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The Extended Hecke Groups as Semi-Direct Products and Related Results

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ABSTRACT

The extended Hecke groups $\overline{H}(\lambda_q)$ have been worked in (Sahin and Bizim, 2003) as amalgamated free products. In this paper, we first show that $\overline{H}(\lambda_q)$ is the semi-direct product (split extension) of the Hecke group $H(\lambda_q)$ by a cyclic group of order 2. Moreover, by considering a presentation $\mathcal{P}_{\overline{H}(\lambda_q)}$ of $\overline{H}(\lambda_q)$, we give the necessary and sufficient conditions of $\mathcal{P}_{\overline{H}(\lambda_q)}$ to be efficient on the minimal number of generators.

Keywords: Extended Hecke groups, semi-direct product, efficiency, minimality.

2000 Mathematics Subject Classification: 11F06; 20F05; 20F32; 20F55; 20H10; 57M05.

1 Introduction

Hecke groups have been studied extensively for many aspects in literature (see, for instance, (Rosen, 1954), (Schmidth and Sheingorn, 1995), (Cangul and Singerman, 1998) and (Ikikardes, Koruoglu and Sahin, 2006)). However, there are still some unsolved problems in this subject, for example, the group structure of some power subgroups of Hecke groups is not known yet. In addition, this problem is also open for the modular group as well (see (Newman, 1962)). Therefore our aim in this work is to get a new sight for solving this kind of problems. In fact we try to put on this sight by using semi-direct products which are equivalent, by (Brown, 1982), to the split extensions of groups. To do that we first give some background about the (extended) Hecke groups and the notion of efficiency on presentations of groups, then we present and prove our main results.

(a) Hecke and Extended Hecke Groups

In (Hecke, 1936), Hecke introduced an infinite class of discrete groups $H(\lambda_q)$ of linear fractional transformations preserving the upper-half plane. The *Hecke group* $H(\lambda_q)$ is the group generated by

$$x(z) = -\frac{1}{z} \text{ and } u(z) = z + \lambda_q$$

where $\lambda_q = 2\cos\pi/q$, for the integer $q \geq 3$. Let

$$y = xu = \frac{-1}{z+\lambda_q}.$$

Then $H(\lambda_q)$ has a presentation

$$\mathcal{P}_{H(\lambda_q)} =< x, y \; ; \; x^2, y^q > \text{ (see (Cangul and Singerman, 1998)).} \tag{1.1}$$

For $q = 3$, the resulting Hecke group $H(\lambda_3) = M$ is the modular group $PSL(2, \mathbb{Z})$. By adding the reflection $r(z) = 1 \,/\, \overline{z}$ to the generators of the modular group, the extended modular group $\overline{H}(\lambda_3) = \overline{M}$ was defined in (Jones and Thornton, 1986). Then the extended Hecke group, denoted by $\overline{H}(\lambda_q)$, was defined in (Sahin and Bizim, 2003) (also see (Huang, 1999), (Sahin, Bizim and Cangul, 2004), (Sahin, Ikikardes and Koruoglu, 2006) and (Sahin, Ikikardes and Koruoglu, 2007)) by adding the reflection

$$r(z) = 1 \,/\, \overline{z}$$

to the generators of $H(\lambda_q)$ similar to the extended modular group $\overline{M}$. The Hecke group $H(\lambda_q)$ is a subgroup of index 2 in $\overline{H}(\lambda_q)$. By (Sahin and Bizim, 2003), we know that the extended Hecke group $\overline{H}(\lambda_q)$ is isomorphic to $D_2 *_{\mathbb{Z}_2} D_q$ and has a presentation

$$\mathcal{P}_{\overline{H}(\lambda_q)} =< x, y, r \; ; \; x^2, \; y^q, \; r^2, \; (xr)^2, \; (yr)^2 > .$$

Again, for $q = 3$, we obtain the extended modular group $\overline{M}$ as introduced in (Jones and Thornton, 1986), (Kulkarni, 1991). Also, by (Jones and Thornton, 1986), it is known that the action of $\overline{M}$ on the modular group M by conjugation induces an isomorphism $\overline{M} \cong Aut(M)$ and then the extended Hecke group $\overline{H}(\lambda_q)$ can be considered as $Aut(H(\lambda_q))$ since $H(\lambda_q)$ has trivial center.

