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Approximation in Morrey-Smirnov classes

D.M. Israfilov*, N.P. Tozman

Abstract. The Morrey-Smirnov classes $E^{p,\alpha}(G)$, $0 < \alpha \leq 2$ and $p > 1$, of the analytic functions in the domain G with a rectifiable Jordan boundary are defined. In these classes the inverse theorem of approximation theory is proved and the constructive characterizations problems of the generalized Lipschitz classes of functions are discussed.

Key Words and Phrases: Morrey space, Morrey-Smirnov classes, Faber series, Inverse theorem, Modulus of smoothness.

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1. Introduction and Main result

In this work we study the inverse theorems of approximation theory and the constructive characterization problems in the Morrey-Smirnov classes, defined on the finite domain G with a sufficiently smooth Jordan boundary Γ. The Morrey spaces, introduced by Morrey in 1938, have been studied intensively by various authors and together with weighted Lebesgue spaces L^p_ω play an important role in the theory of partial equations, especially in the study of local behavior of the solutions of elliptic differential equations (see, for example [24], [33]). They also provide a large class of examples of mild solutions to the Navier-Stokes system [22]. In the context of fluid dynamics, Morrey spaces have been used to model flow when vorticity is a singular measure supported on certain sets in $\mathbb{R}^n$ [11]. Nowadays there are sufficiently wide investigations relating to the fundamental problems in these spaces, in view of the differential equations, potential theory, maximal and singular operator theory and others (see, for example [8] and the references, mentioned above).

Recently by us in [17] have been investigated the approximation problems in the Morrey-Smirnov classes of analytic functions and proved in particular one direct theorem of approximation theory by polynomials in the finite domain G with a sufficiently smooth Jordan boundary Γ, namely when the function $\theta(s)$, the angle between the tangent and the positive real axis expressed as a function of arclength s, has the usual uniform

*Corresponding author.

modulus of continuity $\Omega(\theta, s)$ on $[0, |\Gamma|]$, where $|\Gamma|$ is the linear Lebesgue measure of Γ, satisfying the condition

$$\int_0^\delta \frac{\Omega(\theta, s)}{s} ds < \infty, \qquad \delta > 0. \tag{1}$$

In the current paper we prove the appropriate inverse theorem and obtain the constructive characterization of the generalized Lipschitz classes of functions defined below. To the best of the author's knowledge in the literature there are no results relating to the approximation problems in the Morrey-Smirnov classes, defined on the sets of the real line or complex plane.

Note that the order of approximation by trigonometric polynomials and the constructive description problems of some well-known classes of functions in the weighted and nonweighted Lebesgue spaces, defined on the interval $I_0 := (0, 2\pi)$, have been studied by several authors. The sufficiently wide presentation of the corresponding results can be found in the works [1], [7] and [25]. Afterwards, these results were extended to the complex domains and the analogous theory was also developed on the spaces, defined on the domains with the different geometric properties (see for example [2], [3], [4], [6], [9], [12], [13], [14], [15], [16], [18], [19], [23], [31]).

Let Γ be a rectifiable Jordan curve in the complex plane $\mathbb{C}$. The *Morrey spaces* $L^{p,\alpha}(\Gamma)$, for a given $0 \leq \alpha \leq 2$ and $p \geq 1$, we define as the set of functions $f \in L^p_{loc}(\Gamma)$ such that

$$\|f\|_{L^{p,\alpha}(\Gamma)} := \left\{ \sup_B \frac{1}{|B \cap \Gamma|^{1-\frac{\alpha}{2}}} \int_{B\cap\Gamma} |f(z)|^p \, |dz| \right\}^{1/p} < \infty,$$

where the supremum is taken over all disks B centered on Γ.

In case of $\Gamma = \mathbb{T} := \{w : |w| = 1\}$ the Morrey spaces $L^{p,\alpha}(\mathbb{T})$ for a given $0 \leq \alpha \leq 2$ and $p \geq 1$, can be defined also as the set of functions $f \in L^p_{loc}(\mathbb{T}) \equiv L^p_{loc}(0, 2\pi)$, for which

$$\|f\|_{L^{p,\alpha}(\mathbb{T})} = \|f\|_{L^{p,\alpha}(0,2\pi)} := \left\{ \sup_I \frac{1}{|I|^{1-\frac{\alpha}{2}}} \int_I \left| f(e^{i\theta}) \right|^p d\theta \right\}^{1/p} < \infty,$$

where the supremum is taken over all intervals $I \subset (0, 2\pi)$.

Under this definition $L^{p,\alpha}(\Gamma)$ becomes a Banach space; for $\alpha = 2$ coincides with $L^p(\Gamma)$ and for $\alpha = 0$ with $L^\infty(\Gamma)$. Moreover, $L^{p,\alpha_1}(\Gamma) \subset L^{p,\alpha_2}(\Gamma)$ for $0 \leq \alpha_1 \leq \alpha_2 \leq 2$. If $f \in L^{p,\alpha}(\Gamma)$, then $f \in L^p(\Gamma)$ and hence $f \in L^1(\Gamma)$.

