Tuning of Fractional Order $PI^{\lambda}D^{\mu}$ controller with Response Surface Methodology

Beyza B. İskender
Faculty of Science and Arts
Department of Mathematics
Balikesir University
Balikesir, Turkey
Email: biskender@balikesir.edu.tr

Necati Özdemir
Faculty of Science and Arts
Department of Mathematics
Balikesir University
Balikesir, Turkey
Email: nozdemir@balikesir.edu.tr

Aslan D. Karaoglan
Faculty of Engineering and Architecture
Department of Industrial Engineering
Balikesir University
Balikesir, Turkey
Email: deniz@balikesir.edu.tr

***Abstract*—This paper presents response surface methodology for tuning of fractional order $PI^{\lambda}D^{\mu}$ controller of a fractional order diffusion system subject to input hysteresis which is defined with Riemann-Liouville fractional derivative. Eigenfunction expansion method and the Grünwald-Letnikov numerical technique are used to solve the system. The necessary data for response surface analysis are read from the obtained numerical solution. Then second-order polynomial response surface mathematical model for the experimental design is presented and the optimum controller parameters are predicted from this model. The proposed tuning method is compared with the technique of minimization of integral square error by means of settling time and the results are discussed.**

I. Introduction

Fractional order system has been drawn great interest recently because of their advantages to model systems more accurately than integer order models. Improvement of fractional order systems in different areas of science and technology brought about a new control tool which is called fractional order controllers. CRONE is one of the prior fractional order controller design which was presented by Oustaloup [1]. Then, Podlubny [2] generalized the classical PID controller to the fractional calculus by replacing order of the integral and the derivative controllers with fractional orders λ and μ, respectively. It is called as fractional order $PI^{\lambda}D^{\mu}$ controller. This controller has five parameters to tune while classical PID has only three parameters. Therefore it is more flexible and advantageous. The researches on fractional order controllers were extended to other classical control types, for example fractional order optimal control [3] [6] or fractional order sliding mode control [7].

PID controller is frequently preferred type of controllers due from its ease implementation to industrial systems. Thus, fractional order $PI^{\lambda}D^{\mu}$ controller is also preferable for both integer and fractional order control systems. Recently, many methods have been presented for tuning problem of $PI^{\lambda}D^{\mu}$ controller which is more complex according to tuning problem of classical PID [8]- [12]. In this paper, response surface methodology is presented to tune the fractional order $PI^{\lambda}D^{\mu}$ controller. This method is a collection of mathematical and statistical techniques where a response of interest is influenced by several variables and the objective is to optimize this response [13]. The optimization of the controller parameters using response surface method is achieved by simultaneous testing of limited number of experiments read from the system under control. The controlled system is chosen as the fractional order system subject to input hysteresis which was previously controlled with fractional order $PI^{\lambda}D^{\mu}$ via minimizing of integral square error by Özdemir & İskender [14]. Due to the choice of the system the comparison purpose between these two methods is succeed. Because of the considered fractional order system is mathematical, the necessary data are obtained by solving the partial fractional differential equation constructed by Riemann-Liouville Fractional Derivative (RLFD) defined as

$$ {}_0D_t^{\alpha}x(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_0^t (t-\tau)^{n-\alpha-1}x(\tau)\,d\tau, \tag{1}$$

where $x(\cdot)$ is a time depended function, α is order of derivative such that $n-1 \leq \alpha < n$, $n \in N^{+}$ and $\Gamma(.)$ is Euler's gamma function [15].

The solution is obtained with eigenfunction expansion and Grünwald-Letnikov numerical methods. Detail of this methods and other numerical methods for fractional calculus can be found in the book [16] which has been recently published. Finally, the two methods are compared by simulations.

