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# Finite Derivation Type Property on the Chinese Monoid

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**Abstract**

Squier introduced the notion *finite derivation type* which is a combinatorial condition satisfied by certain rewriting systems. The main result in this paper states that the Chinese monoid has finite derivation type property.

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**Keywords:** Chinese Monoid, Finite Derivation Type, Rewriting Systems, Word Problem

## 1 Introduction

In the last years string-rewriting systems have a lot of attention, both from Theoretical Computer Science and from Mathematics. In particularly, finite and complete (that is, noetherian and conflunet) string-rewriting systems are used to solve word problems among other algebraic problems (see, for example, [1, 13]). This application reveals the importance of such string-rewriting systems. Unfortunately, the property of having finite and complete string-rewriting system is not invariable under monoid presentations (see [7]). For this reason, it would be important to characterize algebraically the finitely presented monoids with solvable word problem that admit a finite and complete string-rewriting system. An important step in that direction was given by Squier ([14]) who worked on some relations, namely *homotopy relations*, between paths in the graph associated with a finite monoid presentation. In the same reference, he also proved that if a monoid $M$ is presented by a finite complete system, then it has *finite homotopy type* (that is also called *finite derivation type*). Further he showed that this finiteness condition is independent on the choice of finite presentations of the given monoid.

Since finite derivation type is a necessary condition for a monoid to be defined by some finite and complete rewriting systems and it is an invariant property of monoid presentations, these give us to think which type monoids and monoid constructions preserve this invariant property. In this sense, by considering the Chinese monoids given in [2], in this paper we show that these monoids has finite derivation type. The proof of our result is based on the method given in [14] that is briefly depends on critical pairs and resolutions.

Let $A = \{x_i : 1 \leq i \leq n\}$ be a well ordered set. The Chinese congruence is the congruence on $A^*$ generated by $T$, where $T$ consists of the following relations:

$$x_i x_j x_k = x_i x_k x_j = x_j x_i x_k \quad \text{for every} \quad i > j > k, \tag{1}$$

$$x_i x_j x_j = x_j x_i x_j, \;\; x_i x_i x_j = x_i x_j x_i \quad \text{for every} \quad i > j. \tag{2}$$

The Chinese monoid $CH(A)$ (of rank $n$) is the quotient monoid of the free monoid $A^*$ by the Chinese congruence [5], i.e., $CH(A) = [A; T]$. Although it is easy to see that (1) and (2) together are equivalent to

$$x_i x_j x_k = x_i x_k x_j = x_j x_i x_k \quad \text{for every} \quad i \geq j \geq k, \tag{3}$$

we will exclusively use equations (1) and (2) instead of (3) in this paper. It is known that every element of $CH(A)$ (of rank $n$) has a unique expression of the form $x = y_1 y_2 \cdots y_n$, where

$$y_1 = x_1^{k_{11}}, \;\; y_2 = (x_2 x_1)^{k_{21}} x_1^{k_{11}}, \;\; y_3 = (x_3 x_1)^{k_{31}} (x_3 x_2)^{k_{32}} x_3^{k_{33}},$$
$$\cdots, \;\; y_n = (x_n x_1)^{k_{n1}} (x_n x_2)^{k_{n2}} \cdots (x_n x_{n-1})^{k_{n(n-1)}} x_n^{k_{nn}},$$

with all exponents non-negative [2]. We call it the canonical form of the element $x \in CH(A)$. The Chinese monoid is related to the so called *plactic monoid* studied by Lascoux et. al. in [10]. Both constructions are strongly related to Young tableaux, and therefore to representation theory and algebraic combinatorics. This monoid appeared in the classification of classes monoids with the growth function coinciding with that of the plactic monoid (see [5]). Then combinatorial properties of this kind monoid were studied in detail in [2]. After that in [6], authors studied the structure of the algebra $K[M]$ of the

Chinese monoid $M$ of rank 3 over a field $K$. Then, in [3] authors simplified some part of the paper [2] by using the Gröbner-Shirshov bases theory for associative algebras. As a last work, in [8] the author showed that the Chinese monoid has a complete rewriting system. We note that this paper can be considered as continuous part of [8].

