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POWER SUBGROUPS OF THE EXTENDED HECKE GROUPS

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Abstract. We consider the extended Hecke groups $\overline{H}(\lambda_q)$ generated by $T(z) = -1\ /\ z$, $S(z) = -1/(z+\lambda_q)$ and $R(z) = 1/\ \overline{z}$ with $\lambda_q = 2\cos(\pi/q)$ for $q \geq 3$ integer. In this article, we study the abstract group structures of the power subgroups $\overline{H}^m(\lambda_q)$ of $\overline{H}(\lambda_q)$ for each positive integer m. Then, we give the relations between commutator subgroups and power subgroups.

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1. INTRODUCTION

In [4], Erich Hecke introduced the groups $H(\lambda)$ generated by two linear fractional transformations

$$T(z) = -\frac{1}{z} \text{ and } S(z) = -\frac{1}{z+\lambda},$$

where λ is a fixed positive real number.

E. Hecke showed that $H(\lambda)$ is discrete if and only if $\lambda = \lambda_q = 2\cos\frac{\pi}{q}$, $q \geq 3$ integer, or $\lambda \geq 2$. We will focus on the discrete case with $\lambda < 2$, i.e., those with $\lambda = \lambda_q$, q an integer ≥ 3. These groups have come to be known as the *Hecke Groups*, and we will denote them $H(\lambda_q)$ for $q \geq 3$. The Hecke group $H(\lambda_q)$ is isomorphic to the free product of two finite cyclic groups of orders 2 and q and it has a presentation

$$H(\lambda_q) = < T, S \mid T^2 = S^q = I > \cong C_2 * C_q.$$

Also $H(\lambda_q)$ has the signature $(0; 2, q, \infty)$, that is, all the groups $H(\lambda_q)$ are triangle groups. The first several of these groups are $H(\lambda_3) = \Gamma = PSL(2,\mathbb{Z})$ (the modular group), $H(\lambda_4) = H(\sqrt{2})$, $H(\lambda_5) = H(\frac{1+\sqrt{5}}{2})$, and $H(\lambda_6) = H(\sqrt{3})$. It is clear that $H(\lambda_q) \subset PSL(2,\mathbb{Z}[\lambda_q])$, for $q \geq 4$. The groups $H(\sqrt{2})$ and $H(\sqrt{3})$ are of particular interest, since they are the only Hecke groups, aside from the modular group, whose elements are completely known.

The extended Hecke group, denoted by $\overline{H}(\lambda_q)$, has been defined by adding the reflection $R(z) = 1/\overline{z}$ to the generators of the Hecke group $H(\lambda_q)$, for $q \geq 3$ integer, in [9, 10] and [6]. Thus, the extended Hecke group $\overline{H}(\lambda_q)$ has a presentation,

$$\overline{H}(\lambda_q) =< T,S,R \mid T^2 = S^q = R^2 = I, RT = TR, RS = S^{q-1}R >\cong D_2 *_{C_2} D_q. \tag{1.1}$$

The Hecke group $H(\lambda_q)$ is a subgroup of index 2 in $\overline{H}(\lambda_q)$.

Now we give some information about the power subgroups of $\overline{H}(\lambda_q)$.

Let m be a positive integer. Let us define $\overline{H}^m(\lambda_q)$ to be the subgroup generated by the m^{th} powers of all elements of $\overline{H}(\lambda_q)$. The subgroup $\overline{H}^m(\lambda_q)$ is called the $m-th$ power subgroup of $\overline{H}(\lambda_q)$. As fully invariant subgroups, they are normal in $\overline{H}(\lambda_q)$.

From the definition one can easily deduce that

$$\overline{H}^{mk}(\lambda_q) \leq \overline{H}^m(\lambda_q)$$

and

$$\overline{H}^{mk}(\lambda_q) \leq \left(\overline{H}^m(\lambda_q)\right)^k.$$

Using the last two inequalities imply that $\overline{H}^m(\lambda_q).\overline{H}^k(\lambda_q) = \overline{H}^{(m,k)}(\lambda_q)$ where (m,k) denotes the greatest common diviser of m and k.