II. Fractional Order $PI^{\lambda}D^{\mu}$ Controller

PID controller which is the combination of proportional, integral and derivative actions represents a basic control structure. It is defined by the following equation

$$u(t) = k_p e(t) + k_i \int e(t)\,dt + k_d \frac{d}{dt}e(t), \tag{2}$$

where t is time variable, $u(t)$ is control and $e(t)$ is error functions, k_p, k_i and k_d are gains of the proportional, integral and derivative controllers, respectively. The error function is the difference between a desired reference value $r(t)$ and the system output $y(t)$. The response of a system can be optimized by tuning of the coefficients of k_p, k_i and k_d. Since this rule

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can be easily applied to most of system the controller is still preferable. Therefore, it is generalized for fractional order systems which is known as fractional order $PI^{\lambda}D^{\mu}$ controller which involves an integrator of order λ and a differentiator of order μ. This controller can be also applied to integer order systems. The $PI^{\lambda}D^{\mu}$ is defined by

$$u(t) = k_p e(t) + k_i I^{\lambda} e(t) + k_d D^{\mu} e(t). \tag{3}$$

It can be clearly seen that selection of $\lambda = 1$ and $\mu = 1$ gives the classical PID controller. As seen between the above equations the $PI^{\lambda}D^{\mu}$ controller has five parameters which are the coefficients of k_p, k_i and k_d, and the orders of λ and μ while the integer order PID has only three parameters. Thus $PI^{\lambda}D^{\mu}$ controller is deduced that more flexible than the integer PID controller. Moreover, it is less sensitive to change of parameters of controlled system, see [15].

Several tuning strategies have been introduced for $PI^{\lambda}D^{\mu}$ in the literature. In this paper we present response surface method and compare this method with another one which is based on minimization of the integral square error.

III. Fractional order diffusion systems subject to input hysteresis

A fractional diffusion process on the one-dimensional spatial domain $[0,1]$, with diffusion coefficient ν and nonlinear control action applied at point $x_b \in (0,1)$ via the SSSL hysteresis operator Φ, is given by the following partial fractional differential equation:

$$\frac{\partial^{\alpha} z(t,x)}{\partial t^{\alpha}} = \nu \frac{\partial^2 z(t,x)}{\partial x^2} + \delta(x - x_b)\Phi(u(t)) \tag{4}$$

with the Dirichlet boundary conditions

$$z(t,0) = z(t,1) = 0, \tag{5}$$

and zero initial condition

$$z(0,x) = 0. \tag{6}$$

The system is observed at a point $x_c \in (x_b, 1)$ such that

$$y(t) = z(t, x_c). \tag{7}$$

The SSSL operator $\omega = \Phi(u)$ is defined by the following differential equation:

$$\frac{d\omega}{dt} = \rho[\zeta u - \omega]\left|\frac{du}{dt}\right| + \eta \frac{du}{dt}. \tag{8}$$

In Eq.(8), the input u and the output ω are real valued functions of time t with piecewise continuous derivatives u and ω, $\left|\frac{du}{dt}\right|$ is the absolute value of $\frac{du}{dt}$. ρ, ζ and η are some constants satisfying the condition $\zeta > \eta$, see [17].

Using separation of variables the general solution of the system is obtained as

$$z(t,x) = \sum_{k=1}^{\infty} q_k(t) \sin(k\pi x). \tag{9}$$

where $\sin(k\pi x)$ are eigenfunctions and $q_k(t)$ are state eigencoordinates. Since the higher order terms do not contribute much, it could be of interest to keep only a finite number of terms denoted by m. After some necessary calculations (where the detailed can be found in [14]) the state space representation of the system is obtained as:

$$\begin{aligned} {}_0D_t^{\alpha} q(t) &= Aq(t) + B\Phi(u(t)) \\ y(t) &= Cq(t) \end{aligned} \tag{10}$$

where $q(t) = \begin{bmatrix} q_1(t) & q_2(t) & \ldots & q_m(t) \end{bmatrix}^T$ is the state variable, $A \in R^{m\times m}$, $B \in R^m$ and $C \in R^{1\times m}$ are the matrices given by

$$\begin{aligned} A &= diagonal\left[-\nu k^2 \pi^2\right], \\ B &= \begin{bmatrix} b_1 & b_2 & \ldots & b_m \end{bmatrix}^T, \\ C &= \begin{bmatrix} c_1 & c_2 & \ldots & c_m \end{bmatrix}, \end{aligned}$$

in which $k = 1,2,...,m$, $b_k = 2\sin(k\pi x_b)$ and $c_k = \sin(k\pi x_c)$. The solution of the System (11) is obtained numerically by Grünwald-Letnikov approximation. For this purpose, the time interval $[0,T]$ is divided N equal parts with size of $h = \frac{1}{N}$ and the nodes are labeled as $0,1,2,...,N$. The Grünwald-Letnikov approximation of the RLFD at node M is