In this work we focus on the Chinese monoid with rank 3 since the general meaning of this case can be considered similarly. Hence we have the Chinese monoid with rank 3 as follows:

$$\begin{aligned} \mathcal{P}_{M_3} = [x_1, x_2, x_3 \quad ; \quad & x_3x_2x_1 = x_2x_3x_1, x_3x_1x_2 = x_2x_3x_1, \\ & x_2x_1x_1 = x_1x_2x_1, x_3x_2x_2 = x_2x_3x_2, \\ & x_3x_1x_1 = x_1x_3x_1, x_2x_2x_1 = x_2x_1x_2, \\ & x_3x_3x_2 = x_3x_2x_3, x_3x_3x_1 = x_3x_1x_3] \end{aligned} \tag{4}$$

where $3 > 2 > 1$.

## 2 Finite Derivation Type Property

The study of finiteness properties is one of the major topics in the theory of (string) rewriting systems (see [12] for a survey). These properties have very interesting connections between themselves and with some other algebraic problems. Among these properties, we have just studied the finite derivation type property for the Chinese monoid.

The proof of our main result is based on the method given in [14] that is briefly depends on critical pairs and resolutions. In fact this will give us that a monoid with a finite complete rewriting system has *FDT*. But to find a finite generating set for a given monoid is important as much as having this property. Some works on *FDT* property can be found in [9, 11, 15, 16]. Now let us give some fundamental notations on this property.

Let $[X; \mathbf{s}]$ be a monoid presentation. We have a graph $\Gamma = \Gamma(X; \mathbf{s})$ associated with $[X; \mathbf{s}]$, where the vertices are the elements of $X^*$, and the edges are the 4-tuples $e = (U, S, \varepsilon, V)$ with $U, V \in X^*$, $S \in \mathbf{s}$ and $\varepsilon = \pm 1$. The initial, terminal and the inversion functions for an edge $e$ as above are given by $\iota(e) = US^{\varepsilon}V$, $\tau(e) = US^{-\varepsilon}V$ and $e^{-1} = (U, S, -\varepsilon, V)$. In fact there is a two-sided action of $X^*$ on $\Gamma$ as follows. If $W, W' \in X^*$, then for any vertex $V$ of $\Gamma$, $W.V.W' = WVW'$ (product in $X^*$), and for any edge $e = (U, S, \varepsilon, V)$ of $\Gamma$, $W.e.W' = (WU, S, \varepsilon, VW')$. This action can be extended to the paths in $\Gamma$.

We let $P(\Gamma)$ denote the set of all paths in $\Gamma$, and let

$$P^2(\Gamma) := \{(p,q) : p,q \in P(\Gamma), \iota(p) = \iota(q), \tau(p) = \tau(q)\}.$$

**Definition 2.1** *An equivalence relation $\simeq \subset P^2(\Gamma)$ is called a homotopy relation if it satisfies the following conditions:*
**(a)** *If $e_1, e_2$ are edges of $\Gamma$, then $(e_1.\iota(e_2))(\tau(e_1).e_2) \simeq (\iota(e_1).e_2)(e_1.\tau(e_2))$.*
**(b)** *If $p \simeq q$ $(p,q \in P(\Gamma))$, then $U.p.V \simeq U.q.V$ for all $U,V \in X^*$.*
**(c)** *If $p, q_1, q_2, r \in P(\Gamma)$ satisfy $\tau(p) = \iota(q_1) = \iota(q_2)$, $\tau(q_1) = \tau(q_2) = \iota(r)$ and $q_1 \simeq q_2$, then $pq_1r \simeq pq_2r$.*
**(d)** *If $q \in P(\Gamma)$, then $pp^{-1} \simeq 1$, where $1$ denotes the empty path (at the vertex $\iota(p)$).*

It is seen that the collection of all homotopy relations on $P(\Gamma)$ is closed under arbitrary intersection, and that $P^{(2)}(\Gamma)$ itself is a homotopy relation. Thus, if $C \subset P^{(2)}(\Gamma)$, then there is a unique smallest homotopy relation $\simeq_C$ on $P(\Gamma)$ that contains $C$. Moreover $[X;\mathbf{s}]$ has *FDT* if there is a finite subset $C \subset P^{(2)}(\Gamma)$ which generates $P^{(2)}(\Gamma)$ as a homotopy relation, that is $\simeq_C = P^{(2)}(\Gamma)$.