Denoting $G := int\, \Gamma$ and $G^- := ext\, \Gamma$, we define the *Morrey-Smirnov classes* $E^{p,\alpha}(G)$, $0 \leq \alpha \leq 2$ and $p \geq 1$, of analytic functions in G as

$$E^{p,\alpha}(G) := \left\{ f \in E^1(G) : f \in L^{p,\alpha}(\Gamma) \right\}.$$

Equipped with the norm

$$\|f\|_{E^{p,\alpha}(G)} := \|f\|_{L^{p,\alpha}(\Gamma)},$$

the class $E^{p,\alpha}(G)$, $0 \leq \alpha \leq 2$ and $p \geq 1$, becomes a Banach space; for $\alpha = 2$ coincides with the classical Smirnov class $E^{p}(G)$, and for $\alpha = 0$ with $E^{\infty}(G)$. It can be easily to verify that, $E^{p,\alpha_1}(G) \subset E^{p,\alpha_2}(G)$ for $0 \leq \alpha_1 \leq \alpha_2 \leq 2$.

In case of $G = D := \{z : |z| < 1\}$, we obtain the space $H^{p,\alpha}(D) := E^{p,\alpha}(D)$, so called Morrey-Hardy space on the unit disk D.

Let φ be the conformal mapping of G^{-} onto $D^{-} := ext\ \mathbb{T}$ with normalization

$$\varphi(\infty) = \infty,\ \lim_{z\to\infty} \frac{\varphi(z)}{z} > 0$$

and let ψ be the inverse mapping of φ. Since Γ is a rectifiable Jordan curve, the derivatives φ' and ψ' exist almost everywhere on Γ and on $\mathbb{T}$, respectively and the boundary functions are integrable on the appropriate sets.

We shall use $c, c_1, c_2, ...$ to denote constants (in general, different in different relations) depending only on numbers that are not important for the questions of interest and denote $\mathbb{N} := \{0, 1, 2, ...\}$, $\mathbb{N}^{+} := \{1, 2, ...\}$.

We define the modulus of smoothness in the Morrey spaces $L^{p,\alpha}(\mathbb{T})$ as following.

Definition 1. *Let $g \in L^{p,\alpha}(\mathbb{T})$, $0 \leq \alpha \leq 2$ and $p \geq 1$. We define the r the modulus of smoothness $\omega^{r}_{p,\alpha}(g, \cdot) : (0, +\infty) \longrightarrow [0, +\infty)$ of order $r \in \mathbb{N}^{+}$ for g as*

$$\omega^{r}_{p,\alpha}(g, t) := \sup_{|h|\leq t} \|\Delta^{r}_{h}(g, \cdot)\|_{L^{p,\alpha}(\mathbb{T})}\ ,$$

where $\Delta^{r}_{h}(g, \cdot) = \sum\limits_{k=0}^{r} \binom{r}{k} (-1)^{r-k} g(\cdot e^{ikh})$.

It is clear that $\lim\limits_{t\longrightarrow 0} \omega^{r}_{p,\alpha}(g, t) = 0$ for every $g \in L^{p,\alpha}(\mathbb{T})$, $0 \leq \alpha \leq 2$ and $p \geq 1$, and by Minkowski's inequality

$$\omega^{r}_{p,\alpha}(g_1 + g_2, \cdot) \leq \omega^{r}_{p,\alpha}(g_1, \cdot) + \omega^{r}_{p,\alpha}(g_2, \cdot)$$

for every $g_1, g_2 \in L^{p,\alpha}(\mathbb{T})$.

Clearly, $\omega^{r}_{p,\alpha}(f, \cdot)$ is an increasing function and has the properties:

a) $\omega^{r}_{p,\alpha}(g, nt) \leq n^{r} \omega^{r}_{p,\alpha}(g, t)\quad , \qquad n \in \mathbb{N}$
b) $\omega^{r}_{p,\alpha}(g, \lambda t) \leq (\lambda + 1)^{r} \omega^{r}_{p,\alpha}(g, t)\quad , \qquad \lambda > 0$
c) $\omega^{r}_{p,\alpha}(g, t) \leq [(n+1)t + 1]^{r}\, \omega^{r}_{p,\alpha}(g, \frac{1}{n+1}), \qquad n \in \mathbb{N}$

which are proved by standard method.

For $f \in E^{p,\alpha}(G)$, $0 \leq \alpha \leq 2$ and $p \geq 1$, we denote by

$$E_n(f)_{E^{p,\alpha}(G)} := \inf \left\{ \|f - p_n\|_{L^{p,\alpha}(\mathbf{\Gamma})} : p_n \textit{ is an algebraic polynomial of degree} \leq n \right\}$$

the minimal error of approximation of f by algebraic polynomials of degree at most n.