We assume in the rest of the paper that $\mathbb{Z}_n$ denotes the cyclic group of order n where n is a positive integer.

In this paper, our aim is to determine the semi-direct product of $H(\lambda_q)$ by $\mathbb{Z}_2$. We expect that this structure gives an alternative approach solving some problems about $\overline{H}(\lambda_q)$ or its subgroups.

(b) Efficiency

Let G be a finitely presented group, and let

$$\mathcal{P} =< \mathbf{x} \; ; \; \mathbf{r} > \tag{1.2}$$

be a finite presentation for G. The *deficiency* of $\mathcal{P}$ is defined by $def(\mathcal{P}) = -|\mathbf{x}| + |\mathbf{r}|$. Let

$$\delta(G) = -rk_{\mathbb{Z}}(H_1(G)) + d(H_2(G)),$$

where $rk_{\mathbb{Z}}(\,.\,)$ denotes the $\mathbb{Z}$-rank of the torsion-free part and $d(\,.\,)$ means the minimal number of generators. Then it is known (see (Baik and Pride, 1993), (Beyl and Tappe, 1982), (Epstein, 1961)) that for the presentation $\mathcal{P}$, it is always true that $def(\mathcal{P}) \geq \delta(G)$. We define

$$def(G) = min\{def(\mathcal{P}) : \mathcal{P} \text{ a finite presentation for } G\}.$$

We say G is *efficient* if $def(G) = \delta(G)$, and a presentation $\mathcal{P}$ such that $def(\mathcal{P}) = \delta(G)$ is then called an *efficient presentation*. A list of citations which is about the known results of efficiency can be found in (Cevik, 2000).

We note that if we can find a minimal presentation $\mathcal{P}$ for a group G such that $\mathcal{P}$ is not efficient then we have

$$def(\mathcal{P}') \geq def(\mathcal{P}) \geq \delta(G),$$

for all presentations $\mathcal{P}'$ defining the same group G. Thus there is no efficient presentation for G, that is to say, G is not an efficient group. Therefore, not all finitely presented groups are efficient. B.H.Neumann (Neumann, 1955) asked whether a finite group G with $\delta(G) = 0$ must be efficient. Swan (Swan, 1965) gave examples (of finite metabelian groups) showing this is not the case. These were the first examples of inefficient groups. In (Wiegold, 1981), Wiegold produced a different construction to the same end, and then Neumann added a slight modification to reduce the number of generators. In (Kovacs, 1995), Kovacs generalized both the above constructions, and he showed how to construct more inefficient finite groups whose Schur multiplicator is trivial. In (Robertson, Thomas and Wotherspoon, 1995), Robertson, Thomas and Wotherspoon examined a class of groups, introduced by Coxeter. By using a symmetric presentation, they showed that groups in this class are inefficient. They also proved that every finite simple group can be embedded into a finite inefficient group. In (Cevik, 2000), Çevik gave the sufficient conditions on the set of all finite groups which have efficient presentations to be closed under the standard wreath product. We note that, by (Ahmad, 1995), there is no algorithm to decide for any finitely presented group whether or not the group is efficient.

2 The Extended Hecke Groups as Semi-Direct Products

Let A and K be any groups, and let θ be a homomorphism defined by

$$\theta : A \to Aut(K), \quad a \to \theta_a$$

for all $a \in A$. Then the semi-direct product $G = K \rtimes_\theta A$ of K by A is defined as follows.
The elements of G are all ordered pairs (a, k) ($a \in A$, $k \in K$) and the multiplication is given by

$$(a, k)(a', k') = (aa', (k\theta_{a'})k').$$

Similar definitions of a semi-direct product can be found in (Baumslag, 1993) or (Rotman, 1988). Also the proof of the following lemma can be found in (Johnson, [Corollary 10.1], 1990).