Now let $\Gamma := \Gamma(A;T)$ be the graph associated with $\mathcal{P}_{M_3}$ given on (4). By $P_+(\Gamma)$ and $P_-(\Gamma)$, we denote the set of all those paths in $\Gamma$ that only contain edges of the form $(x;l,r;y)$ and $(x;r,l;y)$ with $(l,r) \in T$, respectively.

**Theorem 2.2** *The Chinese monoid has finite derivation type property.*

**Proof.** By Theorem 2.2 in [8], $T$ is Noetherian. Thus all paths in $P_+(\Gamma)$ are of finite length. Furthermore, for the case of $T$ is convergent, there exist paths $p_+ \in P_+(\Gamma)$ and $p_- \in P_-(\Gamma)$ such that $\iota(p_+) = \iota(p)$, $\tau(p_+) = \iota(p_-)$, $\tau(p_-) = \tau(p)$ for each path $p \in P(\Gamma)$.

Now let us define a set $B \subseteq P_+^{(2)}(\Gamma)$ that will generate $P^{(2)}(\Gamma)$. (We note that the set $B$ will be formed for the Chinese monoid with three generators since for general numbers of generators can be considered similarly).

**(i)** The following ordered pairs $(e_i, e_{i'})$, for $1 \leq i \leq 10$, of edges are *critical pairs*. In here $\lambda$ is the empty word.

- $e_1 = (\lambda; x_3x_2x_1, x_2x_3x_1; x_1)$ and $e_{1'} = (x_3; x_2x_1x_1, x_1x_2x_1; \lambda)$,
- $e_2 = (\lambda; x_3x_1x_2, x_2x_3x_1; x_1x_1)$ and $e_{2'} = (x_3x_1; x_2x_1x_1, x_1x_2x_1; \lambda)$,
- $e_3 = (\lambda; x_3x_1x_2, x_2x_3x_1; x_2x_1)$ and $e_{3'} = (x_3x_1; x_2x_2x_1, x_2x_1x_2; \lambda)$,
- $e_4 = (\lambda; x_2x_2x_1, x_2x_1x_2; x_1)$ and $e_{4'} = (x_2; x_2x_1x_1, x_1x_2x_1; \lambda)$,
- $e_5 = (\lambda; x_3x_2x_2, x_2x_3x_2; x_1x_1)$ and $e_{5'} = (x_3x_2; x_2x_1x_1, x_1x_2x_1; \lambda)$,

- $e_6 = (\lambda; x_3x_2x_2, x_2x_3x_2; x_1)$ and $e_{6'} = (x_3; x_2x_2x_1, x_2x_1x_2; \lambda)$,
- $e_7 = (\lambda; x_3x_3x_2, x_3x_2x_3; x_1x_1)$ and $e_{7'} = (x_3x_3; x_2x_1x_1, x_1x_2x_1; \lambda)$,
- $e_8 = (\lambda; x_3x_2x_1, x_2x_3x_1; x_2x_1)$ and $e_{8'} = (x_3; x_2x_2x_1, x_2x_1x_2; \lambda)$,
- $e_9 = (\lambda; x_3x_3x_2, x_2x_3x_2; x_1)$ and $e_{9'} = (x_3; x_3x_2x_1, x_2x_3x_1; \lambda)$,
- $e_{10} = (\lambda; x_3x_3x_1, x_3x_1x_3; x_2)$ and $e_{10'} = (x_3; x_3x_2x_1, x_2x_3x_1; \lambda)$.

Since $T$ is complete, a resolution always exists for each critical pair of edges. Now let us check each of the resolutions.