$${}_0D_t^{\alpha} q(hM) = \frac{1}{h^{\alpha}} \sum_{j=0}^{M} w_j^{(\alpha)} q(hM - jh), \tag{11}$$

where the coefficients $w_j^{(\alpha)}$ are computed by the following recurrence relationships

$$\begin{aligned} w_0^{(\alpha)} &= 1; \\ w_j^{(\alpha)} &= \left(1 - \frac{\alpha+1}{j}\right) w_{j-1}^{(\alpha)} \end{aligned}$$

for $j = 1,2,...,N$. Using Eq.(11), numerical solution of the System (11) is obtained as

$$\begin{aligned} q(hM) &= \left(\frac{1}{h^{\alpha}} w_0^{(\alpha)} I - A\right)^{-1} \\ &\times \left(B\Phi(u(hM)) - \frac{1}{h^{\alpha}} \sum_{j=1}^{M} w_j^{(\alpha)} q(hM - jh)\right). \end{aligned} \tag{12}$$

Similarly, the $PI^{\lambda}D^{\mu}$ controller can be computed by the Grünwald-Letnikov approximation. Note that, integrator of order λ is also approximated with Eq.(11) by replacing α with $-\lambda$. Therefore, the $PI^{\lambda}D^{\mu}$ controller at node M can be numerically calculated as

$$\begin{aligned} u(Mh) &= k_p e(Mh) + k_i \frac{1}{h^{-\lambda}} \sum_{j=0}^{M} w_j^{(-\lambda)} e(Mh - jh) \\ &\quad + k_d \frac{1}{h^{\mu}} \sum_{j=0}^{M} w_j^{(\mu)} e(Mh - jh). \end{aligned} \tag{13}$$

Control objective of the system is to reach a fixed reference input $r(t) = r$ with a minimum settling time and no overshoot.

For this purpose the method of minimizing integral square error has been used in [14] defined as

$$J(p) = \int_0^\infty [e(t,p)]^2 dt, \tag{14}$$

where p is the vector of control parameters:

$$p = \begin{bmatrix} k_p & k_i & k_d & \lambda & \mu \end{bmatrix}, \tag{15}$$

and $e(t,p)$ is the error function between reference input function $r(t)$ and the system response $y(t)$. Then the following algorithm has been given to calculate the optimum control parameters

Step 1. *Initialization*

- *Choose time interval,*
- *Choose convergent tolerance* ϵ*,*
- *Set loop counter* $k = 0$*,*
- *Choose the initial controller parameter vector* $p(k)$*.*

Step 2. *Gradient Calculation*

- *Calculate gradient of J. If the gradient satisfies the following condition*

$$\left|\frac{\partial J}{\partial p}(k)\right| < \epsilon,$$

then stop.

Step 3. *Update calculation*

- *Compute the update parameters* γ_k *and* R_k*, and compute*

$$p(k+1) = p(k) - \gamma_k R_k^{-1} \frac{\partial J}{\partial p}(k), \tag{16}$$

- *Update* $k = k+1$ *and go to Step 2.*

Here, R_k^{-1} *is chosen as Hessian of J and* γ_k *is a positive real scalar that determines the step size.*

For $\alpha = 0.8$, $\nu = 1$, $x_b = 0.25$ and $x_c = 0.375$. the optimum parameters have been obtained according to the above algorithm as $k_p = 0.2022, k_i = 0.1915, k_d = 0.1958, \lambda = 0.1921$ and $\mu = 0.1904$ in [14]. Then output of the system is $y(t) = 0.997$ and settling time is $t = 13.8$. Another tuning strategies for the System (11) is proposed in the following section.