Lemma 2.1. *Suppose that $\mathcal{P}_K =< \mathbf{y} \ ; \ \mathbf{s} >$ and $\mathcal{P}_A =< \mathbf{x} \ ; \ \mathbf{r} >$ are presentations for the groups K and A, respectively under the maps*

$$y \mapsto k_y \ (y \in \boldsymbol{y}), \qquad x \mapsto a_x \ (x \in \boldsymbol{x}).$$

Then we have a presentation

$$\mathcal{P} =< \mathbf{y}, \mathbf{x} \ ; \ \mathbf{s}, \mathbf{r}, \mathbf{t} >$$

for $G = K \rtimes_\theta A$, where $\boldsymbol{t} = \{yx\lambda_{yx}^{-1}x^{-1} \mid y \in \boldsymbol{y}, x \in \boldsymbol{x}\}$ and λ_{yx} is a word on $\mathbf{y}$ representing the element $(k_y)\theta_{a_x}$ of K ($a \in A$, $k \in K$, $x \in \boldsymbol{x}$, $y \in \boldsymbol{y}$).

Let us take A to be $\mathbb{Z}_2$ and K to be $H(\lambda_q)$. Then one of the main result of this paper is the following.

Theorem 2.2. $\overline{H}(\lambda_q) \cong H(\lambda_q) \rtimes_\theta \mathbb{Z}_2$.

Proof. Let us take the Hecke group $H(\lambda_q)$ with the associated presentation $\mathcal{P}_{H(\lambda_q)}$, as in (1.1), and let $\mathbb{Z}_2$ be generated by the element r. Also let θ be a homomorphism, defined by

$$\mathbb{Z}_2 \longrightarrow Aut(H(\lambda_q)), \quad r \longmapsto \theta_r.$$

As an easy consequence of the result in (Jones and Thornton, 1986), the action of $\overline{H}(\lambda_q)$ on $H(\lambda_q)$ by conjugation can be defined by

$$x \overset{\theta_r}{\longmapsto} rxr^{-1},\ y \overset{\theta_r}{\longmapsto} ry^{-1}r^{-1} \text{ and } (xy) \overset{\theta_r}{\longmapsto} \theta_r(xy),$$

where

$$\begin{aligned}\theta_r(xy) &= \theta_r(x)\theta_r(y)\\ &= rxr^{-1}ry^{-1}r^{-1} = rxy^{-1}r^{-1} = r(yx)^{-1}r^{-1}.\end{aligned}$$

Thus we have a semi-direct product $G = H(\lambda_q) \rtimes_\theta \mathbb{Z}_2$ and, by Lemma 2.1, have a presentation

$$\mathcal{P}_G =< y, r, x\ ;\ y^q, r^2, x^2, \mathbf{t} >, \tag{2.1}$$

where $\mathbf{t}$ denotes the set of relators of the form

$$xr(rxr^{-1})^{-1}r^{-1},\ yr(ry^{-1}r^{-1})^{-1}r^{-1} \text{ and } (xy)r(r(yx)^{-1}r^{-1})^{-1}r^{-1}.$$

In the set $\mathbf{t}$, we can rearrange the relators by the meaning of conjugacy. In other words, since x and rxr^{-1} are conjugate, their inverses are conjugate as well, thus we get the commutator of x and r as follows:

$$xr(rxr^{-1})^{-1}r^{-1} \sim xrrx^{-1}r^{-1}r^{-1} \sim xrx^{-1}r^{-1}.$$

Also in $\mathcal{P}_G$, since $r^2 = 1$ and $x^2 = 1$ then we get $r = r^{-1}$ and $x = x^{-1}$, respectively. Thus we get a new relator of the form $(xr)^2$ in $\mathcal{P}_G$. Then, by Tietze transformation (Magnus, Karras and Solitar, 1966), we can delete the relator $xr(rxr^{-1})^{-1}r^{-1}$.
Similarly, for the relator $yr(ry^{-1}r^{-1})^{-1}r^{-1} \sim yrryr^{-1}r^{-1}$, we get a new relator $(yr)^2$ in $\mathcal{P}_G$ since each element is conjugate to itself, that is, y and ryr^{-1} are conjugate to each other and $r^2 = 1$ implies that $r = r^{-1}$. Again by Tietze transformation we delete the relator $yr(ry^{-1}r)^{-1}r$ from $\mathcal{P}_G$.
Also, the last relator $(xy)r(r(yx)^{-1}r^{-1})^{-1}r^{-1}$ is equivalent to

