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Expanded Lie Group Method Applied to Generalized Boussinesq Equation

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Abstract

In this study, Generalized Bossinesq Equation reduced to previously unknown target ordinary differential equation by applying the Expanded Lie group transformation and similarity reduction. Morover, obtained target ordinary equation is used to find the exact soution of Generalized Boussinesq Equation.

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1 Introduction

In applied group analysis, Lie theory of symmetry group for differential equations, constituted by Sophus Lie, is the most important solution method for the nonlinear problems in the field of applied maths. The fundamentals of Lie's theory are based on the invariance of the equation under transformation groups of independent and dependent variables, so called Lie groups. This approach is used to analyisis the symmetries of the differential equations and may be apoint, a contact, and a generalized or nonlocal symmetry. In the last century, the application of the Lie group method has been developed by a number of mathematicians. Ovsiannikov [15], Olver [14], Ibragimov [10], Baumann [1] and Bluman and Anco [3] are some of the mathematicians who have enormous amount of studies in this field.

The existence of symmetries of diferential equations under Lie group of transformations often allows those equations to be reduced to simpler equations. One of the major accomlishment of Lie was to identify that the properties of global transformations of the group are completely and uniquely determined by the infinitesimal transformations around the identity transformation. This allows the nonlinear relations for the identification of invariance groups to be dealing with global transformation equations, we use differential operators, called the group generators, whose exponentiation generates the action of the group. The collection of these differentail operators forms the basis for the Lie algebra. There is a one-to-one correspondence between the Lie groups and the associated Lie algebras.

Among those transformation groups, an expanded Lie group transformation of a partial differential equation is a continuious group transformation which is acting on expanded space of variables that includes the equation parameters in addition to independent and dependant variables.An expanded group of transformations represents a particular case of the equivalence group that presrves the class of partial differential equations which holds the same structure. The approach to find these equivalence transformation groups with th use of the Lie infinitesimal tecnique was introduced by Ovasiannikov [15] who suggested using the Lie infinitesimal principle in the properly extend space of variables which include dependent and independent variables, arbitrary functions and their derivatives. More recently, Burde [4] used the Lie groups of transformations in the expanded space of variables including equation parameters enables one to enrich the cocept of similarity reductions as applied to partial differential equations. And also he used these groups for finding changes of variables that remove some terms from the original equation.

In this paper, we have used an Expande Lie group transformation and similarity reduction to obtain the exact analytical solution of Generalized Boussinesq Equation.

2 Expanded Lie group transformation

Consider the Generalised Boussinesq (GBQ) Equation

$$u_{xxxx} + pu_t u_{xx} + qu_x u_{xt} + ru_x^2 u_{xx} + u_{tt} = 0 \tag{1}$$

where p,q and r are constants such that $r \neq 0$ and subscripts denote partial derivatives.

Clasical symmetry reductions of some special cases of equation (1) have been discussed by Schwarz [17], Clarkson [7], Kawamoto [12], Lou [13], Paquin and Winternitz [16]. Clasical symmetrys of some different type of equation (1) have been investigated by Clarkson and Priestly [8], Gandarias and Bruzon [9]. Clarkson and Kruskal [5] developed a Direct method (in the sequel referred as the Direct method) for finding symmetry reductions which is used to obtain previously unknown reductions of the Boussinesq Equation and Clarkson and Ludlow [6] derived smmetry reductions of GBQ by using the Direct method and said that those derived by using the Lie group method with one illustration. Recently Burde [4] showed that the use the Lie groups of transformation in the expanded space of variables including parameters equation improved the concept of similarity reductions as applied to partial differential equations.

In this paper our main motivation and starting point based on Burde's paper [4], is to demonstrate that the procedure of symmetry reduction implemented in the expanded space which reduces GBQ systematically to a previously unknown target ordinary differential equation by the suitable choice of expanded group transformaton.

To illustrate the process we introduce a cofficient (parameter) into the equation "artifically", for example, in front of the last term equation (1), i.e.

$$u_{xxxx} + pu_t u_{xx} + qu_x u_{xt} + ru_x^2 u_{xx} + au_{tt} = 0. \tag{2}$$

It may appear that introducing this kind of coeffient makes useless the physics of the problem, but one may always set a=1 in the final stage. Currently for convenience we choose the coefficients

$$p = q, \quad r = \frac{q^2}{2a}$$

and rewrite equation (2) as

$$u_{xxxx} + qu_t u_{xx} + qu_x u_{xt} + \frac{q^2}{2a} u_x^2 u_{xx} + au_{tt} = 0. \tag{3}$$

To apply the classical Lie symmetry group method to equation (3), we perform symmetry analysis. Let us consider a one-parameter Lie group of infinitesimal transformation

$$x \to x + \varepsilon \xi_1(x, t, u, q, a) + O(\varepsilon^2)$$

$$t \to t + \varepsilon \xi_2(x, t, u, q, a) + O(\varepsilon^2) \tag{4}$$

$$u \to u + \varepsilon \eta(x, t, u, q, a) + O(\varepsilon^2)$$

where ε is group parameter in the expanded (x, t, u, q, a) space. The vector field associated with the above group of transformations can be written as

$$X = \xi_1 \frac{\partial}{\partial x} + \xi_2 \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial u} + Q(q) \frac{\partial}{\partial q} + A(a) \frac{\partial}{\partial a}. \tag{5}$$

This is symmetry generator and invariance of equation (3) under transformation (4). The associated Lie algebra of the infinitesimal system involves the set of vector fields of this form.