The power subgroups of the Hecke groups $H(\lambda_q)$ have been studied and classified in [2,3] and [5]. For $q \geq 3$ prime, the power subgroups of the extended Hecke groups $\overline{H}(\lambda_q)$ were studied by Sahin, Ikikardes and Koruoğlu in [11, 12] and [13].

The aim of this paper is to study the power subgroups $\overline{H}^m(\lambda_q)$ of the extended Hecke groups $\overline{H}(\lambda_q)$, $q \geq 3$ integer. For each positive integer m, we determine the abstract group structures and generators of $\overline{H}^m(\lambda_q)$. Also, we give the signatures of $\overline{H}^m(\lambda_q)$. To get all these results, we use the techniques of combinatorial group theory (Reidemeister-Schreier method, permutation method and Riemann-Hurwitz formula). Finally, we give the relations between commutator subgroups and power subgroups.

2. The group structure of power subgroups of $\overline{H}(\lambda_q)$

Now we consider the presentation of the extended Hecke group $\overline{H}(\lambda_q)$ given in (1.1):

$$\overline{H}(\lambda_q) =< T,S,R \mid T^2 = S^q = R^2 = I, TR = RT, RS = S^{-1}R >$$

Firstly, we find a presentation for the quotient $\overline{H}(\lambda_q)/\overline{H}^m(\lambda_q)$ by adding the relation $X^m = I$ for all $X \in \overline{H}(\lambda_q)$ to the presentation of $\overline{H}(\lambda_q)$. The order of $\overline{H}(\lambda_q)/\overline{H}^m(\lambda_q)$ gives us the index. We have,

$$\begin{aligned}\overline{H}(\lambda_q)/\overline{H}^m(\lambda_q) <T,S,R \mid T^2 = S^q = R^2 = (TR)^2 = (RS)^2 = T^m = S^m \\ = R^m = (TS)^m = (RS)^m = (TR)^m = \ldots = I > .\end{aligned} \tag{2.1}$$

Thus we use the Reidemeister-Schreier process to find the generators and the presentations of the power subgroups $\overline{H}^m(\lambda_q)$, $q \geq 3$ integer (for the method, please see [2] and [5]).

Firstly, we now discuss the group theoretical structure of these subgroups for $q \geq 3$ odd integer. We start with the case $m = 2$.

Theorem 1. 1) *If $q \geq 3$ is an odd integer, then $\overline{H}^2(\lambda_q)$ is the free product of two finite cyclic groups of order q, i.e.,*

$$\overline{H}^2(\lambda_q) = \langle S, TST \mid S^q = (TST)^q = I \rangle \cong C_q * C_q.$$

2) *If $q > 3$ is an even integer, then $\overline{H}^2(\lambda_q)$ is the free product of the infinite cyclic group and two finite cyclic groups of order $q/2$, i.e.,*

$$\begin{aligned}\overline{H}^2(\lambda_q) = &\left\langle S^2, TS^2T, TSTS^{-1} \mid (S^2)^{q/2} = \left(TS^2T\right)^{q/2} = (TSTS^{-1})^{\infty} = I \right\rangle,\\ &\cong C_{q/2} * C_{q/2} * \mathbb{Z}.\end{aligned}$$

Proof. 1) By (2.1), we have

$$\overline{H}(\lambda_q)/\overline{H}^2(\lambda_q) \cong \langle T, R \mid T^2 = R^2 = (TR)^2 = I \rangle \cong C_2 \times C_2,$$

since $S^2 = S^q = I$ and $(2, q) = 1$. Now we can choose $\{I, T, R, TR\}$ as a Schreier transversal for $\overline{H}^2(\lambda_q)$. According to the Reidemeister-Schreier method (see [8]), we get the generators of $\overline{H}^2(\lambda_q)$ as the followings :

$$\begin{array}{lll} I.T.(T)^{-1} = I, & I.S.(I)^{-1} = S, & I.R.(R)^{-1} = I,\\ T.T.(I)^{-1} = I, & T.S.(T)^{-1} = TST^{-1}, & T.R.(TR)^{-1} = I,\\ R.T.(TR)^{-1} = RTRT, & R.S.(R)^{-1} = RSR^{-1}, & R.R.(I)^{-1} = I,\\ TR.T.(R)^{-1} = TRTR, & TR.S.(TR)^{-1} = TRSR^{-1}T^{-1}, & TR.R.(T)^{-1} = I. \end{array}$$