$$xyr(r(xy)^{-1}r^{-1})r^{-1} \sim xyrry^{-1}x^{-1}r^{-1}r^{-1} \sim xyy^{-1}x^{-1} \sim 1.$$

Hence we can delete the relator $(xy)r(r(yx)^{-1}r^{-1})^{-1}r^{-1}$ from $\mathcal{P}_G$.
Therefore the presentation $\mathcal{P}_G$, as in (2.1), becomes

$$\mathcal{P}'_{\overline{H}} = \langle\ r, y, x\ ;\ r^2, y^q, x^2, (xr)^2, (yr)^2\ \rangle \tag{2.2}$$

for the group G. In fact, by (Sahin and Bizim, 2003), since $\mathcal{P}'_{\overline{H}}$ presents the group $\overline{H}(\lambda_q)$, we have $\overline{H}(\lambda_q) \cong H(\lambda_q) \rtimes_\theta \mathbb{Z}_2$, as required. □

As a consequence of this theorem, we can give the following result.

Corollary 2.3. $H(\lambda_q) \rtimes_\theta \mathbb{Z}_2 \cong D_2 *_{\mathbb{Z}_2} D_q \cong \overline{H}(\lambda_q)$.

Proof. By (Sahin and Bizim, 2003), we know that $\overline{H}(\lambda_q) \cong D_2 *_{\mathbb{Z}_2} D_q$ and then, by Theorem 2.2, we get the result as required. □

3 Minimality on Efficiency of the Group $\overline{H}(\lambda_q)$

In this section we will present some applications of $\overline{H}(\lambda_q)$, given in Theorem 2.2, in the name of efficiency. So let us take the group $\overline{H}(\lambda_q)$ which is presented by the presentation $\mathcal{P}'_{\overline{H}}$, as in (2.2). Since $x^2 = 1 = r^2$ and $y^q = 1$, we get $x = x^{-1}$, $r = r^{-1}$ and $y = y^{q-1}$. So if we replace these equalities in (2.2), we have a presentation

$$\mathcal{P}'_{\overline{H}(\lambda_q)} = \langle\ x, y, r\ ;\ x^2, y^q, r^2, [x, r], yry^{q-1}r\ \rangle \tag{3.1}$$

which is equal to the presentation (2.2) for the extended Hecke group $\overline{H}(\lambda_q)$. Then we have the following theorem as another main result of this work.
For $q \geq 3$, let $\mathcal{P}'_{\overline{H}(\lambda_q)}$ be a presentation, as in (3.1), for the group $\overline{H}(\lambda_q) \cong H(\lambda_q) \rtimes_\theta \mathbb{Z}_2$. Thus;

Theorem 3.1. *$\mathcal{P}'_{\overline{H}(\lambda_q)}$ is efficient if and only if $(q, 2) \neq 1$. Moreover $\mathcal{P}'_{\overline{H}(\lambda_q)}$ is efficient on 3-generators.*

Remark 3.1. The reason for us keeping track of the number of generators in Theorem *3.1* is that there is interest not just finding efficient presentations, but in finding presentations that are efficient on the minimal number of generators (see (Wamsley, 1973)). Therefore, we are going to prove $\mathcal{P}'_{\overline{H}(\lambda_q)}$ is minimal separately, that is $def(\mathcal{P}'_{\overline{H}(\lambda_q)}) = def(\overline{H}(\lambda_q))$.
We can obtain the following result as a quick consequence of Corollary 2.3 and Theorem 3.1.