IV. Response Surface Method

For determining the desired values of the output y and the settling time t the optimum controller parameters of k_p, k_i, k_d, λ, and μ are calculated by using response surface methodology. First of all the mathematical relationships between the responses (y and t) and the tuning parameters (k_p, k_i, k_d, λ, and μ) are established. By using this mathematical equation optimum parameters are determined for $y(t) = 1$ and minimum t. The general second-order polynomial response surface mathematical model (full quadratic model) for the experimental design presented in the present study ([13], [18], [19], [20]) is

$$Y = \beta_0 + \sum_{i=1}^{n} \beta_i X_i + \sum_{i=1}^{n} \beta_{ii} {X_i}^2 + \sum_{j=1}^{n}\sum_{j<i} \beta_{ij} X_i X_j + E, \tag{17}$$

TABLE I

Factor levels

factors	minimum	high
k_p	0.15	0.30
k_i	0.15	0.45
k_d	0.25	0.30
λ	0.15	0.45
μ	0.05	0.25

in which $Y = \begin{bmatrix} y & t \end{bmatrix}^T$ is the response and the β's are parameters whose values are to be determined. X_i and X_j are the factors and the E is the random error term. The model in terms of the observations may be written in matrix notation as

$$Y = \beta X + E, \tag{18}$$

where Y is the output matrix and X is the input matrix and ϵ is the residuals (random error term). The least square estimator of β matrix that composes of coefficients of the regression equation calculated by the given formula:

$$\beta = (X^T X)^{-1} X^T Y. \tag{19}$$

To reduce the number of tests, an L_{32} orthogonal array that only needs 32 experimental runs was adopted. Because of using nonrandom system one center point is used in the design of experiment and by this way the number of experiments are reduced to 27 runs. MINITAB 16 statistical package is used to establish mathematical models for achieving the target value of 1 for y, while minimizing t at a desired confidence interval (95%). Table 1 and Table 2 displays the factor levels and the design of experiment matrix respectively. According to the results of the experiments given in Table 2, mathematical models based on response surface method for correlating responses such as the y and t have been established which are represented by Eqs. (20)-(21).

$$\begin{aligned} y \;=\; & 0.0876 + 4.7234k_p + 1.3262k_i - 1.2102k_d \\ & +0.2525\lambda + 2.3687\mu - 0.7383{k_p}^2 - 0.2512{k_i}^2 \\ & +9.3552{k_d}^2 + 0.8376\lambda^2 + .1347\mu^2 - 3.1383k_pk_i \\ & -10.8300k_pk_d + 1.1117k_p\lambda - 1.4882k_p\mu \\ & +1.3317k_ik_d - 1.2308k_i\lambda - 2.3288k_i\mu \\ & -3.1850k_d\lambda\mu - 1.1725k_d\mu - 0.4746\lambda \end{aligned} \tag{20}$$

$$\begin{aligned} t \;=\; & -300.83 + 341.23k_p - 90.10k_i + 2267.08k_d \\ & +0.38\lambda - 26.29\mu - 87.68{k_p}^2 + 54.75{k_i}^2 \\ & -3429.09{k_d}^2 + 216.97\lambda^2 - 6.82\mu^2 + 259.44k_pk_i \\ & -1533.33k_pk_d + 256.67k_p\mu - 108.33k_ik_d \\ & +11.39k_i\lambda + 173.75k_i\lambda + 30.00k_p\lambda - 500k_d\lambda \\ & +725k_d\mu + 57.50\lambda\mu \end{aligned} \tag{21}$$

TABLE II
DESIGN OF EXPERIMENT MATRIX

Ex.no	k_p	k_i	k_d	λ	μ	y	t
1	0.150	0.150	0.250	0.150	0.250	1.130	17.75
2	0.300	0.150	0.250	0.150	0.050	1.028	15.95
3	0.150	0.450	0.250	0.150	0.050	1.052	10.20
4	0.300	0.450	0.250	0.150	0.250	1.224	28.50
5	0.150	0.150	0.300	0.150	0.050	0.930	24.40
6	0.300	0.150	0.300	0.150	0.250	1.290	16.35
7	0.150	0.450	0.300	0.150	0.250	1.320	23.20
8	0.300	0.450	0.300	0.150	0.050	1.147	3.90
9	0.150	0.150	0.250	0.450	0.050	0.805	29.00
10	0.300	0.150	0.250	0.450	0.250	1.280	30.00
11	0.150	0.450	0.250	0.450	0.250	1.048	26.65
12	0.300	0.450	0.250	0.450	0.050	1.023	24.00
13	0.150	0.150	0.300	0.450	0.250	1.117	29.15
14	0.300	0.150	0.300	0.450	0.050	1.024	6.50
15	0.150	0.450	0.300	0.450	0.050	0.988	10.30
16	0.300	0.450	0.300	0.450	0.250	1.089	29.15
17	0.150	0.300	0.275	0.300	0.150	1.036	6.35
18	0.300	0.300	0.275	0.300	0.150	1.108	30.00
19	0.225	0.150	0.275	0.300	0.150	1.056	12.20
20	0.225	0.450	0.275	0.300	0.150	1.085	27.60
21	0.225	0.300	0.250	0.300	0.150	1.067	5.45
22	0.225	0.300	0.300	0.300	0.150	1.097	27.60
23	0.225	0.300	0.275	0.150	0.150	1.142	29.80
24	0.225	0.300	0.275	0.450	0.150	1.048	17.30
25	0.225	0.300	0.275	0.300	0.050	1.007	7.20
26	0.225	0.300	0.275	0.300	0.250	1.148	30.00
27	0.225	0.300	0.275	0.300	0.150	1.085	4.55