Since $TRTR = RTRT = I$, $RSR^{-1} = S^{-1}$ and $TRSR^{-1}T^{-1} = (TST)^{-1}$, the generators of $\overline{H}^2(\lambda_q)$ are S and TST. Thus $\overline{H}^2(\lambda_q)$ has a presentation

$$\overline{H}^2(\lambda_q) = \langle S, TST \mid S^q = (TST)^q = I \rangle \cong C_q * C_q.$$

Also, using the permutation method (see [14]) and the Riemann-Hurwitz formula, we get the signature of $\overline{H}^2(\lambda_q)$ as $(0; q, q, \infty) = (0; q^{(2)}, \infty)$.

2) By 2.1, the quotient group $\overline{H}(\lambda_q)/\overline{H}^2(\lambda_q)$ is

$$\begin{aligned}\overline{H}(\lambda_q)/\overline{H}^2(\lambda_q) &\cong \langle T, S, R \mid T^2 = S^2 = R^2 = (TR)^2 = (RS)^2 = (TS)^2 = I \rangle\\ &\cong C_2 \times C_2 \times C_2,\end{aligned}$$

since $S^2 = S^q = I$. Now we can choose Schreier transversal as $\{I, T, S, R, TS, TR, SR, TSR\}$. According to the Reidemeister-Schreier method, all possible products are

$$\begin{array}{ll}
I.T.(T)^{-1} = I, & TS.T.(S)^{-1} = TSTS^{-1}, \\
T.T.(I)^{-1} = I, & TR.T.(R)^{-1} = I, \\
S.T.(TS)^{-1} = STS^{-1}T, & SR.T.(TSR)^{-1} = SRTRS^{-1}T, \\
R.T.(TR)^{-1} = I, & TSR.T.(SR)^{-1} = TSRTRS^{-1}, \\
I.S.(S)^{-1} = I, & TS.S.(T)^{-1} = TS^2T, \\
T.S.(TS)^{-1} = I, & TR.S.(TSR)^{-1} = I, \\
S.S.(I)^{-1} = S^2, & SR.S.(R)^{-1} = I, \\
R.S.(SR)^{-1} = RSRS^{-1}, & TSR.S.(TR)^{-1} = TSRSRT, \\
I.R.(R)^{-1} = I, & TS.R.(TSR)^{-1} = I, \\
T.R.(TR)^{-1} = I, & TR.R.(T)^{-1} = I, \\
S.R.(SR)^{-1} = I, & SR.R.(S)^{-1} = I, \\
R.R.(I)^{-1} = I, & TSR.R.(TS)^{-1} = I,
\end{array}$$

Since $SRTRS^{-1}T = STS^{-1}T$, $TSRTRS^{-1} = TSTS^{-1}$, $TSTS^{-1} = (STS^{-1}T)^{-1}$, $RSRS^{-1} = S^{-2}$ and $TSRSRT = I$, the generators of $\overline{H}^2(\lambda_q)$ are S^2,TS^2T and $TSTS^{-1}$. Thus $\overline{H}^2(\lambda_q)$ has a presentation

$$\overline{H}^2(\lambda_q) = \left\langle S^2, TS^2T, TSTS^{-1} \mid (S^2)^{q/2} = \left(TS^2T\right)^{q/2} = (TSTS^{-1})^{\infty} = I \right\rangle.$$

Therefore $\overline{H}^2(\lambda_q)$ has the signature $\left(0; (q/2)^{(2)}, \infty^{(2)}\right)$. □