Corollary 3.2. *The group $D_2 *_{\mathbb{Z}_2} D_q$ is efficient on 3 generators if and only if $(q, 2) \neq 1$.*

Now, let us cover some basic material for helping to prove Theorem 3.1.
Let $\mathcal{P}$ be a presentation, as given in (1.2), for a finitely presented group G. If we regard $\mathcal{P}$ as a 2-complex with one 0-cell, a 1-cell for each $x \in \mathbf{x}$, and a 2-cell for each $R \in \mathbf{r}$ in the standard way, then G is just the fundamental group of $\mathcal{P}$. There is also, of course, the second homotopy module $\pi_2(\mathcal{P})$ of $\mathcal{P}$, which is a left $\mathbb{Z}G$-module. The elements of $\pi_2(\mathcal{P})$ can be represented by geometric configurations called *spherical pictures* which are usually labelled by $\mathbb{P}$. These are described in detail in (Pride, 1991), and we refer the reader these for details. We should note that we need only one base point on each discs of our pictures in this paper so that we will actually use $*$-pictures, as described in (Pride, [Section 2.4], 1991).
For any picture $\mathbb{P}$ over $\mathcal{P}$ and for any $R \in \mathbf{r}$, the *exponent sum* of R in $\mathbb{P}$, denoted by $\exp_R(\mathbb{P})$ is the number of discs of $\mathbb{P}$ labelled by R, minus the number of discs labelled by R^{-1}. Thus, for a non-negative integer n, $\mathcal{P}$ is said to be n-*Cockcroft* if $\exp_R(\mathbb{P}) \equiv 0$ (mod n) where congruence (mod 0) is taken to be equality, for all $R \in \mathbf{r}$ and for all spherical pictures $\mathbb{P}$ over $\mathcal{P}$. A group G is said to be n-*Cockcroft* if it admits an n-Cockcroft presentation. We note that, by (Pride, 1991), to verify the n-Cockcroft property holds, it is enough to check for pictures $\mathbb{P} \in \mathbf{X}$ where

X is a set of generating pictures. One can find the listed examples which hold Cockcroft and p-Cockcroft properties in (Cevik, 2001).
The following result which is essentially due to Epstein (Epstein, 1961) can also be found in (Kilgour and Pride, 1996).

Theorem 3.3. *Let $\mathcal{P}$ be a group presentation as in* (1.2)*. Then $\mathcal{P}$ is efficient if and only if it is p-Cockcroft for some prime p.*

By (Bogley and Pride, 1993) (and (Pride, 1991)), there is an embedding μ of $\pi_2(\mathcal{P})$ into the free module $\underset{R\in\mathbf{r}}{\oplus}\mathbb{Z}Ge_R$ defined as follows. Let $\langle\mathbb{P}\rangle \in \pi_2(\mathcal{P})$ and suppose that $\mathbb{P}$ has discs $\triangle_1, \triangle_2, ..., \triangle_n$ with the labels $R_1^{\varepsilon_1}, R_2^{\varepsilon_2}, ..., R_n^{\varepsilon_n}$, respectively $(R_i \in$ **r**, $\varepsilon_i = \pm 1,\, 1 \leq i \leq n)$. Let $\gamma = (\gamma_1, ..., \gamma_n)$ be a spray for $\mathbb{P}$. Also let $W(\gamma_i)$ be the label on γ_i which represents an element of G. Then

$$\mu(\langle\mathbb{P}\rangle) = \sum_{i=1}^{n}\varepsilon_i\overline{W(\gamma_i)}e_{R_i}.$$

For each spherical picture $\mathbb{P}$ over $\mathcal{P}$ and for each $R \in \mathbf{r}$, let $\lambda_{\mathbb{P},R}$ be the coefficients of e_R in $\mu(\langle\mathbb{P}\rangle)$. Let $I_2(\mathcal{P})$ be the 2-sided ideal in $\mathbb{Z}G$ generated by the set

$$\{\lambda_{\mathbb{P},R} : \mathbb{P} \text{ is a spherical picture, } R \in \mathbf{r}\}.$$

