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4 authors:

Sebahattin Ikikardes
Balikesir University

Özden Koruoğlu
Balikesir University

Recep Sahin
Balikesir University

Ismail naci Cangul
Uludag University

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S. Ikikardes, Ö. Koruoglu, R. Sahin and I. N. Cangül

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ONE RELATOR QUOTIENTS OF THE EXTENDED MODULAR GROUP[1]

by
Sebahattin İkikardes*, Özden Koruoğlu, Recep Sahin*,**
and
Ismail Naci Cangül***

*Balıkesir Üniversitesi, Fen-Edebiyat Fakültesi, Matematik Bölümü
10145 Çağış Kampüsü, Balıkesir, Turkey
skardes@balikesir.edu.tr & rsahin@balikesir.edu.tr

**Balıkesir Üniversitesi, Necatibey Eğitim Fakültesi, İlköğretim Bölümü,
Matematik Eğitimi, 10100 Balıkesir, Turkey
ozdenk@balikesir.edu.tr

***Uludag Üniversitesi, Fen-Edebiyat Fakültesi, Matematik Bölümü,
16059, Bursa, Turkey
cangul@uludag.edu.tr

Abstract

In this paper, we obtain one relator quotients of the extended modular group $\overline{\Gamma}$ by adding an extra relation to the existing two relations. Then, we show that some of one-relator quotients of $\overline{\Gamma}$ are M^*-groups.

1. Introduction

The modular group $\Gamma = PSL(2,\mathbb{Z})$ is the discrete subgroup of $PSL(2,\mathbb{R})$ generated by two linear fractional transformations

$$t(z) = -\frac{1}{z} \qquad \text{and} \qquad s(z) = -\frac{1}{z+1},$$

and it has a presentation

$$\Gamma = < t, s \mid t^2 = s^3 = I \ > \cong C_2 * C_3.$$

The extended modular group $\overline{\Gamma} = PGL(2,\mathbb{Z})$ is defined by adding the reflection $r(z) = 1/\bar{z}$ to the generators of the modular group Γ and it has a presentation

$$\overline{\Gamma} = < t, s, r \mid t^2 = s^3 = r^2 = I, \ rt = tr, rs = s^{-1}r >,$$

or equivalently,

$$\overline{\Gamma} = < t, s, r \mid t^2 = s^3 = r^2 = (tr)^2 = (rs)^2 = I > \cong D_2 *_{C_2} D_3.$$

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The modular group Γ is a subgroup of index 2 in $\overline{\Gamma}$, (see, [1] and [7]).

In [5], [6] and [20], the authors studied one relator quotients of the modular group Γ by adding extra relations to the existing two relations. In the relevant literature, there is a huge number of papers on one-relator quotients of free products and free products with amalgamation. For examples of these studies, see [9] and [11]. Also, the book [10] provides an excellent reference for the work on one-relator products.

In this paper, we obtain one relator quotients of the extended modular group $\overline{\Gamma}$ by adding an extra relation to the existing relations. We came across an interesting case when we were obtaining one relator quotients. Some of one-relator quotients of the extended modular group $\overline{\Gamma}$ are the M^*−groups. M^*-groups were studied by May [15] first. Then these groups were investigated intensively [3, 4, 8, 12, 14−19]. The article in [2] contains a nice survey of known results about M^*−groups. Now, we briefly mention about the M^*-groups.

Let X be a compact bordered Klein Surface of algebraic genus $g \geq 2$. May proved in [14] that the automorphism group G of X is finite, and the order of G is at most $12(g-1)$. Groups isomorphic to the automorphism group of such a compact bordered Klein surface with this maximal number of automorphisms are called M^*-groups. Thus a finite group G is called an M^*-group if it is generated by three distinct non-trivial elements α, β, γ which satisfy the relations

$$\alpha^2 = \beta^2 = \gamma^2 = (\alpha\beta)^2 = (\alpha\gamma)^3 = I$$

and other relations which make the group finite [15]. In [12, p.277], Greenleaf and May proved that there is a relationship between the extended modular group and M^*-groups which says a finite group of order at least twelve is an M^*-group if and only if it is the homomorphic image of the extended modular group. In fact, by using known results about normal subgroups of the extended modular group, they found an infinite family of M^*−groups. Thus, if the order of one relator quotient group of the extended modular group $\overline{\Gamma}$ is greater than 6, then the homomorphic image of it is an M^*-group [12, p.277]. From [15, p.7], we give a table of M^*−groups of low order,