Corollary 1. 1) *If $q \geq 3$ is an odd integer and if m is a positive even integer such that $(m,q) = 1$, then $\overline{H}^m(\lambda_q) = \overline{H}^2(\lambda_q)$.*

2) *If $q \geq 3$ is an integer and if m is a positive odd integer, then $\overline{H}^m(\lambda_q) = \overline{H}(\lambda_q)$.*

Proof. 1) If $q \geq 3$ is an odd integer and if m is a positive even integer such that $(m,q) = 1$, then by (2.1), we get

$$\overline{H}(\lambda_q)/\overline{H}^m(\lambda_q) \cong < T, R \mid T^2 = R^2 = (TR)^2 = I > \cong D_2,$$

from the relations

$$R^2 = R^m = I, \ S^q = S^m = I \text{ and } T^2 = T^m = I.$$

Since $\overline{H}^2(\lambda_q)$ is the only normal subgroup of index 4, we have $\overline{H}^m(\lambda_q) = \overline{H}^2(\lambda_q)$.

2) If $q \geq 3$ is an integer and if m is a positive odd integer, then by (2.1), we obtain

$$\overline{H}^m(\lambda_q) = \overline{H}(\lambda_q),$$

from the relations

$$R^2 = R^m = I, \ T^2 = T^m = I \text{ and } (RS)^2 = (RS)^m = I.$$

□

Theorem 2. *Let $q > 3$ be an even integer and let m be a positive even integer such that $(m,q) = 2$. The normal subgroup $\overline{H}^m(\lambda_q)$ is the free product of finite cyclic groups m of order $q/2$ and the infinite cyclic group $\mathbb{Z}$, i.e.,*

$$\overline{H}^m(\lambda_q) = < \underbrace{(TS)(TS)...(TS)}_{(m-1)\ times} TS^{-1} > * < S^2 > * < TS^2T > *$$
$$< TSTS^2TS^{-1}T > * ... * < \underbrace{(TS)(TS)...(TS)}_{(m-2)\ times} TS^2T \underbrace{(S^{-1}T)(S^{-1}T)...(S^{-1}T)}_{(m-2)\ times} > .$$

Proof. By (2.1), we have

$$\overline{H}(\lambda_q)/\overline{H}^m(\lambda_q) \cong \langle T,S,R \mid T^2 = S^2 = R^2 = (TR)^2 = (RS)^2 = (TS)^m = I \rangle,$$
$$\cong C_2 \times D_m,$$

since $S^q = S^m = I$. Now we can choose $\{I,\ T,\ S,\ TS,\ TST,\ TSTS,\ ...,\ \underbrace{(TS)(TS)...(TS)}_{(m-1)\text{ times}},\ R,\ TR,\ SR,\ TSR,\ TSTR,\ TSTSR,\ ...,\ \underbrace{(TS)(TS)...(TS)}_{(m-1)\text{ times}}R\}$ as a Schreier transversal for $\overline{H}^m(\lambda_q)$. According to the Reidemeister-Schreier method, we find the generators generators of $\overline{H}^m(\lambda_q)$ as

$$a_1 = \underbrace{(TS)(TS)...(TS)}_{(m-1)\text{ times}} TS^{-1},\ a_2 = S^2,\ a_3 = TS^2T,\ a_4 = TSTS^2TS^{-1}T, ...,$$
$$a_{m+1} = \underbrace{(TS)(TS)...(TS)}_{(m-2)\text{ times}} TS^2T \underbrace{(S^{-1}T)(S^{-1}T)...(S^{-1}T)}_{(m-2)\text{ times}}.$$

Thus $\overline{H}^m(\lambda_q)$ has a presentation $\overline{H}^m(\lambda_q) =$

$$\Big\langle a_1,a_2,a_3,a_4,...,a_{m+1} \mid (a_2)^{q/2} = (a_3)^{q/2} = (a_4)^{q/2} = \cdots = (a_{m+1})^{q/2} = I \Big\rangle.$$

Also the signature of $\overline{H}^m(\lambda_q)$ is $\Big(0; (q/2)^{(m)}, \infty^{(2)}\Big)$. □

We are only left to consider the case where $(m,2) = 2$ and $(m,q) = d > 2$. In this case, the above techniques do not say much about $\overline{H}^m(\lambda_q)$. But, we can say something about $\overline{H}^m(\lambda_q)$ some special cases of q. To do these, we need the following results about the commutator subgroups of $\overline{H}(\lambda_q)$ in [9] and [10] .

