Time-Fractional Boundary Optimal Control Of Thermal Stresses

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***Abstract*—In this paper, a temperature field described by a fractional heat subconduction equation with a boundary temperature control is considered. The foundation of an optimal boundary control to take the thermal stress under constraints is purposed. Problem is formulated in terms of Caputo time-fractional derivative. The solution is found by applying Laplace and finite Fourier sine transforms. In addition, linear approximation is used to get the numerical solution. Consequently, the graphics of numerical results obtained by MATLAB are illustrated.**

I. Introduction

The classical thermoelasticity researches the stresses properties of parabolic heat conduction equation based on the Fourier law which gives the relation between heat flux and temperature gradient and leads to the classical diffusion equation

$$\rho\frac{\partial c}{\partial t} = \kappa\Delta c. \tag{1}$$

In non-classical theory of thermoelasticity, the heat conduction equation is generalized in terms of fractional derivative of order α $(0 < \alpha < 2)$ and the stresses caused by the temperature field occurring due to fractional heat conduction are investigated. This generalized type equation is also called anomalous diffusion and is characterized by the time-fractional differential equation

$$\rho\frac{\partial^\alpha c}{\partial t^\alpha} = \kappa\Delta c \tag{2}$$

with the particular cases:

$$0 < \alpha < 1, \text{ subdiffusion (weak diffusion)},$$

$$\alpha = 1, \text{ normal diffusion},$$

$$1 < \alpha < 2, \text{ superdiffusion (strong diffusion)},$$

$$\alpha = 2, \text{ ballistic diffusion}.$$

Fractional heat conduction equation in the case of $0 < \alpha < 1$, which is called as 'heat subconduction', is taken under consideration in this work. The theory of thermoelasticity based on anomalous heat conduction equation was proposed and the stresses corresponding to the fundamental solutions to the Cauchy problem for the one and two-dimensional fractional heat conduction equation were studied in [1]. The central-symmetric thermal stresses in an infinite medium with a spherical [2] and cylindrical [3] cavity for different boundary conditions were analyzed. As a further generalization, a quasistatic uncoupled theory of thermal stresses for space-time fractional heat conduction equation was introduced [4]. In recent years, thermal stresses for a fractional telegraph equation have also been researched ([5], [6]) . Further discussion on generalized thermoelasticity should be found in [7]-[11].

In the present paper, we analyze the thermal stresses corresponding to the heat subconduction equation defined in terms of fractional Caputo term. In classical sense, the optimal heating mode with respect to stress over the thickness of a spherical shell in the absence of external force loading and with the zero initial condition was proposed [12]. Another work was presented for stress optimization of the thermal conditions of heating after a minimum time to the tempering temperature with specified constraints on the temperature of the heaters . For more detailed researches on this subject, see [13]-[19]. Recently, a mathematical model which is defined by the standard parabolic heat conduction equation describing the temperature field and assuring the stress under control with the linear boundary heating has been studied [20]. Here, we investigate the fractional generalization of the consideration of [20].

II. Preliminaries

In this section, we briefly give basic definitions and relations necessary for problem formulation. It is well known in the fractional calculus literature that several definitions of a fractional derivative have been proposed such as Riemann-Liouville, Grünwald-Letnikov, Weyl, Caputo, Riesz, etc (see [21], [22]). Here, we consider the main problem in terms of the Caputo time-fractional derivative. Note that, from the physical point of view, one-dimensional model of heat conduction equation with the Caputo time-fractional derivative is a good description to study thermal stress in a large radius cylindrical hole or globe. The Caputo derivative of the fractional order α $(n-1<\alpha\leq n)$ is defined by

$$\frac{d^{\alpha}f(t)}{dt^{\alpha}}=\frac{1}{\Gamma(n-\alpha)}\int_{0}^{t}(t-\tau)^{n-\alpha-1}\frac{d^{n}f(\tau)}{d\tau^{n}}d\tau,$$

and the Laplace transform rule for this operator has the form

$$\mathcal{L}\left\{\frac{d^{\alpha}f(t)}{dt^{\alpha}}\right\}=s^{\alpha}\mathcal{L}\{f(t)\}-\sum_{k=0}^{n-1}f^{(k)}\left(0^{+}\right)s^{\alpha-1-k}.$$

This operator has wide applications because the initial conditions of fractional differential equations with Caputo derivatives should be expressed in terms of a given function and its derivatives of integer order. This gives us physically interpretable initial conditions for fractional differential equations.