$C_2 \times S_3$	A_5
S_4	$C_2 \times A_5$
$S_3 \times S_3$	$S_3 \times S_4$
$C_2 \times S_4$	$C_2 \times C_2 \times A_5$

Table I

where, S_n denotes the symmetric group on n letters, A_n is the alternating group on n letters, and C_n stands for the cyclic group of order n. Therefore, one relator quotients of the extended modular group $\overline{\Gamma}$ are M^*-groups if they obey the group structures given in the Table I. Such results are important in the study of bordered Klein surfaces.

In this paper, we deal with the general problem : What are the possible homomorphic images of $\overline{\Gamma}$, the kernels of which are normally generated by a single non-trivial element

R? Since the modular group Γ is a subgroup of index 2 in $\overline{\Gamma}$, every element R in $\overline{\Gamma}\setminus\Gamma$ can be uniquely written as $R = ur$, where $u \in \Gamma$ and r is an element of order 2 outside Γ such that $\overline{\Gamma}$ is the semidirect product of Γ and $< r >$.

The first main statement of the work is that if $R = sr$ then $\overline{\Gamma}$ has only one homomorphic image, (up to isomorphism) which is C_2, the cycle group of order 2.

The second main statement is that (as a corollary of the first statement) if $R \in \Gamma$ then the corresponding images are the semidirect products of $\Gamma / < R >$ and C_2, where $< R >$ is the normal closure of R in Γ.

2. One Relator Quotients of $\overline{\Gamma}$

The extended Modular group $\overline{\Gamma}$ has a presentation

$$\overline{\Gamma} = < t, s, r : t^2 = s^3 = r^2 = (tr)^2 = (sr)^2 = I > .$$

We now add an extra relation

$$w = R(t, s, r) = I$$

where $R(s, t, r)$ is a cyclically reduced word of the form

$$R_1(t, s, r) = ts^{\varepsilon_1} r t s^{\varepsilon_2} r ... t s^{\varepsilon_n} r \tag{2.1}$$

or

$$R_2(t, s, r) = t r s^{\varepsilon_1} t r s^{\varepsilon_2} ... t r s^{\varepsilon_n} \tag{2.2}$$

with $1 \leq \varepsilon_i \leq 2$. Here, the two expressions for $R(t, s, r)$ represent the same relation. Since $I = (tr)^2 = trtr = trt^{-1}r^{-1}$, we have $tr = rt$. Therefore, $r(R_1)r^{-1} = R_2$, where R_1 and R_2 are the expressions given in (2.1) and (2.2), respectively. Therefore, it is enough to consider R_1.

Also, since $\Gamma = \Gamma . < r >$ with $\Gamma \lhd \overline{\Gamma}$ and $|r| = 2$, every element of $\overline{\Gamma}$is either in Γ or has form gr, for a uniquely defined element $g \in \Gamma$. In this work the case when $w = gr$, $g \in \Gamma$ is considered. Since Γ is a free product,

$$w = R(t, s, r) = gr = (ts^{\varepsilon_1} t s^{\varepsilon_2} ... t s^{\varepsilon_n})r.$$

We denote the number of t's, r's and s's in w by $e_t(w)$, $e_r(w)$ and $e_s(w)$, respectively. Here note that $e_s(w) = \sum_{i=1}^{n} \varepsilon_i$. It is clear that $e_r(w) = 1$ or $e_r(w) = 0$. If $e_t(w) = 0$, then $1 \leq e_s(w) \leq 2$ and if $e_t(w) = n$, then $n \leq e_s(w) \leq 2n$.