Since Γ is smooth and satisfies the condition (1), by [32] (see also, [29], pp.140-141)

$$0 < c_1 \leq \left|\psi'(w)\right| \leq c_2 < \infty, \tag{2}$$

almost everywhere on $\mathbb{T}$ and hence, for any disk $B \subset \mathbb{C}$, with sufficiently small diameter, there exists a disk $B_0 \subset \mathbb{C}$ such that

$$|B \cap \Gamma| \leq c_2 |B_0 \cap \mathbb{T}| \leq c_3 |B \cap \Gamma| . \tag{3}$$

Indeed, let B be a disk with $B \cap \Gamma \neq \emptyset$ and let $\gamma_z := B \cap \Gamma$, $\gamma_w := \varphi(\gamma_z)$. For this disk B with a sufficiently small diameter, the set γ_z consists only of one arc lying on Γ. Then γ_w also consists only of one arc and lies on $\mathbb{T}$. Denoting by B_0 the disk containing γ_w and having minimal radius, we have $\int\limits_{\gamma_w} |dw| = |B_0 \cap \mathbb{T}|$ and hence

$$\begin{aligned} |B \cap \Gamma| &= \int\limits_{\gamma_z} |dz| = \int\limits_{\gamma_w} \left|\psi'(w)\right| |dw| \leq c_2 \int\limits_{\gamma_w} |dw| = c_2 |B_0 \cap \mathbb{T}| = \\ &= c_2 \int\limits_{\gamma_z} \left|\varphi'(z)\right| |dz| \leq \frac{c_2}{c_1} \int\limits_{\gamma_z} |dz| = c_3 |B \cap \Gamma| . \end{aligned}$$

The relation (3) implies that $f_0 := f \circ \psi \in L^{p,\alpha}(\mathbb{T})$, as soon as $f \in L^{p,\alpha}(\Gamma)$.

Moreover, if $f \in E^{p,\alpha}(G)$, then by Corollary 1 from [17] the function

$$f_0^+(w) := \frac{1}{2\pi i} \int\limits_{\mathbb{T}} \frac{f_0(\tau)\, d\tau}{\tau - w}, \qquad w \in D, \tag{4}$$

and

$$f_0^-(w) := \frac{1}{2\pi i} \int\limits_{\mathbb{T}} \frac{f_0(\tau)\, d\tau}{\tau - w}, \qquad w \in D^-, \tag{5}$$

belong to $H^{p,\alpha}(D)$ and $H^{p,\alpha}(D^-)$, respectively. Defining the rth modulus of smoothness of $f \in E^{p,\alpha}(G)$ by

$$\Omega^r_{\Gamma,p,\alpha}(f,\delta) := \omega^r_{p,\alpha}(f_0^+,\delta), \qquad \delta > 0,$$

and taking the proof of Theorem 1 from [17] into account, we deduce the following direct theorem of approximation theory in $E^{p,\alpha}(G)$, $0 < \alpha \leq 2$ and $p > 1$.

Theorem A *Let $G \subset \mathbb{C}$ be a finite simply connected domain with a boundary Γ, satisfying the condition* (1) *and $f \in E^{p,\alpha}(G)$, $0 < \alpha \leq 2$ and $1 < p < \infty$. Then for a given $r \in \mathbb{N}^+$*

$$E_n(f)_{E^{p,\alpha}(G)} \leq c\, \Omega^r_{\Gamma,p,\alpha}(f, 1/(n+1)), \qquad n \in \mathbb{N},$$

with a constant $c > 0$ independent of n.

Our main result in this work is the following inverse theorem.

Theorem 1. *Let $G \subset \mathbb{C}$ be a finite simply connected domain with a boundary Γ, satisfying the condition (1). If $f \in E^{p,\alpha}(G)$, $0 < \alpha \leq 2$ and $1 < p < \infty$, then for a given $r \in \mathbb{N}^+$*

$$\Omega^r_{\Gamma,p,\alpha}(f, 1/n) \leq \frac{c}{n^r} \sum_{k=1}^{n} k^{r-1} E_k(f)_{E^{p,\alpha}(G)}, \qquad n \in \mathbb{N}^+,$$

with a constant $c > 0$ independent of n.

This result in case of $\alpha = 2$, in the Lebesgue spaces $L^p(\mathbb{T})$ was proved in [30] (for detailed information see also [1], pp. 331-335). The similar results under the condition (1) in the nonweighted and weighted Smirnov classes $E^p(G)$ and $E^p(G, \omega)$ were obtained in [18] and [16], respectively.

From theorem 1 after simply computations, we deduce the following result.

Corollary 1. *If*

$$E_n(f)_{E^{p,\alpha}(G)} = \mathcal{O}\left(n^{-\beta}\right), \qquad \beta > 0, \qquad n \in \mathbb{N}^+,$$

for $0 < \alpha \leq 2$ and $1 < p < \infty$, then $f \in E^{p,\alpha}(G)$ and

$$\Omega^r_{\Gamma,p,\alpha}(f, \delta) = \begin{cases} \mathcal{O}\left(\delta^\beta\right), & r > \beta, \\ \mathcal{O}\left(\delta^\beta \log(1/\delta)\right), & r = \beta, \\ \mathcal{O}\left(\delta^r\right), & r < \beta, \end{cases}$$

for a given $r \in \mathbb{N}^+$ and $\delta > 0$.

Setting here $r := [\beta] + 1$ for a given $\beta > 0$ and defining the generalized Lipschitz class $Lip_{\alpha,p}(\beta)$, $0 \leq \alpha \leq 2$ and $p > 1$, as

$$Lip_{\alpha,p}(\beta) := \left\{ f \in E^{p,\alpha}(G) : \ \Omega^r_{\Gamma,p,\alpha}(f, \delta) = \mathcal{O}\left(\delta^\beta\right), \ \delta > 0 \right\},$$

we obtain the following.