Lemma 1. 1) *For an odd number $q \geq 3$:*

$$\overline{H}'(\lambda_q) = \langle S, TST \mid S^q = (TST)^q = I \rangle \cong C_q * C_q.$$

2) $\overline{H}'(\lambda_q)/\ \overline{H}''(\lambda_q) = < S, TST \mid S^q = (TST)^q = (S.TST)^q = I > \cong C_q \times C_q$.
3) *For an even integer $q > 3$:*

$$\overline{H}'(\lambda_q) = < S^2, TS^2T, TSTS^{-1} \mid (S^2)^{q/2} = \left(TS^2T\right)^{q/2} = (TSTS^{-1})^{\infty} = I >$$
$$\cong C_{q/2} * C_{q/2} * \mathbb{Z}.$$

By using Lemma 1 and Theorem 1 we get the following

Corollary 2. *Let $q \geq 3$ be an integer. Then $\overline{H}'(\lambda_q) \cong \overline{H}^2(\lambda_q)$.*

Theorem 3. *Let $q \geq 3$ be an odd integer and let m be a positive integer. The groups $\overline{H}^{2q}(\lambda_q)$ are the subgroups of the second commutator subgroup $\overline{H}''(\lambda_q)$.*

Proof. Since $\overline{H}'(\lambda_q) = \overline{H}^2(\lambda_q)$, we get $\left(\overline{H}^2(\lambda_q)\right)^q \leq \overline{H}^2(\lambda_q)$ and $\left(\overline{H}'(\lambda_q)\right)^q \leq \overline{H}'(\lambda_q)$. Also we know that

$$\overline{H}'(\lambda_q)/\left(\overline{H}'(\lambda_q)\right)^q = < S, TST \mid S^q = (TST)^q = (S.TST)^q = \cdots = I > .$$

Therefore the index $\left|\overline{H}'(\lambda_q) : \left(\overline{H}'(\lambda_q)\right)^q\right|$ is greater than or equal to the index $\left|\overline{H}'(\lambda_q) : \overline{H}''(\lambda_q)\right| = q^2$. Thus we have

$$\overline{H}^{2q}(\lambda_q) \subset \overline{H}''(\lambda_q).$$

□

By means of this results, we are going to be able to investigate the subgroups $\overline{H}^{2qm}(\lambda_q)$. We have by Schreier's theorem the following theorem:

Theorem 4. *Let $q \geq 3$ be an odd integer. The groups $\overline{H}^{2qm}(\lambda_q)$ are free.*

Finally, we can only say something the case $q = 4$. This Hecke group is very important and studied by many authors, see [1] and [7].

Theorem 5. *i)* $\left|\overline{H}^2(\lambda_4) : (\overline{H}^2)^2(\lambda_4)\right| = 8.$
ii) The group $(\overline{H}^2)^2(\lambda_4)$ is a free group of rank 9.

Proof. i) If we take $k_1 = S^2$, $k_2 = TS^2T$ and $k_3 = TSTS^3$, then the quotient group $\overline{H}^2(\lambda_4)/(\overline{H}^2)^2(\lambda_4)$ is the group obtained by adding the relation $k_i^2 = I$ to the relations of $(\overline{H}^2)^2(\lambda_4)$, for $i \in \{1,2,3\}$. Thus we have

$$\overline{H}^2(\lambda_4)/(\overline{H}^2)^2(\lambda_4) \cong C_2 \times C_2 \times C_2.$$