**(ii)** Let $(e_i, e_{i'})$ $(1 \le i \le 10)$ be a critical pair of edges. An ordered pair $(p_i, p_{i'})$ of paths $p_i, p_{i'} \in P_+(\Gamma)$ is called a resolution of $(e_i, e_{i'})$ if $\iota(p_i) = \tau(e_i)$, $\iota(p_{i'}) = \tau(e_{i'})$ and $\tau(p_i) = \tau(p_{i'})$ hold. For

- $e_1 = (\lambda; x_3x_2x_1, x_2x_3x_1; x_1)$ and $e_{1'} = (x_3; x_2x_1x_1, x_1x_2x_1; \lambda)$,

$\iota(p_1) = \tau(e_1) = x_2x_3x_1x_1$ and $\iota(p_{1'}) = \tau(e_{1'}) = x_3x_1x_2x_1$. Since $\tau(p_1) = \tau(p_{1'})$ will be hold, we get $p_1 = (x_2; x_3x_1x_1, x_1x_3x_1; \lambda)$ and $p_{1'} = (\lambda; x_3x_1x_2, x_2x_3x_1; x_1)$.

- $e_2 = (\lambda; x_3x_1x_2, x_2x_3x_1; x_1x_1)$ and $e_{2'} = (x_3x_1; x_2x_1x_1, x_1x_2x_1; \lambda)$,

$\iota(p_2) = \tau(e_2) = x_2x_3x_1x_1x_1$ and $\iota(p_{2'}) = \tau(e_{2'}) = x_3x_1x_1x_2x_1$. Since $\tau(p_2) = \tau(p_{2'})$ will be hold, we get $p_2 = (x_2; x_3x_1x_1, x_1x_3x_1; x_1)$ and $p_{2'} = (\lambda; x_3x_1x_1, x_1x_3x_1; x_2x_1)$.

- $e_3 = (\lambda; x_3x_1x_2, x_2x_3x_1; x_2x_1)$ and $e_{3'} = (x_3x_1; x_2x_2x_1, x_2x_1x_2; \lambda)$,

$\iota(p_3) = \tau(e_3) = x_2x_3x_1x_2x_1$ and $\iota(p_{3'}) = \tau(e_{3'}) = x_3x_1x_2x_1x_2$. Since $\tau(p_3) = \tau(p_{3'})$ will be hold, we get $p_3 = (x_2; x_3x_1x_2, x_2x_3x_1; x_1)$ and $p_{3'} = (\lambda; x_3x_1x_2, x_2x_3x_1; x_1x_2)$.

- $e_4 = (\lambda; x_2x_2x_1, x_2x_1x_2; x_1)$ and $e_{4'} = (x_2; x_2x_1x_1, x_1x_2x_1; \lambda)$,

$\iota(p_4) = \tau(e_4) = x_2x_1x_2x_1$ and $\iota(p_{4'}) = \tau(e_{4'}) = x_2x_1x_2x_1$. So $p_4$ and $p_{4'}$ are empty paths.

- $e_5 = (\lambda; x_3x_2x_2, x_2x_3x_2; x_1x_1)$ and $e_{5'} = (x_3x_2; x_2x_1x_1, x_1x_2x_1; \lambda)$,

$\iota(p_5) = \tau(e_5) = x_2x_3x_2x_1x_1$ and $\iota(p_{5'}) = \tau(e_{5'}) = x_3x_2x_1x_2x_1$. Since $\tau(p_5) = \tau(p_{5'})$ will be hold, we get $p_5 = (x_2; x_3x_2x_1, x_2x_3x_1; x_1)$ and $p_{5'} = (\lambda; x_3x_2x_1, x_2x_3x_1; x_2x_1)$.

- $e_6 = (\lambda; x_3x_2x_2, x_2x_3x_2; x_1)$ and $e_{6'} = (x_3; x_2x_2x_1, x_2x_1x_2; \lambda)$,

$\iota(p_6) = \tau(e_6) = x_2x_3x_2x_1$ and $\iota(p_{6'}) = \tau(e_{6'}) = x_3x_2x_1x_2$. Since $\tau(p_6) = \tau(p_{6'})$ will be hold, we get $p_6 = (x_2; x_3x_2x_1, x_2x_3x_1; \lambda)$ and $p_{6'} = (\lambda; x_3x_2x_1, x_2x_3x_1; x_2)$.