This ideal is called the *second Fox ideal* of $\mathcal{P}$. The concept of Fox ideals has been discussed in (Lustig, 1993).
Now suppose **X** is a collection of spherical pictures over $\mathcal{P}$. Then, by (Pride, 1991), one can define the certain operations on spherical pictures. Allowing this operations lead to the notion of *equivalence (rel **X**) of spherical pictures*. Then, again by (Pride, 1991), *the elements $\langle\mathbb{P}\rangle$ ($\mathbb{P}$ ∈**X**) generate $\pi_2(\mathcal{P})$ as a module if and only if every spherical picture is equivalent (rel **X**) to the empty picture*. If the elements $\langle\mathbb{P}\rangle$ ($\mathbb{P}$ ∈**X**) generate $\pi_2(\mathcal{P})$ then we say that **X** generates $\pi_2(\mathcal{P})$. Moreover if **X** is a set of generating pictures, then $I_2(\mathcal{P})$ is generated by $\{\lambda_{\mathbb{P},R} : \mathbb{P}$ ∈**X**, $R \in$ **r**$\}$.
The next result, due to Lustig (Lustig, 1993) (see also (Kilgour and Pride, 1996)) gives a method of showing that a presentation is minimal.

Theorem 3.4 (Lustig)**.** *Let G be a group with a presentation $\mathcal{P}$. If there is a ring homomorphism ϕ from $\mathbb{Z}G$ into the matrix ring of all $k \times k$-matrices* $(k \geq 1)$ *over some commutative ring A with 1, such that $\phi(1) = 1$, and if ϕ maps the second Fox ideal $I_2(\mathcal{P})$ to 0, then $\mathcal{P}$ is minimal.*

Proof of Theorem 3.1

Let us take the presentation $\mathcal{P}'_{\overline{H}(\lambda_q)}$ as in (3.1). By (Baik, 1992) and (Pride, 1991), the generating pictures of $\mathcal{P}'_{\overline{H}(\lambda_q)}$ can be defined as in Figure 1. In these pictures, we have

$$\exp_{x^2}(\mathbb{P}_1) = \exp_{y^q}(\mathbb{P}_2) = \exp_{r^2}(\mathbb{P}_3) = \exp_{x^2}(\mathbb{P}_4) = \exp_{r^2}(\mathbb{P}_5) = 1 - 1 = 0$$

and

$$\exp_{[x,r]}(\mathbb{P}_4) = 2 = -\exp_{[x,r]}(\mathbb{P}_5),\;\; \exp_{y^q}(\mathbb{P}_6) = q - 2 \text{ and } \exp_{yry^{q-1}r^{-1}}(\mathbb{P}_6) = q.$$