We define $M_{k,l,m}$ as the total number of words w with $e_t(w) = k$, $e_s(w) = l$ and $e_r(w) = m = 0$. Then it follows that the number of normal forms is

$$\alpha_{k,l,m} = \begin{pmatrix} k \\ l-k \end{pmatrix}. \tag{2.3}$$

The following formula for the number of all normal forms having $e_t(w) = k$, $e_s(w) = l$ and $e_r(w) = m = 0$ is given in [20] as follows:

$$M_{k,l,0} = \frac{1}{k} \sum_{d|(k,l-k)} \left[\varphi(d) \begin{pmatrix} k/d \\ (l-k)/d \end{pmatrix} \right]. \tag{2.4}$$

By mean the formula (2.4), we obtain $M_{n,n,0} = M_{n,n+1,0} = M_{n,2n,0} = M_{n,2n-1,0} = 1$. Also, it is clear that if n is odd, then $M_{n,n+1,0} = M_{n,2n-1,0}$, $M_{n,n+2,0} = M_{n,2n-2,0}, \cdots$, $M_{n,\frac{3n-1}{2},0} = M_{n,\frac{3n+1}{2},0}$ and if n is even, then $M_{n,n+1,0} = M_{n,2n-1,0}$, $M_{n,n+2,0} = M_{n,2n-2,0}, \cdots$, $M_{n,\frac{3n}{2}-1,0} = M_{n,\frac{3n}{2}+1,0}$. Thus, we only investigate the following two cases. The first case is $k = n$ odd and $l=\frac{3n-1}{2}$ and the second case is $k = n$ even and $l=\frac{3n}{2} - 1$. Once having this formula, we can check whether we have all words of the required property or not.

Example 2.1. *Let $k = 5$, $l = 7$ and $m = 0$. Then by the formula (2.3), the number of normal forms $\alpha_{5,7,0}$ is 10. By formula (2.4), there are only 2 non-equivalent cyclically reduced ones. These are*

$$tststs^2ts^2$$

and

$$tsts^2tststs^2.$$

We don't consider the words such as $ts^2ts^2tststs$ or $tsts^2ts^2tsts$ as both are equivelent to the first word.

Let us add the first relation

$$tststs^2ts^2 = I,$$

to $t^2 = s^3 = r^2 = (tr)^2 = (rs)^2 = I$. Then as

$$tst = ststs^2ts^2,$$

we have

$$(tst)^2 = ts^2t = I,$$

and equivalently $s^2 = I$. Then $t^2 = s^3 = s^2 = r^2 = (tr)^2 = (rs)^2 = I$ gives $t^2 = s = r^2 = (tr)^2 = (rs)^2 = I$. Also if we put $s = I$ in the first relation $tststs^2ts^2ts^2 = I$, we get $t = I$. This means that the quotient group is C_2, the cyclic group of order two.

Notice that if $k = 5$, $l = 8$ and $m = 0$, then by the formula (2.3), the number of normal forms $\alpha_{5,8,0}$ is 10. Also, by formula (2.4), there are only 2 non-equivalent cyclically reduced ones. These are $ts^2ts^2ts^2tsts$ and $ts^2ts^2tststs^2$. In fact these words are inverses of non-equivalent cyclically reduced words in the case $k = 5$, $l = 7$ and $m = 0$, since the words $(tststs^2ts^2)^{-1} = ststs^2ts^2ts^2t$ and $(tsts^2tststs^2)^{-1} = sts^2ts^2tsts^2t$ are equivalent to the words $ts^2ts^2ts^2tsts$ and $ts^2ts^2tststs^2$, respectively.

Now we give the following theorem.

Theorem 2.1. *If $1 \le k \le 7$ and $1 \le l \le 10$ and $m = 1$, then the quotient group $\overline{\Gamma}/R(t, s, r)$ is isomorphic to C_2, i.e. $\overline{\Gamma}/R(t, s, r) \cong C_2$.*

Proof. Firstly, we find a presentation for the quotient $\overline{\Gamma}/R(t,s,r)$ by adding the relation $ts^{\varepsilon_1}ts^{\varepsilon_2}...ts^{\varepsilon_n}r = I$ with $1 \le \varepsilon_i \le 2$ to the presentation of $\overline{\Gamma}$. We have

$$\overline{\Gamma}/R(t,s,r) \cong \langle t,s,r; t^2 = s^3 = r^2 = (tr)^2 = (rs)^2 = ts^{\varepsilon_1}ts^{\varepsilon_2}...ts^{\varepsilon_n}r = I \rangle.$$