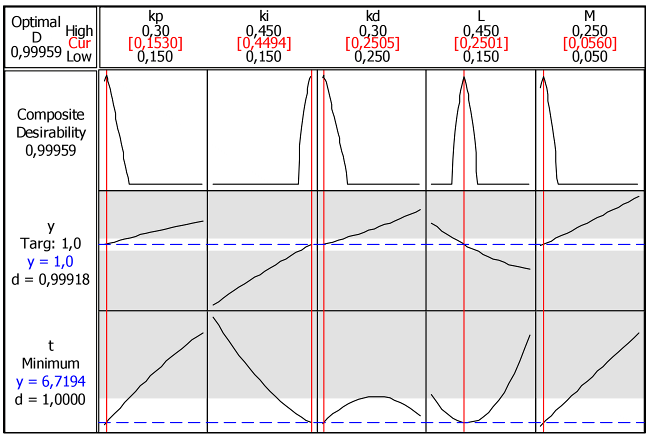

Fig. 1. Optimum levels of parameters obtained from response optimizer module of MINITAB Package.

By using the response optimizer module of MINITAB the optimum parameter levels are determined as $k_p = 0.1530$ $k_i = 0.4494$, $k_d = 0.2505$ $\lambda = 0.2501$ and $\mu = 0.0560$. As shown is Figure 1, by using the given parameter combination y is predicted as 1.00 while t is predicted as 6.7194. After the confirmation tests for the given optimum parameter levels by using MATLAB, $y = 1.00$ and $t = 6.15$ are obtained. Results are displayed in Figure 2. Therefore it can be concluded that the settling time is decreased by the response surface method.

Results points out that the simulated results obtained from fitted model and the real results of experiments calculated by using MATLAB are close to each other. Real system results are better for the response t when it is compared with its expected value from MINITAB.

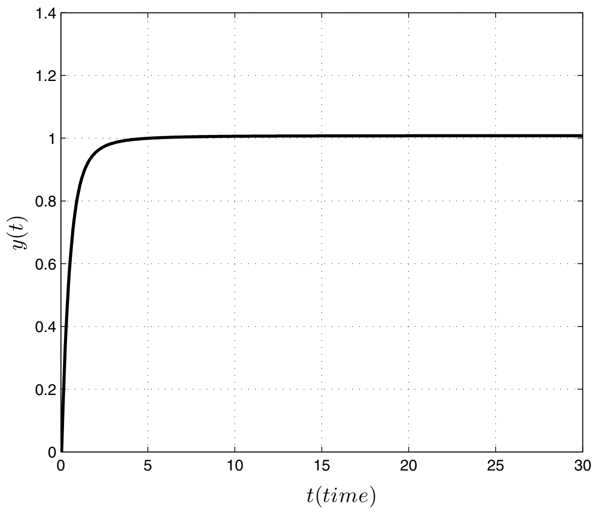

Fig. 2. Response of the system with tuning parameters via response surface method

V. CONCLUSION

The tuning strategies of fractional order $PI^{\lambda}D^{\mu}$ controller for a fractional order diffusion process subject to input hysteresis is developed by response surface methodology. To reach the fixed desired output with minimum time 27 experiment data are read on numerical solution of the system and so the orthogonal design of experiment matrix is constructed. The mathematical relation between the response values y and t and the fractional order controller parameters k_p, k_i, k_d, λ, and μ are obtained by a full quadratic model. When comparing output of the system according to response surface method and minimizing integral square error strategy, it can be concluded that the settling time is decreased.

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