Corollary 2. *If*

$$E_n(f)_{E^{p,\alpha}(G)} = \mathcal{O}\left(n^{-\beta}\right), \beta > 0, n \in \mathbb{N},$$

for $0 < \alpha \leq 2$ and $1 < p < \infty$, then $f \in Lip_{\alpha,p}(\beta)$.

Combining this corollary with Theorem A, we obtain the constructive characterization of the classes $Lip_{\alpha,p}(\beta)$.

Theorem 2. *Let $G \subset \mathbb{C}$ be a finite simply connected domain with a boundary Γ, satisfying the condition (1). Let also $0 < \alpha \leq 2$, $\beta > 0$ and $1 < p < \infty$.The following statements are equivalent:*

(1) $f \in Lip_{\alpha,p}(\beta)$,

(2) $E_n(f)_{E^{p,\alpha}(G)} = \mathcal{O}\left(n^{-\beta}\right)$, $\forall n \in \mathbb{N}^+$.

In case of $\alpha = 2$ and $\beta \in (0, 1)$ the last result coincides with Alper's result obtained in [2].

The similar results in the Smirnov classes of analytic functions were proved in [16].

2. Auxiliary results

Let I be any subinterval of $I_0=(0,2\pi)$ with the characteristic function χ_I. As usual we define the maximal function of χ_I, setting

$$M\chi_I(x):=\sup_{J\ni x}\frac{1}{|J|}\int_J \chi_I(y)dy,$$

where *sup* is taken over all intervals $J\ni x$. Then the following useful relation of equivalence holds:

$$M\chi_I(x)\approx \chi_I(x)+\sum_{k=0}^{\infty}2^{-k}\chi_{\left(2^{k+1}I\backslash 2^kI\right)\cap I_0}(x),\qquad x\in I_0. \tag{6}$$

Indeed, if $x\in I$, then

$$M\chi_I(x):=\sup_{J\ni x}\frac{1}{|J|}\int_J \chi_I(y)dy=\frac{1}{|I|}\int_I \chi_I(y)dy=1,$$

and also

$$\chi_I(x)+\sum_{k=0}^{\infty}2^{-k}\chi_{\left(2^{k+1}I\backslash 2^kI\right)\cap I_0}(x)=1.$$

If $x\in I_0\backslash\, I$, then there is a number $k_0\in\mathbb{N}$ such that $x\in\left(2^{k_0+1}I\backslash 2^{k_0}I\right)\cap I_0$. Let x poses after the interval $2^{k_0}I$ and b is the endpoint of I (when x poses before the interval $2^{k_0}I$ the assertion proves by similar way), then denoting $I_+:=\{y: b\leq y<x\}$ and $J_+:=I\cup I_+$ we have $I_+\cap\ I=\emptyset$, $|I_+|=dist(x,\ I)$ and $|J_+|=\ |I|+dist(x,\ I)$. Hence,

$$M\chi_I(x)=\sup_{J\ni x}\frac{1}{|J|}\int_J \chi_I(y)dy=\sup_{J\ni x}\frac{|J\cap I|}{|J|}=\frac{|J_+\cap I|}{|J_+|}=$$

$$=\frac{|\ I|}{|J_+|}=\frac{|\ I|}{|I|+dist(x,\ I)}\approx\frac{|\ I|}{|I|+2^{k_0}\,|I|}\approx\frac{1}{2^{k_0}}.$$

Also for the right side of (6) we have

$$\chi_I(x)+\sum_{k=0}^{\infty}2^{-k}\chi_{\left(2^{k+1}I\backslash 2^kI\right)\cap I_0}(x)=\frac{1}{2^{k_0}}.$$

Thus the relation (6) is true.

We begin with the Bernstein inequality concerning trigonometric polynomials T_n of degree $\leq n$ in the Morrey spaces $L^{p,\alpha}(0,2\pi)$, $0<\alpha\leq 2$ and $1<p<\infty$.

Lemma 1. *Let $L^{p,\alpha}(0,2\pi)$ be a Morrey spaces with $0<\alpha\leq 2$ and $1<p<\infty$. Then for every trigonometric polynomial T_n of degree n and $k\in\mathbb{N}^+$ the inequality*

$$\left\|T_n^{(k)}\right\|_{L^{p,\alpha}(0,2\pi)}\leq cn^k\left\|T_n\right\|_{L^{p,\alpha}(0,2\pi)},\qquad n\in\mathbb{N},$$

holds with a constant c independent of n.

Proof. We prove the inequality in case of $k=1$. The general case can be proved by iteration.