Then we obtain the relations

$$t^2 = s^3 = (ts^{\varepsilon_1}ts^{\varepsilon_2}...ts^{\varepsilon_n})^2 = (s^{\varepsilon_1}ts^{\varepsilon_2}...ts^{\varepsilon_n})^2 = (ts^{\varepsilon_1}ts^{\varepsilon_2}...ts^{\varepsilon_n+1})^2 = I$$

as $r = ts^{\varepsilon_1}ts^{\varepsilon_2}...ts^{\varepsilon_n}$. Using the these relations, we find $s = I$. If k is even, we have $r = I$, since $t^2 = I$. If k is odd, we obtain the relation $t = r$. Thus, the quotient group is isomorphic to C_2, i.e.,

$$\overline{\Gamma}/R(t,s,r) \cong \langle t; t^2 = I \rangle.$$

□

Example 2.2. *Let $R(t,s,r) = tststs^2tsts^2ts^2r$. Here $k = 6$, $l = 9$ and $m = 1$. Then we get a presentation for the quotient group*

$$\overline{\Gamma}/R(t,s,r) \cong \langle t,s,r; t^2 = s^3 = r^2 = (tr)^2 = (rs)^2 = tststs^2tsts^2ts^2r = I \rangle.$$

Thus we find the relations

$$t^2 = s^3 = (tststs^2tsts^2ts^2)^2 = (ststs^2tsts^2ts^2)^2 = (tststs^2tsts^2t)^2 = I$$

as $r = tststs^2tsts^2ts^2$. Using $(tststs^2tsts^2t)^2 = I$, we obtain $tststs^2tsts^2t.tststs^2tsts^2t = tststs^2ts^2ts^2tsts^2t = I$. Then we have $sts^2ts^2ts^2ts = I$ and $sts^2ts^2ts = s^2ts^2$. If we put the last relation to $(tststs^2tsts^2ts^2)^2 = tststs^2tsts^2ts^2tststs^2tsts^2ts^2 = I$, we find $tststs^2ts^2ts^2tsts^2tsts^2ts^2 = I$. Using $tststs^2ts^2ts^2tsts^2t = I$ and $tststs^2ts^2ts^2tsts^2tsts^2ts^2 = I$, we obtain $sts^2ts^2 = I$. Furthermore, we get $s^2 = I$ and so $s = I$. If we write $s = I$ in $R(t,s,r)$, we have $r = I$, since k is even and $t^2 = I$. Finally, we find a presentation for the quotient group $\overline{\Gamma}/R(t,s,r) \cong \langle t; t^2 = I \rangle \cong C_2$.

Now we can give the following result.

Corollary 2.2. *If $1 \le k \le 7$ and $1 \le l \le 10$ and $m = 0$, then the quotient group $\overline{\Gamma}/R(t,s,r)$ is isomorphic to $C_2 \rtimes \Gamma/R(t,s)$.*

Proof. Since $m = 0$, there is no r in the word $R(t,s,r)$. Thus $R(t,s,r) = R(t,s)$. Also the extended modular $\overline{\Gamma}$ is the semidirect product of Γ and C_2 (see, [1]), i.e. $\overline{\Gamma} \cong C_2 \rtimes \Gamma$. Since $\overline{\Gamma}/R(t,s,r) \cong \overline{\Gamma}/R(t,s)$, we have $\overline{\Gamma}/R(t,s,r) \cong C_2 \rtimes \Gamma/R(t,s)$. □

Here, we consider all possibilities for $k \le 7$, $l \le 14$ and $m = 0$. Also, for $m = 1$, we consider the cases $k = 0$ and $l \le 2$ or $k = 1$ and $l = 0$, seperately. Thus we identify the quotient as being either finite (give its structure) or infinite.

One-relator quotients of the modular group have been studied by Conder in [5] for $2 \le k \le 12$ and $2 \le l \le 24$ and by Ulutas and Cangul in [20] for $1 \le k \le 7$ and $1 \le l \le 10$. Thus, if $k \ge 1$, $l \ge 1$ and $m = 0$, then the quotient group $\overline{\Gamma}/R(t,s,r)$ can be easily found using the their results.

Notation for the groups themselves is standard, except that we let $X \sim Y$ denote an extension of the group X by the group Y, that is, a group Z with a normal subgroup X

such that $Z/X \cong Y$. Such an extension is not necessarily a split extension, nor do we claim it to be unique.