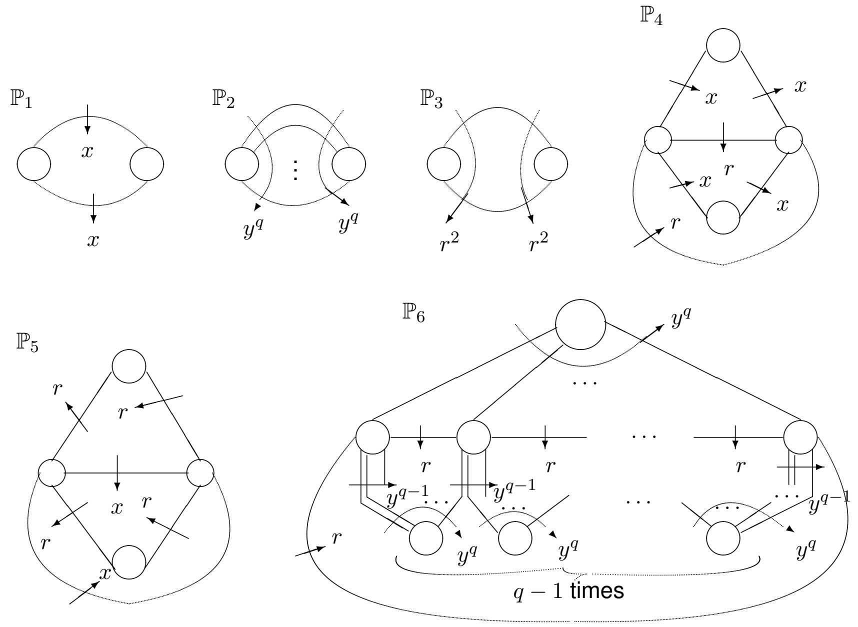

Figure 1

Now if q (for ≥ 3) is an *even* positive integer, that is $(q,2) \neq 1$, then we always have $\mathcal{P}'_{\overline{H}(\lambda_q)}$ is 2-Cockcroft and so, by Theorem 3.3, it is efficient. Otherwise, if q is an *odd* positive integer then we get that $(q-2,q)=1$ so $\mathcal{P}'_{\overline{H}(\lambda_q)}$ can not be p-Cockcroft for any prime p or 0, and then not be efficient.

By considering the pictures, as depicted in Figure 1, the converse part of the efficiency case is quite clear.

This part of the proof, we will conclude that $\mathcal{P}'_{\overline{H}(\lambda_q)}$ is efficient on 3-generators. To do that we use Remark 3.1. In other words, we will show that $\mathcal{P}'_{\overline{H}(\lambda_q)}$ is minimal when $(q,2) \neq 1$ and then we can say that the group $\overline{H}(\lambda_q)$ with presentation $\mathcal{P}'_{\overline{H}(\lambda_q)}$ is efficient on just 3-generators.

By the pictures shown in Figure 1, $I_2(\mathcal{P}'_{\overline{H}(\lambda_q)})$ is generated, as a 2-sided ideals, by the set

$$\begin{aligned} N &= \{1-\overline{x},\ 1-\overline{y},\ 1-\overline{r},\ -1+\overline{r},\ 1+\overline{x},\ -1+\overline{x},\ \overline{r}+\overline{r}^2, \\ &\quad (q-1)\overline{r}-1,\ 1+\overline{y}+\overline{y}^2+...+\overline{y}^{q-1}\}. \end{aligned}$$

Let $\langle c \rangle$ be an infinite cyclic group and consider the ring homomorphism

$$\mathbb{Z}\overline{H}(\lambda_q) \to \mathbb{Z}\langle c \rangle$$

arising from the group homomorphism defined by

$$r \to 1, \quad y \to 1, \quad x \to c.$$

If we consider

$$\mathbb{Z}\langle c \rangle \to \mathbb{Z}_2$$

by sending all integer coefficients to their congruence modulo 2 and sending c to the just congruence class of 0 in $\mathbb{Z}_2$. Then the mapping

$$\mathbb{Z}\overline{H}(\lambda_q) \rightarrow \mathbb{Z}\langle c\rangle \rightarrow \mathbb{Z}_2$$

sends N to 0 and 1 to 1. Hence, by Theorem 3.4, $\overline{H}(\lambda_q)$ is minimal, in other words,

$$def(\mathcal{P}'_{\overline{H}(\lambda_q)}) = def(\overline{H}(\lambda_q)),$$

as required. $\diamondsuit$

By taking $q = 3$, we get the modular group M with a presentation $\mathcal{P}_M = \langle x, y; x^2, y^3\rangle$ and then the semi-direct product of M by $\mathbb{Z}_2$ gives that the extended modular group $\overline{M}$ with a presentation

$$\mathcal{P}_{\overline{M}} = \langle x, y, r\ ;\ x^2, y^3, r^2, (xr)^2, (yr)^2\rangle. \quad (3.2)$$

Then, as a consequence of Theorem 3.1, we have

Corollary 3.5. *The extended modular group $\overline{M}$ with a presentation $\mathcal{P}_{\overline{M}}$ as in (3.2) is always inefficient but not minimal.*

Remark 3.2. By using the above corollary, we suspect but can not prove that there is still some chance either to get an efficient presentation for the extended modular group $\overline{M}$ or to show that the presentation (3.2) is always minimal and then there is no efficient presentation for the group $\overline{M}$.

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