Let I be any subinterval of $I_0=(0,2\pi)$ with the characteristic function $\chi_{_I}$. As was noted in the proof of Theorem 3 from [5] (referring to [27]), the maximal function $M\chi_{_I}$ satisfies the A_1 condition of Muckenhoupt, i.e. $M\left(M\chi_{_I}\right)\leq cM\chi_{_I}$. Then it satisfies also the A_p, $p>1$, Muckenhoupt condition on I_0 and using the Bernstein inequality for the trigonometric polynomials in the weighted Lebesgue spaces $L^p\left(I_0,\omega\right)$ with $\omega\in A_p\left(0,2\pi\right)$, proved in [21], we have

$$\int\limits_I\left|T_n^{'}(t)\right|^p dt=\int\limits_{I_0}\left|T_n^{'}(t)\right|^p\chi_{_I}(t)dt\leq\int\limits_{I_0}\left|T_n^{'}(t)\right|^p M\chi_{_I}(t)dt\leq c_4 n^p\int\limits_{I_0}\left|T_n(t)\right|^p M\chi_{_I}(t)dt,$$

in case of $\omega:=M\chi_{_I}$. Applying here the equivalence (6) we get

$$\left\|T_n^{'}\right\|^p_{L^{p,\alpha}(I_0)}=\sup_I\frac{1}{|I|^{1-\frac{\alpha}{2}}}\int\limits_I\left|T_n^{'}(t)\right|^p dt=$$

$$\leq c_5\sup_I\frac{n^p}{|I|^{1-\frac{\alpha}{2}}}\int\limits_{I_0}\left|T_n(t)\right|^p\left(\chi_{_I}(t)+\sum_{k=0}^{\infty}2^{-k}\chi_{\left(2^{k+1}I\,\setminus\,2^kI\right)\cap I_0}(t)\right)dt=$$

$$=c_5\sup_I\frac{n^p}{|I|^{1-\frac{\alpha}{2}}}\int\limits_I\left|T_n(t)\right|^p|dt|+c_5\sup_I\frac{n^p}{|I|^{1-\frac{\alpha}{2}}}\sum_{k=0}^{\infty}2^{-2k}\int\limits_{\left(2^{k+1}I\,\setminus\,2^kI\right)\cap I_0}\left|T_n(t)\right|^p dt\leq$$

$$\leq c_5 n^p\left(\left\|T_n\right\|^p_{L^{p,\alpha}(I_0)}+\sum_{k=0}^{\infty}2^{-k}\sup_I\frac{1}{|I|^{1-\frac{\alpha}{2}}}\int\limits_{2^{k+1}I\,\cap I_0}\left|T_n(t)\right|^p dt\right)\leq$$

$$\leq c_5 n^p\left\|T_n\right\|^p_{L^{p,\alpha}(0,2\pi)}+c_5\sum_{k=0}^{\infty}2^{-k+(k+1)(1-\frac{\alpha}{2})}\sup_I\frac{n^p}{\left|2^{k+1}I\cap I_0\right|^{1-\frac{\alpha}{2}}}\int\limits_{2^{k+1}I\,\cap I_0}\left|T_n(t)\right|^p dt\leq$$

$$\leq c_5 n^p\left\|T_n\right\|^p_{L^{p,\alpha}(I_0)}+c_6\sum_{k=0}^{\infty}2^{-k+(k+1)(1-\frac{\alpha}{2})}\sup_I\frac{n^p}{|I|^{1-\frac{\alpha}{2}}}\int\limits_I\left|T_n(t)\right|^p dt\leq$$

$$\leq c_7 n^p\left\|T_n\right\|^p_{L^{p,\alpha}(I_0)}+c_8 n^p\left\|T_n\right\|^p_{L^{p,\alpha}(I_0)}\leq cn^p\left\|T_n\right\|^p_{L^{p,\alpha}(I_0)},$$

because $\sum\limits_{k=0}^{\infty}2^{-k+(k+1)(1-\frac{\alpha}{2})}<\infty$. ◄

The following inverse theorem is the model version of the main result and its proof, taking Lemma 1 and the above emphasized properties of modulus of smoothness $\omega^r_{p,\alpha}(f,\cdot)$ into account, realizes by repeating step by step the proof of the appropriate result in the spaces $L^p\left(0,2\pi\right)$, due to A. F. Timan and M. F. Timan [1, pp. 331-335] (see also, [7], p. 208).

Theorem 3. *Let $L^{p,\alpha}(\mathbb{T})$ be a Morrey spaces with $0<\alpha\leq 2$ and $1<p<\infty$. Then for a given $f\in L^{p,\alpha}(\mathbb{T})$ and $r\in\mathbb{N}^{+}$ the estimate*

$$\omega_{p,\alpha}^{r}(f,1/n)\leq c\ n^{-r}\sum_{k=1}^{n}k^{r-1}E_{k}(f)_{L^{p,\alpha}(\mathbb{T})},\qquad n=1,2,...,$$

holds with a constant $c>0$ independent of n.

Definition 2. *A Jordan curve Γ is said to be a regular (or Carleson) curve, if*

$$mes\{t\in\Gamma\ :\ |t-z|\ <r\}\leq cr,$$

for all $z\in\Gamma$ and $r>0$, where c> 0 does not depend on z and r.

In particular, the curves satisfying the condition *(1)* are regular.

The below mentioned result, on the boundedness of the singular integral

$$S(f)(z):=\lim_{\varepsilon\to 0}\frac{1}{2\pi i}\int\limits_{\Gamma\setminus\overline{D}(z,\varepsilon)}\frac{f(\zeta)}{\zeta-z}d\varsigma,\qquad z\in\Gamma,\quad D(z,\varepsilon):=\{\zeta:|\zeta-z|<\varepsilon\},$$

was proved in [17] (see also, [20] and [28]).