Note that one relator quotients of the extended modular group $\overline{\Gamma}$ in the Table II are M^*-groups if they obey the given in the Table I group structures. Also in [8], it was shown that A_n is an M^*-group, for all but finitely many values of n. It was also shown in [4] that $C_2 \times A_n$ is an M^*-group. Thus the ones in these forms are M^*-groups.

k	l	m	Possible Cyclically Reduced Words	Presentation of Quotient Group	Abstract Structure of Quotient Group
0	0	1	r	$\langle t,s,r; t^2=s^3=r^2=(tr)^2=(rs)^2=r=I \rangle$	C_2
0	1	0	s	$\langle t,s,r; t^2=s^3=r^2=(tr)^2=(rs)^2=s=I \rangle$	D_2
0	1	1	sr	$\langle t,s,r; t^2=s^3=r^2=(tr)^2=(rs)^2=sr=I \rangle$	C_2
0	2	0	s^2	$\langle t,s,r; t^2=s^3=r^2=(tr)^2=(rs)^2=s^2=I \rangle$	D_2
0	2	1	s^2r	$\langle t,s,r; t^2=s^3=r^2=(tr)^2=(rs)^2=s^2r=I \rangle$	C_2
1	0	0	t	$\langle t,s,r; t^2=s^3=r^2=(tr)^2=(rs)^2=t=I \rangle$	S_3
1	0	1	tr	$\langle t,s,r; t^2=s^3=r^2=(tr)^2=(rs)^2=tr=I \rangle$	S_3
1	1	0	ts	$\langle t,s,r; t^2=s^3=r^2=(tr)^2=(rs)^2=ts=I \rangle$	C_2
1	2	0	ts^2	$\langle t,s,r; t^2=s^3=r^2=(tr)^2=(rs)^2=ts^2=I \rangle$	C_2
2	2	0	$tsts$	$\langle t,s,r; t^2=s^3=r^2=(tr)^2=(rs)^2=tsts=I \rangle$	$C_2 \rtimes S_3$
2	3	0	$tsts^2$	$\langle t,s,r; t^2=s^3=r^2=(tr)^2=(rs)^2=tsts^2=I \rangle$	$C_2 \rtimes C_6$
3	3	0	$tststs$	$\langle t,s,r; t^2=s^3=r^2=(tr)^2=(rs)^2=tststs=I \rangle$	$C_2 \rtimes A_4$
3	4	0	$tststs^2$	$\langle t,s,r; t^2=s^3=r^2=(tr)^2=(rs)^2=tststs^2=I \rangle$	C_2
4	4	0	$tstststs$	$\langle t,s,r; t^2=s^3=r^2=(tr)^2=(rs)^2=tstststs=I \rangle$	$C_2 \rtimes S_4$
4	5	0	$tstststs^2$	$\langle t,s,r; t^2=s^3=r^2=(tr)^2=(rs)^2=tstststs^2=I \rangle$	D_2
4	6	0	$tststs^2ts^2$	$\langle t,s,r; t^2=s^3=r^2=(tr)^2=(rs)^2=tststs^2ts^2=I \rangle$	$S_3 \times S_3$
4	6	0	$tsts^2tsts^2$	$\langle t,s,r; t^2=s^3=r^2=(tr)^2=(rs)^2=tsts^2tsts^2=I \rangle$	$C_2 \rtimes (C_2 \times A_4)$
4	7	0	$tsts^2ts^2ts^2$	$\langle t,s,r; t^2=s^3=r^2=(tr)^2=(rs)^2=tsts^2ts^2ts^2=I \rangle$	D_2
4	8	0	$ts^2ts^2ts^2ts^2$	$\langle t,s,r; t^2=s^3=r^2=(tr)^2=(rs)^2=ts^2ts^2ts^2ts^2=I \rangle$	$C_2 \rtimes S_4$
5	5	0	$tststststs$	$\langle t,s,r; t^2=s^3=r^2=(tr)^2=(rs)^2=tststststs=I \rangle$	$C_2 \rtimes A_5$
5	6	0	$tststststs^2$	$\langle t,s,r; t^2=s^3=r^2=(tr)^2=(rs)^2=tststststs^2=I \rangle$	S_3
5	7	0	$tstststs^2ts^2$	$\langle t,s,r; t^2=s^3=r^2=(tr)^2=(rs)^2=tstststs^2ts^2=I \rangle$	C_2
5	7	0	$tststs^2tsts^2$	$\langle t,s,r; t^2=s^3=r^2=(tr)^2=(rs)^2=tststs^2tsts^2=I \rangle$	C_2
6	6	0	$tstststststs$	$\langle t,s,r; t^2=s^3=r^2=(tr)^2=(rs)^2=tstststststs=I \rangle$	*Infinite*
6	7	0	$tstststststs^2$	$\langle t,s,r; t^2=s^3=r^2=(tr)^2=(rs)^2=tstststststs^2=I \rangle$	D_2
6	8	0	$tststststs^2ts^2$	$\langle t,s,r; t^2=s^3=r^2=(tr)^2=(rs)^2=tststststs^2ts^2=I \rangle$	$C_2 \rtimes S_3$
6	8	0	$tstststs^2tsts^2$	$\langle t,s,r; t^2=s^3=r^2=(tr)^2=(rs)^2=tstststs^2tsts^2=I \rangle$	D_2
6	8	0	$tststs^2tststs^2$	$\langle t,s,r; t^2=s^3=r^2=(tr)^2=(rs)^2=tststs^2tststs^2=I \rangle$	$C_2 \rtimes (C_2 \times S_4)$
6	9	0	$tstststs^2ts^2ts^2$	$\langle t,s,r; t^2=s^3=r^2=(tr)^2=(rs)^2=tstststs^2ts^2ts^2=I \rangle$	$C_2 \rtimes (C_4 \sim A_4)$
6	9	0	$tststs^2tsts^2ts^2$	$\langle t,s,r; t^2=s^3=r^2=(tr)^2=(rs)^2=tststs^2tsts^2ts^2=I \rangle$	*Infinite*