Therefore, we obtain $\left|\overline{H}^2(\lambda_4) : (\overline{H}^2)^2(\lambda_4)\right| = 8.$

ii) Let $\Sigma = \{I, k_1, k_2, k_3, k_1k_2, k_1k_3, k_2k_3, k_1k_2k_3\}$ be a Schreier transversal for $(\overline{H}^2)^2(\lambda_4)$. Using the Reidemeister-Schreier method, we obtain the generators of $(\overline{H}^2)^2(\lambda_4)$ as follows:

$I.k_1.(k_1)^{-1} = I,$
$k_1.k_1.(I)^{-1} = I,$
$k_2.k_1.(k_1k_2)^{-1} = k_2k_1k_2k_1,$
$k_3.k_1.(k_1k_3)^{-1} = k_3k_1k_3^{-1}k_1,$
$k_1k_2.k_1.(k_2)^{-1} = k_1k_2k_1k_2,$
$k_1k_3.k_1.(k_3)^{-1} = k_1k_3k_1k_3^{-1},$
$k_2k_3.k_1.(k_1k_2k_3)^{-1} = k_2k_3k_1k_3^{-1}k_2k_1,$
$k_1k_2k_3.k_1.(k_2k_3)^{-1} = k_1k_2k_3k_1k_3^{-1}k_2,$

$I.k_2.(k_2)^{-1} = I,$
$k_1.k_2.(k_1k_2)^{-1} = I,$
$k_2.k_2.(I)^{-1} = I,$
$k_3.k_2.(k_2k_3)^{-1} = k_3k_2k_3^{-1}k_2,$
$k_1k_2.k_2.(k_1)^{-1} = I,$
$k_1k_3.k_2.(k_1k_2k_3)^{-1} = k_1k_3k_2k_3^{-1}k_2k_1,$
$k_2k_3.k_2.(k_3)^{-1} = k_2k_3k_2k_3^{-1},$
$k_1k_2k_3.k_2.(k_1k_3)^{-1} = k_1k_2k_3k_2k_3^{-1}k_1,$

$I.k_3.(k_3)^{-1} = I,$
$k_1.k_3.(k_1k_3)^{-1} = I,$
$k_2.k_3.(k_2k_3)^{-1} = I,$
$k_3.k_3.(I)^{-1} = k_3^2,$
$k_1k_2.k_3.(k_1k_2k_3)^{-1} = I,$
$k_1k_3.k_3.(k_1)^{-1} = k_1k_3^2k_1,$
$k_2k_3.k_3.(k_2)^{-1} = k_2k_3^2k_2,$
$k_1k_2k_3.k_3.(k_1k_2)^{-1} = k_1k_2k_3^2k_2k_1.$

After some calculations, we get the generators of $(\overline{H}^2)^2(\lambda_4)$ *as*

$$\begin{array}{lll} k_1k_2k_3k_2k_3^{-1}k_1, & k_1k_2k_1k_2, & k_3^2, \\ k_1k_2k_3k_1k_3^{-1}k_2, & k_1k_3k_1k_3^{-1}, & k_1k_3^2k_1, \\ k_1k_2k_3^2k_2k_1, & k_2k_3k_2k_3^{-1}, & k_2k_3^2k_2. \end{array}$$

Also, we find the signature of $(\overline{H}^2)^2(\lambda_4)$ *as* $(1; \underbrace{\infty, \infty, \cdots, \infty}_{8 \text{ times}}) = (1; \infty^{(8)})$. □

Notice that the group $(\overline{H}^2)^2(\lambda_4) = (H^2)^2(\lambda_4)$ is the principal congruence subgroup $H_4(\lambda_4)$ of $H(\lambda_4)$.

Since $\overline{H}^{4k}(\lambda_4) \leq \overline{H}^4(\lambda_4) \leq (\overline{H}^2)^2(\lambda_4)$, we are going to be able to investigate the subgroups $\overline{H}^{4k}(\lambda_4)$. We have by Schreier's theorem the following theorem:

Corollary 3. *The groups* $\overline{H}^{4k}(\lambda_4)$ *are free.*

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