Lemma 2. *Let Γ be a Jordan regular curve and let $L^{p,\alpha}(\Gamma)$ be a Morrey space with $0<\alpha\leq 2$ and $1<p<\infty$. Then for every $f\in L^{p,\alpha}(\Gamma)$ the estimate*

$$\|S(f)\|_{L^{p,\alpha}(\Gamma)}\leq c\,\|f\|_{L^{p,\alpha}(\Gamma)},$$

holds with a constant $c=c(p,\alpha,\Gamma)>0$ independent of f.

Now we construct a linear operator from $H^{p,\alpha}(D)$ to $E^{p,\alpha}(G)$, which play an important role for the investigations of the approximation problems in the classes $E^{p,\alpha}(G)$, starting from the solutions of the similar problems in $H^{p,\alpha}(D)$. For this purpose we remind some necessary knowledges on the Faber polynomials for $\overline{G}$, which can be found in [29].

The Faber polynomials F_k, $k\in\mathbb{N}$, for $\overline{G}$ are defined through the expansion

$$\frac{\psi'(w)}{\psi(w)-z}=\sum_{k=0}^{\infty}\frac{F_k(z)}{w^{k+1}},\qquad z\in G\text{ and }w\in D^{-},\tag{6}$$

and for every $k\in\mathbb{N}$ the inequalities

$$F_k(z)=\frac{1}{2\pi i}\int\limits_{\mathbb{T}}\frac{w^k\psi'(w)}{\psi(w)-z}dw,\qquad z\in G,\qquad F_k(z)=\varphi^k(z)+\frac{1}{2\pi i}\int\limits_{\Gamma}\frac{\varphi^k(\zeta)}{\zeta-z}d\varsigma,\qquad z\in G^{-},$$

hold.

If $f \in E^{p,\alpha}(G)$, then by definition $f \in E^p(G)$ and hence

$$f(z) = \frac{1}{2\pi i}\int\limits_{\Gamma} \frac{f(\zeta)}{\zeta - z} d\varsigma = \frac{1}{2\pi i}\int\limits_{\mathbb{T}} \frac{f(\psi(w))\psi'(w)}{\psi(w) - z} dw,$$

for every $z \in G$.

This representation together with (6) imply that we can associate with f the formal series

$$f(z) \sim \sum_{k=0}^{\infty} a_k F_k(z), \quad z \in G,$$

where

$$a_k = a_k(f) := \frac{1}{2\pi i}\int\limits_{\mathbb{T}} \frac{f_0(w)}{w^{k+1}} dw, \qquad k \in \mathbb{N}.$$

This formal series is called the Faber series of f, and the coefficients a_k, $k \in \mathbb{N}$, are said to be the Faber coefficients of f. By $S_n(f, z) := \sum\limits_{k=0}^{n} a_k F_k(z)$ we denote the nth partial sum of $f \in E^{p,\alpha}(G)$.

Let

$$\mathcal{P} := \{\textit{the set of all polynomials (with no restrictions on the degree)}\}$$

and

$$\mathcal{P}(D) := \{\textit{traces of all members of } \mathcal{P} \textit{ on } D\}.$$

We define the operator $T(P)$ on $\mathcal{P}(D)$ as:

$$T(P)(z) := \frac{1}{2\pi i}\int\limits_{\mathbb{T}} \frac{P(w)\psi'(w)}{\psi(w) - z} dw = \frac{1}{2\pi i}\int\limits_{\Gamma} \frac{P(\varphi(\varsigma))}{\varsigma - z} d\varsigma, \qquad z \in G. \tag{7}$$

Then

$$T\left(\sum_{k=0}^{n} b_k w^k\right) = \frac{1}{2\pi i}\sum_{k=0}^{n} b_k \int\limits_{\mathbb{T}} \frac{w^k\psi'(w)}{\psi(w) - z} dw = \sum_{k=0}^{n} b_k F_k(z).$$

If $z' \in G$, then taking limit $z' \to z \in \Gamma$ over all non-tangential paths inside Γ in (7), we get

$$T(P)(z) = S_\Gamma (P \circ \varphi)(z) + \frac{1}{2}(P \circ \varphi)(z),$$

a. e. on Γ. Hence applying Lemma 2 and relation (2)we conclude that

$$\|T(P)\|_{L^{p,\alpha}(\Gamma)} \le c_{10} \|(P \circ \varphi)\|_{L^{p,\alpha}(\Gamma)} \le c \|P\|_{L^{p,\alpha}(\mathbb{T})}.$$

Therefore, we obtain the following result.

Lemma 3. *If* Γ *satisfies the condition (*1*), then the linear operator* $T : \mathcal{P}(D) \to E^{p,\alpha}(G)$, *with* $0 < \alpha \leq 2$ *and* $1 < p < \infty$*, is bounded.*

Extending the operator $T : P(D) \to E^{p,\alpha}(G)$,$0 \leq \alpha \leq 2$ and $1 < p < \infty$, from $\mathcal{P}(D)$ to the space $H^{p,\alpha}(D)$ as a linear and bounded operator, for the extension $T : H^{p,\alpha}(D) \to E^{p,\alpha}(G)$, $0 \leq \alpha \leq 2$ and$1 < p < \infty$, we have the representation

$$T(g)(z) := \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{g(w)\psi'(w)}{\psi(w) - z} dw, \;\; z \in G, \;\; g \in H^{p,\alpha}(D).$$

Theorem 4. *If* Γ *satisfies the condition (1), then the linear operator*

$$T : H^{p,\alpha}(D) \to E^{p,\alpha}(G), \;\; 0 \leq \alpha \leq 2 \;\; \textit{and} \;\; 1 < p < \infty,$$

is one-to one and onto.