6	9	0	$tststs^2ts^2tsts^2$	$\langle t,s,r;t^2=s^3=r^2=(tr)^2=(rs)^2=tststs^2ts^2tsts^2=I\rangle$	$C_2\rtimes(C_7\sim C_6)$
6	9	0	$tsts^2tsts^2tsts^2$	$\langle t,s,r;t^2=s^3=r^2=(tr)^2=(rs)^2=tsts^2tsts^2tsts^2=I\rangle$	$Infinite$
7	7	0	$tststststststs$	$\langle t,s,r;t^2=s^3=r^2=(tr)^2=(rs)^2=tststststststs=I\rangle$	$Infinite$
7	8	0	$tststststststs^2$	$\langle t,s,r;t^2=s^3=r^2=(tr)^2=(rs)^2=tststststststs^2=I\rangle$	C_2
7	9	0	$tstststststs^2ts^2$	$\langle t,s,r;t^2=s^3=r^2=(tr)^2=(rs)^2=tstststststs^2ts^2=I\rangle$	S_3
7	9	0	$tststststs^2tsts^2$	$\langle t,s,r;t^2=s^3=r^2=(tr)^2=(rs)^2=tststststs^2tsts^2=I\rangle$	$C_2\rtimes A_4$
7	9	0	$tststststs^2tststs^2$	$\langle t,s,r;t^2=s^3=r^2=(tr)^2=(rs)^2=tststststs^2tststs^2=I\rangle$	S_3
7	10	0	$tststststs^2ts^2ts^2$	$\langle t,s,r;t^2=s^3=r^2=(tr)^2=(rs)^2=tststststs^2ts^2ts^2=I\rangle$	C_2
7	10	0	$tststsls^2tsts^2ts^2$	$\langle t,s,r;t^2=s^3=r^2=(tr)^2=(rs)^2=tststsls^2tsts^2ts^2=I\rangle$	C_2
7	10	0	$tststs^2ts^2tststs^2$	$\langle t,s,r;t^2=s^3=r^2=(tr)^2=(rs)^2=tststs^2ts^2tststs^2=I\rangle$	C_2
7	10	0	$tststsls^2ts^2tsts^2$	$\langle t,s,r;t^2=s^3=r^2=(tr)^2=(rs)^2=tststsls^2ts^2tsts^2=I\rangle$	C_2
7	10	0	$tststs^2tsts^2tsts^2$	$\langle t,s,r;t^2=s^3=r^2=(tr)^2=(rs)^2=tststs^2tsts^2tsts^2=I\rangle$	C_2

Table II

References

[1] F.R. Beyl and G. Rosenberger, *Efficient presentations of GL(2,ℤ) and PGL(2,ℤ)*, Proceedings of groups-St. Andrews 1985, 135–137, London Math. Soc. Lecture Note Ser., 121, Cambridge Univ. Press, Cambridge, 1986.