The symmetry condition

$$pr^{(4)} X \Delta \mid_{\triangle} = 0$$

yields an overdetermined system of PDE for the unknown functions ξ_1, ξ_2 and η where Δ is the manifold defined by (3) in jet space $J^{(3)}$ and $pr^{(4)}X$ is the fourth prolongation of X. We obtain this system by using package MathLie [1] and this system can solve if provide following system

$$(\xi_1)_u = 0, (\xi_2)_u = 0, (\xi_2)_x = 0, (\eta)_{uu} = 0$$

$$-6(\xi_1)_{xx} + 4(\eta)_{xu} = 0$$

$$-q(\xi_1)_{xx} + 3q(\eta)_{xu} = 0.$$

Then we obtain $(\xi_1)_{xx} = (\eta)_{xu} = 0$ and so $(\eta)_t = (\xi_1)_{tt} = 0$. Thus the determining equations can be obtained as:

$$\frac{A}{a} - 2(\xi_2)_t + (\eta)_u = 0$$

$$\frac{Q}{q} - 2(\xi_1)_x - (\xi_2)_t + 2(\eta)_u = 0$$

$$-\frac{A}{2a}+\frac{Q}{q}-2(\xi_1)_x+\frac{3}{2}(\eta)_u=0$$

$$-4(\xi_1)_x+(\eta)_u=0$$

$$-2a(\xi_1)_t+q(\eta)_x=0$$

$$-a(\xi_2)_{tt}+q(\eta)_{xx}=0$$

$$-2a(\xi_1)_{xt}+q(\eta)_{xx}=0$$

The resulting system of equations easily be solved to give the infinitesimals

$$\xi_1=C_1xt+C_0x+C_3t+C_4$$

$$\xi_2=C_1+t^2+3C_0t+C_2$$

$$\eta=4C_0u+\frac{a}{q}C_1x^2+\frac{2a}{q}C_3x+C_5 \tag{6}$$

$$Q=-3qC_0$$

$$A=2aC_0$$

which includes the determination of the generators A and Q. Here it is worth to note that not only the generators A and Q but also the coefficients C_0, C_1, C_2, C_3, C_4 and C_5 are depended on the parameters a and q. Under conditions which are $C_1=C_2=C_3=C_4=C_5=0$ and $C_0=\frac{1}{4a}$ the infinitesimals (6) are

$$\xi_1=\frac{x}{4a},\quad \xi_2=\frac{3t}{4a},\ \eta=\frac{u}{a},\ Q=-\frac{3q}{4a},\ A=\frac{1}{2}.$$

Thus the subgroup of one-parameter Lie group of infinitesimal transformation (4) obtained to be

$$\overline{x}=x(1+\frac{\varepsilon}{2a})^{1/2},\quad \overline{t}=t(1+\frac{\varepsilon}{2a})^{3/2},\quad \overline{u}=u(1+\frac{\varepsilon}{2a})^{2},\quad \overline{q}=q(1+\frac{\varepsilon}{2a})^{-3/2},\quad \overline{a}=a+\frac{1}{2}\varepsilon. \tag{7}$$

Consequently the generator of this subgroup can be given by

$$X_0=\frac{x}{4a}\frac{\partial}{\partial x}+\frac{3t}{4a}\frac{\partial}{\partial t}+\frac{u}{a}\frac{\partial}{\partial u}-\frac{3q}{4a}\frac{\partial}{\partial q}+\frac{1}{2}\frac{\partial}{\partial a}$$

and the subalgebra of the Lie algebra admited by (3) is $L_{1,0}=\{X_0\}$.

3 Reduction to ODE and exact solution

In this section we use the method of characteristics to determine the invariants and reduced ODE corresponding to the subalgebra given by $L_{1,0}$.From equation $X_0\ v(x,t,u,a,q)=0$, similarity variables $z=\frac{x}{\sqrt{a}}$, $\widetilde{r}=a^3q^2$, $u=v(z)+\frac{\lambda}{2}\ln t$ is found by solving characteristic equation

$$\frac{dx}{x}=\frac{dt}{2t}=\frac{du}{\lambda}.$$

If reduction of equation (1) is done by these similarity variables, then

$$W'-\frac{1}{2}z(p+q)WW'+(\frac{\lambda}{2}p+\frac{1}{4}z^2)W'-\frac{q}{2}W^2+rW^2W'+\frac{3}{4}zW-\frac{\lambda}{2}=0$$

where $v'=W$. if $q=0$ and $r=-\frac{1}{2}p^2$ and we make the transformation

$$W(z)=\frac{1}{p}(-3^{\frac{3}{4}}y+z),\qquad x=-\frac{1}{2}3^{\frac{1}{4}}z$$

then $y(x)$ satisfies the fourth Painleve equation (PIV) [6].

$$y''=\frac{1}{2y}(y')^2+\frac{3}{2}y^3+4xy^2+2(x^2-A)+\frac{B}{y}$$

with $A=\frac{\lambda p}{6}$ and B a constant of integration.

2. Reduction by using algebra $L_{1,2}^{\lambda,\delta}$:

Similarity variables $z=x-t$, $u=v(z)+\frac{\lambda}{\delta}t=v(x-t)+\frac{\lambda}{\delta}t$ is obtained by solving equation $(\lambda X_1+\delta X_2+\delta X_3\ )\ w(x,t,u)=0$. Thus, reduction of equation (1) done by these similarity variables is

$$W'+r\frac{W^3}{3}-(p+q)\frac{W^2}{2}+(\frac{\lambda}{\delta}p+1)W=C$$

with C a constant of integration where $v'=W$. This equation is solved by using elliptic integral.

4 Concluding remarks

In this paper, we have determined an optimal system for Generalised Boussinesq (GBQ) equation (p,q,r are constant). Thus, one classification of the similarity solutions has been obtained. One reduction of equation (1) can be done by using two-dimensional subalgebras.

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