vector fields.

2 Preliminaries

A non-flat Riemannian manifold (M^n, g) is called pseudo-Ricci symmetric (see [1]) and R-harmonic (see [9]) if the Ricci tensor S of M^n satisfy the following conditions

$$(\nabla_X S)(Y,Z) = 2\alpha(X)S(Y,Z) + \alpha(Y)S(X,Z) + \alpha(Z)S(X,Y) \tag{2.1}$$

and

$$(\nabla_X S)(Y,Z) = (\nabla_Z S)(X,Y), \tag{2.2}$$

respectively, where α is a one form, X, Y, Z are vector fields on M^n and ∇ is the Levi-Civita connection of M^n.

3 Main Results

In this section, we consider generalized quasi-Einstein manifolds under the condition that U and V are parallel vector fields.

Suppose that M^n is a generalized quasi-Einstein manifold and the vector fields U and V are parallel. Then $\nabla_X U = 0$ and $\nabla_X V = 0$, which implies $R(X,Y)U = 0$ and $R(X,Y)V = 0$. Hence contracting these equations with respect to Y we see that $S(X,U) = 0$ and $S(X,V) = 0$. So from (1.1)

$$S(X,U) = (a+b)A(X) = 0$$

and

$$S(X,V) = (a+c)B(X) = 0,$$

which implies that $a = -b = -c$. Then the equation (1.1) turns the form

$$S(X,Y) = a\left(g(X,Y) - A(X)A(Y) - B(X)B(Y)\right), \tag{3.1}$$

(for more details see [5]). On the other hand, it is well-known that

$$(\nabla_X S)(Y,Z) = \nabla_X S(Y,Z) - S(\nabla_X Y, Z) - S(Y, \nabla_X Z). \tag{3.2}$$

Since M^n is a generalized quasi-Einstein manifold, by the use of (3.1) and (3.2) we can write

$$(\nabla_X S)(Y,Z) = X[a]\left(g(Y,Z) - A(Y)A(Z) - B(Y)B(Z)\right), \tag{3.3}$$

where $X[a]$ denotes the derivative of a with respect to the vector field X. Since M^n is pseudo Ricci-symmetric, by the use of (2.1) and (3.3), we can write

$$\begin{aligned} &X[a]\left(g(Y,Z)-A(Y)A(Z)-B(Y)B(Z)\right)\\ &=2a\alpha(X)\left(g(Y,Z)-A(Y)A(Z)-B(Y)B(Z)\right)\\ &+a\alpha(Y)\left(g(X,Z)-A(X)A(Z)-B(X)B(Z)\right)\\ &+a\alpha(Z)\left(g(X,Y)-A(X)A(Y)-B(X)B(Y)\right). \end{aligned} \tag{3.4}$$

Taking $X=U$ and $X=V$ in (3.4) we find

$$U[a]=2a\alpha(U) \tag{3.5}$$

and

$$V[a]=2a\alpha(V), \tag{3.6}$$

respectively.

Putting $Z=U$ and $Z=V$ in (3.4) we have

$$\alpha(U)=0 \tag{3.7}$$

and

$$\alpha(V)=0, \tag{3.8}$$

respectively. So in view of (3.5), (3.6), (3.7) and (3.8) we obtain

$$U[a]=0, \qquad V[a]=0,$$

which implies a is constant along the vector fields U and V.

Hence we can state the following theorem:

Theorem 3.1. *Let M^n be a generalized quasi-Einstein manifold under the condition that U, V are parallel vector fields. If M^n is pseudo Ricci-symmetric then the scalar function a is constant along the vector fields U and V.*

Assume that M^n is a R-harmonic generalized quasi-Einstein manifold. If U and V are parallel vector fields then from (2.2) and (3.3) we have

$$\begin{aligned} &(\nabla_X S)(Y,Z)-(\nabla_Z S)(X,Y)\\ &=X[a]\left(g(Y,Z)-A(Y)A(Z)-B(Y)B(Z)\right)\\ &-Z[a]\left(g(X,Y)-A(X)A(Y)-B(X)B(Y)\right)=0. \end{aligned} \tag{3.9}$$

Then taking $X=U$ and $X=V$ in (3.9) we find

$$U[a]=0 \quad \textit{and} \quad V[a]=0,$$

respectively, which implies that a is constant along the vector fields U and V.

So we have proved the following theorem:

Theorem 3.2. *Let M^n be a generalized quasi-Einstein manifold under the condition that U, V are parallel vector fields. If M^n is R-harmonic then the scalar function a is constant along the vector fields U and V.*

Now assume that M^n has cyclic parallel Ricci tensor. Then

$$(\nabla_X S)(Y,Z) + (\nabla_Y S)(X,Z) + (\nabla_Z S)(X,Y) = 0, \tag{3.10}$$

holds on M^n. If M^n is a generalized quasi-Einstein manifold under the condition that U and V are parallel vector fields then from (3.10) and (3.3) we get

$$\begin{aligned} 0 &= X[a](g(Y,Z) - A(Y)A(Z) - B(Y)B(Z)) \\ &\quad + Y[a](g(X,Z) - A(X)A(Z) - B(X)B(Z)) \\ &\quad + Z[a](g(X,Y) - A(X)A(Y) - B(X)B(Y). \end{aligned} \tag{3.11}$$

Taking $X = U$ in (3.11) we have $U[a] = 0$. Putting $X = V$ in (3.11) we find $V[a] = 0$. So we have the following theorem:

Theorem 3.3. *Let M^n be a generalized quasi-Einstein manifold under the condition that U, V are parallel vector fields. If M^n has cyclic parallel Ricci tensor then the scalar function a is constant along the vector fields U and V.*

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