Proof. Let $g \in H^{p,\alpha}(D)$ with the Taylor expansion

$$g(w) = \sum_{k=0}^{\infty} \alpha_k w^k, \qquad w \in D.$$

Setting $g_r(w) := g(rw)$, $0 < r < 1$, we have

$$\frac{1}{|B \cap \mathbb{T}|^{1-\frac{\alpha}{2}}} \int_{B \cap \mathbb{T}} |g_r(w) - g(w)|^p |dw| = \frac{1}{|B \cap \mathbb{T}|^{1-\frac{\alpha}{2}}} \int_{\mathbb{T}} |g_r(w) - g(w)|^p \chi_{B \cap \mathbb{T}}(w) |dw| \leq$$

$$\leq \frac{1}{|B \cap \mathbb{T}|^{1-\frac{\alpha}{2}}} \int_{\mathbb{T}} |g_r(w) - g(w)|^p M\chi_{B \cap \mathbb{T}}(w) |dw|. \tag{8}$$

for every disk $B \subset \mathbb{C}$. As was emphasized above, $M\chi_{B \cap \mathbb{T}} \in A_1$. Moreover, the function $g \in H^{p,\alpha}(D)$ is the Poisson integral of its boundary function $g \in L^{p,\alpha}(\mathbb{T})$. Taking these arguments and [26, Theorem 10] into account, we have

$$\lim_{r \to 1} \int_{\mathbb{T}} |g_r(w) - g(w)|^p M\chi_{B \cap \mathbb{T}}(w) |dw| = 0.$$

Hence from (8) we get

$$\lim_{r \to 1} \|g_r - g\|_{L^{p,\alpha}(\mathbb{T})} = 0,$$

which by the boundedness of the operator $T : H^{p,\alpha}(D) \to E^{p,\alpha}(G), \; 0 < \alpha \leq 2$ and $1 < p < \infty$, implies that

$$\lim_{r \to 1} \|T(g_r) - T(g)\|_{L^{p,\alpha}(\Gamma)} = 0. \tag{9}$$

The series $\sum\limits_{k=0}^{\infty} \alpha_k r^k w^k$ converges uniformly on $\mathbb{T}$, because the series $\sum\limits_{k=0}^{\infty} \alpha_k w^k$ is uniformly convergent on $|w| = r < 1$. Hence,

$$T_p(g_r)(z) = \frac{1}{2\pi i}\int\limits_{\mathbb{T}} \frac{g_r(w)\psi'(w)}{\psi(w)-z}dw = \sum_{k=0}^{\infty}\alpha_k r^k \frac{1}{2\pi i}\int\limits_{\mathbb{T}} \frac{w^k\psi'(w)}{\psi(w)-z}dw = \sum_{k=0}^{\infty}\alpha_k r^k F_k(z),$$

for $z \in G$. Taking the limit as $z^{'} \to z \in \Gamma$ along all non-tangential paths inside Γ, we have

$$T(g_r)(z) = \sum_{k=0}^{\infty}\alpha_k r^k F_k(z), \qquad z \in \Gamma.$$

From this equality by Lemma 3 of [10, p. 43], for the Faber coefficients $a_k\left(T_p(g_r)\right)$ of $T_p(g_r)$, we have

$$a_k\left(T(g_r)\right) = \frac{1}{2\pi i}\int\limits_{\mathbb{T}} \frac{T(g_r)\circ\psi(w)}{w^{k+1}}dw = \sum_{k=0}^{\infty}\alpha_k r^k \frac{1}{2\pi i}\int\limits_{\mathbb{T}} \frac{F_k\circ\psi(w)}{w^{k+1}}dw = \alpha_k r^k, \qquad k \in \mathbb{N},$$

and hence

$$a_k\left(T(g_r)\right) \to \alpha_k, \quad \text{as} \quad r \to 1^-. \tag{10}$$

Now by (2) and Hölder's inequality,

$$|a_k\left(T(g_r)\right) - a_k\left(T(g)\right)| = \left|\frac{1}{2\pi i}\int\limits_{\mathbb{T}} \frac{[T(g_r)-T(g)]\circ\psi(w)}{w^{k+1}}dw\right| \leq$$

$$\leq \frac{1}{2\pi}\int\limits_{\mathbb{T}} |[T(g_r)-T(g)]\circ\psi(w)|\,|dw|\,,$$

$$\frac{1}{2\pi}\int\limits_{\Gamma} |[T(g_r)-T(g)]\,(z)|\,\left|\varphi'(z)\right|\,|dz| \leq$$

$$\leq \frac{c}{2\pi}\int\limits_{\Gamma} |[T(g_r)-T(g)]\,(z)|\,|dz| \leq c_9\,\|T(g_r)-T(g)\|_{L^p(\Gamma)} \leq c_{10}\,\|T(g_r)-T(g)\|_{L^{p,\alpha}(\Gamma)}\,.$$