- $e_7 = (\lambda; x_3x_3x_2, x_3x_2x_3; x_1x_1)$ and $e_{7'} = (x_3x_3; x_2x_1x_1, x_1x_2x_1; \lambda)$,

$\iota(p_7) = \tau(e_7) = x_3x_2x_3x_1x_1$ and $\iota(p_{7'}) = \tau(e_{7'}) = x_3x_3x_1x_2x_1$. Since $\tau(p_7) = \tau(p_{7'})$ will be hold, we get $p_7 = (x_3x_2; x_3x_1x_1, x_1x_3x_1; \lambda)$ and $p_{7'} = (x_3; x_3x_1x_2, x_2x_3x_1; x_1)$.

- $e_8 = (\lambda; x_3x_2x_1, x_2x_3x_1; x_2x_1)$ and $e_{8'} = (x_3; x_2x_2x_1, x_2x_1x_2; \lambda)$,

$\iota(p_8) = \tau(e_8) = x_3x_2x_3x_2x_1$ and $\iota(p_{8'}) = \tau(e_{8'}) = x_3x_3x_2x_1x_2$. Since $\tau(p_8) = \tau(p_{8'})$ will be hold, we get $p_8 = (x_3x_2; x_3x_2x_1, x_2x_3x_1; \lambda)$ and $p_{8'} = (x_3; x_3x_2x_1, x_2x_3x_1; x_2)$.

- $e_9 = (\lambda; x_3x_3x_2, x_2x_3x_2; x_1)$ and $e_{9'} = (x_3; x_3x_2x_1, x_2x_3x_1; \lambda)$,

$\iota(p_9) = \tau(e_9) = x_3x_2x_3x_1$ and $\iota(p_{9'}) = \tau(e_{9'}) = x_3x_2x_3x_1$. So $p_9$ and $p_{9'}$ are empty paths.

- $e_{10} = (\lambda; x_3x_3x_1, x_3x_1x_3; x_2)$ and $e_{10'} = (x_3; x_3x_2x_1, x_2x_3x_1; \lambda)$,

$\iota(p_{10}) = \tau(e_{10}) = x_3x_1x_3x_2$ and $\iota(p_{10'}) = \tau(e_{10'}) = x_3x_2x_3x_1$. Since we have the relator $x_3x_2x_3x_1 = x_3x_1x_3x_2$, we have $\tau(e_{10}) = \tau(e_{10'})$ and hence empty paths $p_{10}$ and $p_{10'}$. After all above, we let

$$B = \{(e_i \circ p_i, e_{i'} \circ p_{i'}) \ : \ (e_i, e_{i'}) \text{ is a critical pair of edges and} (p_i, p_{i'}) \text{ is the choosen resolution of } (e_i, e_{i'}) \text{ for } 1 \leq i \leq 10\}.$$

Since $T$ is finite then the set $B$ is finite. Then $\simeq_B = P^{(2)}(\Gamma)$.

Hence the result. $\diamondsuit$

# 3 Summary of Results

For a future project, we may express briefly some other algebraic properties that the Chinese monoid does not hold. A monoid $M$ is called *cancellative* if $uw = vw$ always implies $u = v$, and also $wu = wv$ always implies $u = v$, for all $u, v, w \in M$. Since the Chinese monoid does not commutative, it is easily seen that it is not cancellative and so not a group-embeddable (e.g. the relation $x_ix_jx_k = x_ix_kx_j$ does not require $x_jx_k = x_kx_j$). In addition to that, although in [2] the authors studied conjugacy classes, results in that paper does not completely answer the conjugacy problem. In other words, there are still a lot problems left to be revealed (such as *efficiency* (equivalently, *p-Cockcroft property* for some prime $p$)) on the Chinese monoid. We note that [4] is an important paper on the efficiency of monoids. After all we can summarize some of the results on this special monoid as in Table 1.

Table 1: The Chinese Monoid

| *Property* | *Yes(+)/No(-)/Unknown(?)* |
|---|---|
| having solvable word problem | + |
| having *FDT* | + |
| having solvable conjugacy problem | ? |
| group-embeddability | − |
| cancellativity | − |
| efficiency | ? |

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