[2] E. Bujalance, F. J. Cirre and P. Turbek, *Groups acting on bordered Klein surfaces with maximal symmetry*, Proceedings of Groups St. Andrews 2001 in Oxford. Vol. I, 50–58, London Math. Soc. Lecture Note Ser., 304, Cambridge Univ. Press, Cambridge, 2003.

[3] E. Bujalance, F. J. Cirre and P. Turbek, *Subgroups of M^*-groups*, Q. J. Math., 54 (2003), no. 1, 49–60.

[4] E. Bujalance, F. J. Cirre and P. Turbek, *Automorphism criteria for M^*-groups*, Proc. Edinb. Math. Soc., (2) 47 (2004), no. 2, 339–351.

[5] M. Conder, *Three-relator quotients of the modular group*, Quart. J. Math., Oxford (2), 38 (1987), 427-447.

[6] M. Conder and P. Dobcsanyi, *Normal subgroups of low index in the modular group and other Hecke groups*, to appear Combinatorial Group Theory, Number Theory and Discrete Groups" (ed. B. Fine, A. Gaglione & D. Spellman), Contemporary Mathematics series, American Mathematical Society

[7] H.S.M. Coxeter and W.O.J. Moser *Generators and relations for discrete groups* : Springer-Berlin, 1957.

[8] J.J. Etayo and C. Perez-Chirinos, *Bordered and unbordered Klein surfaces with maximal symmetry*, J.Pure Appl. Algebra 42 (1986), 29-35.

[9] B. Fine, J. Howie and G. Rosenberger, *One-relator quotients and free products of cyclics.* Proc. Amer. Math. Soc. 102 (1988), no. 2, 249–254.

[10] B. Fine and G. Rosenberger, *Algebraic generalizations of discrete groups: a path to combinatorial group theory through one-relator products*, Pure and Applied Mathematics, Marcel Dekker. 223. New York, NY: Marcel Dekker, (1999).

[11] B. Fine, F. Roehl and G. Rosenberger, *A Freiheitssatz for certain one-relator amalgamated products*, Combinatorial and geometric group theory (Edinburgh, 1993), 73–86, London Math. Soc. Lecture Note Ser., 204, Cambridge Univ. Press, Cambridge, 1995.

[12] N. Greenleaf and C. L. May, *Bordered Klein surfaces with maximal symmetry*, Trans. Amer. Math. Soc., 274 (1982), no. 1, 265–283.

[13] G. A. Jones and J. S. Thornton, *Automorphisms and congruence subgroups of the extended modular group*, J. London Math. Soc. (2) 34 (1986) 26-40.

[14] C. L. May, *Automorphisms of compact Klein surfaces with boundary*, Pacific J. Math. 59 (1975), 199-210.
[15] C. L. May, *Large automorphism groups of compact Klein surfaces with boundary*, Glasgow Math. J., 18 (1977), 1-10.
[16] C. L. May, *A family of M^*-groups*, Canad. J. Math., 38 (1986), no. 5, 1094–1109.
[17] C. L. May, *Supersolvable M^*-groups*, Glasgow Math. J., 30 (1988), no. 1, 31–40.
[18] R. Sahin, S. Ikikardes and Ö. Koruoğlu, *Note on M^*-groups*, Adv. Stud. Contemp. Math. (Kyungshang) 14 (2007), no. 2, 311–315.
[19] D. Singerman, *PSL(2,q) as an image of the extended modular group with applications to group actions on surfaces*, Proc. Edinb. Math. Soc. 30 (1987), 143-151.
[20] Y.T.Ulutas and I. N. Cangul, *One relator quotients of the modular group*, Bull. Inst. Math. Acad. Sinica, 32 (2004), no.4, 291-196.