From here, by virtue of (9)

$$a_k\left(T(g_r)\right) \to a_k\left(T(g)\right) \quad \text{as} \quad r \to 1^-.$$

This and the relation (10) yield that

$$a_k\left(T(g)\right) = \alpha_k, \qquad k \in \mathbb{N}.$$

Hence, if $T(g)=0$, then $\alpha_k = a_k(T(g)) = 0$ for $k=0,1,2,...$, and thus $g=0$. This proves that the operator

$$T_p : H^{p,\alpha}(D) \to E^{p,\alpha}(G),$$

is one-to-one.

Now let $f \in E^{p,\alpha}(G)$. Consider the function $f_0 = f \circ \psi \in L^{p,\alpha}(\mathbb{T})$. The non-tangential boundary values of the functions f_0^+ and f_0^-, defined respectively by the representations (4) and (5), have the representations

$$f_0^+(w) = S(f_0)(w) + f_0(w)/2,$$

$$f_0^-(w) = S(f_0)(w) - f_0(w)/2,$$

almost everywhere on $\mathbb{T}$ and hence

$$f_0(w) = f_0^+(w) - f_0^-(w),$$

almost everywhere on $\mathbb{T}$. Then for the Faber coefficients $a_k(f)$, $k \in \mathbb{N}$, we get

$$a_k(f) = \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{f_0(w)}{w^{k+1}} dw = \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{f_0^+(w)}{w^{k+1}} dw - \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{f_0^-(w)}{w^{k+1}} dw = \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{f_0^+(w)}{w^{k+1}} dw = a_k(f_0^+),$$

because $f_0^- \in H^1(D)$ and $f_0^-(\infty) = 0$. This means that the Faber coefficients $a_k(f)$, $k \in \mathbb{N}$, of f also becomes the Taylor coefficients $a_k(f_0^+)$, $k \in \mathbb{N}$, of f_0^+ at the origin, namely

$$f_0^+(w) = \sum_{k=0}^{\infty} a_k w^k, \qquad w \in D.$$

On the other hand, from the first part of the proof we have

$$T(f_0^+) \sim \sum_{k=0}^{\infty} a_k(f) F_k.$$

Since there are no two different functions in $E^p(G)$ that have the same Faber coefficients [3], we conclude that $T(f_0^+) = f$. Therefore, the operator T is onto. ◄

3. Proof of Main result

Proof. (of Theorem 1.) Let $f \in E^{p,\alpha}(G)$. Then $T_p(f_0^+) = f$, by the proof of Theorem 4.

Since $T_p : H^{p,\alpha}(D) \to E^{p,\alpha}(G)$ is linear, bounded, one-to-one and onto, the operator

$$T_p^{-1} : E^{p,\alpha}(G) \to H^{p,\alpha}(D),$$

is also bounded.

Let $P_k^* \in P_k$, $k \in \mathbb{N}$, be the polynomials of best approximation to f in $E^{p,\alpha}(G)$, i.e.,

$$E_k(f)_{L^{p,\alpha}(G)} = \|f - P_k^*\|_{L^{p,\alpha}(\Gamma)}.$$

It is clear that $T_p^{-1}(P_k^*) \in \mathcal{P}_k(D)$ and therefore,

$$E_k(f_0^+)_{H^{p,\alpha}(D)} \leq \left\|f_0^+ - T_p^{-1}(P_k^*)\right\|_{L^{p,\alpha}(\mathbb{T})} = \left\|T_p^{-1}(f) - T_p^{-1}(P_k^*)\right\|_{L^{p,\alpha}(\mathbb{T})} \leq$$

$$\leq \left\|T_p^{-1}\right\| \|f - P_k^*\|_{L^{p,\alpha}(\Gamma)} = \left\|T_p^{-1}\right\| E_k(f)_{E^{p,\alpha}(G)}.$$

Hence, applying Theorem 3 in case of $H^{p,\alpha}(D)$ and the last relation, we have

$$\Omega^r_{\Gamma,p,\alpha}(f, 1/n) = \omega^r_{p,\alpha}(f_0^+, 1/n) \leq c_{11} n^{-r} \sum_{k=1}^{n} k^{r-1} E_k(f_0^+)_{H^{p,\alpha}(D)} \leq$$

$$\leq c_{11} n^{-r} \left\|T_p^{-1}\right\| \sum_{k=1}^{n} k^{r-1} E_k(f)_{E^{p,\alpha}(G)} \leq c\, n^{-r} \sum_{k=1}^{n} k^{r-1} E_k(f)_{E^{p,\alpha}(G)},$$

which proves Theorem 1. ◄

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Daniyal M. Israfilov
Department of Mathematics, Faculty of Art and Science,
Balikesir University, 10145, Balikesir, Turkey
E-mail: mdaniyal@balikesir.edu.tr

Pınar N. Tozman
Department of Mathematics, Faculty of Art and Science,
Balikesir University, 10145, Balikesir, Turkey
E-mail: ptuzkaya@hotmail.com

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