The following formula for the inverse Laplace transform

$$\mathcal{L}^{-1}\left\{\frac{s^{\alpha-\beta}}{s^{\alpha}+b}\right\}=t^{\beta-1}E_{\alpha,\beta}(-bt^{\alpha})$$

will be applied in the problem formulation, where $E_{\alpha,\beta}(z)$ is the two-parameter Mittag-Leffler function defined as

$$E_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{z^{n}}{\Gamma(\alpha n+\beta)},\quad \alpha>0,\quad \beta>0.$$

The finite Fourier sine transform

$$S[f]=S_{n}=\frac{2}{L}\int_{0}^{L}f(x)\sin\left(\frac{n\pi x}{L}\right)dx,\quad (n=1,2,...)$$

will be used for transform of the spatial coordinate x. If $f(x,t)$ is a function of two variables, then we transform the x variable as follows

$$S[f]=S_{n}(t)=\frac{2}{L}\int_{0}^{L}f(x,t)\sin\left(\frac{n\pi x}{L}\right)dx.$$

Below we present a few of useful laws

$$S[f_{t}]=\frac{dS[f]}{dt},$$

$$S[f_{tt}]=\frac{d^{2}S[f]}{dt^{2}},$$

$$S[f_{xx}]=-[n\pi/L]^{2}S[f]+\frac{2n\pi}{L^{2}}\left[f(0,t)+(-1)^{n+1}f(L,t)\right].$$

III. Problem Formulation

Let $T(x,t)$ be the temperature distribution, the thermoelastic stress is proportional to the distance from the average temperature:

$$\sigma_{yy}(x,t)=-\frac{\alpha E}{1-\gamma}\left[T(x,t)-T_{average}(x,t)\right] \tag{3}$$

where

$$T_{average}(x,t)=\frac{1}{L}\int_{0}^{L}T(x,t)\,dx. \tag{4}$$

Here, α is the linear thermal expansion coefficient, E is the elasticity modulus and γ denotes the Poisson ratio. In our description, we consider that the temperature is symmetric with respect to x. The temperature field $T(x,t)$ satisfies a time-fractional heat subconduction equation as follows

$$\frac{\partial^{\alpha}T(x,t)}{\partial t^{\alpha}}=a\Delta T(x,t),\quad 0<x<L \text{ and } 0<t<\infty \tag{5}$$

where $0<\alpha<1$ and time-fractional derivative is defined in the sense of Caputo. To use the nondimensional quantities only for convenience in the calculations, we make the following changing of variables

$$\xi=\frac{x}{L},\quad t=t_{0}\tau \tag{6}$$

and then Eq. (5) reduces to

$$\frac{\partial^{\alpha}T(\xi,\tau)}{\partial\tau^{\alpha}}=\kappa^{2}\frac{\partial^{2}T(\xi,\tau)}{\partial\xi^{2}} \tag{7}$$

where $0<\xi<1, 0<\tau<\infty$ and

$$\kappa^{2}=\frac{at_{0}^{\alpha}}{L^{2}}.$$

In addition, we adopt the following initial

$$T(x,0)=0, \tag{8}$$

and boundary conditions

$$\begin{aligned} x=0: &\quad T=g(t)T_{0},\\ x=L: &\quad T=g(t)T_{0}, \end{aligned} \tag{9}$$

where $g(t)$ is the boundary control function which we motivate to find the optimal one to keep the thermal stress under constraint. Using the nondimensional temperature quantity $\overline{T}=\frac{T}{T_{0}}$, the main problem is as follows:

$$\frac{\partial^{\alpha}\overline{T}(\xi,\tau)}{\partial\tau^{\alpha}}=\kappa^{2}\frac{\partial^{2}\overline{T}(\xi,\tau)}{\partial\xi^{2}}, \tag{10}$$

$$\tau=0:\quad \overline{T}=0, \tag{11}$$

$$\xi=0:\quad \overline{T}=g(\tau), \tag{12}$$

$$\xi=1:\quad \overline{T}=g(\tau). \tag{13}$$

To solve this problem, the Laplace transform with respect to time τ and the finite Fourier sine transform with respect to the spatial coordinate ξ are used, respectively. Applying the integral transforms, we obtain

$$\overline{T}^{**} = \frac{1}{s^{\alpha} + \kappa^2 \xi_n^2} \kappa^2 \xi_n g_n^* (s) \left[1 - (-1)^n\right] \tag{14}$$

where $\xi_n = n\pi$ and taking the inverse Fourier and Laplace transforms leads to

$$\overline{T} = 2\kappa^2 \sum_{n=1}^{\infty} \xi_n \left[1 - (-1)^n\right] \sin(x\xi_n)$$

$$\times \int_0^{\tau} (\tau - u)^{\alpha-1} E_{\alpha,\alpha} \left[-\kappa^2 \xi_n^2 (\tau - u)^{\alpha}\right] g(u)\, du. \tag{15}$$

Similarly, we calculate the $\overline{T}_{average}(\xi, \tau)$ using Eq. (15)

$$\overline{T}_{average}(\xi, \tau) = 2\kappa^2 \sum_{n=1}^{\infty} \left[1 - (-1)^n\right]^2$$

$$\times \int_0^{\tau} (\tau - u)^{\alpha-1} E_{\alpha,\alpha} \left[-\kappa^2 \xi_n^2 (\tau - u)^{\alpha}\right] g(u)\, du. \tag{16}$$

Now, nondimensional stress should be given as

$$\overline{\sigma}_{yy}(\xi, \tau) = \frac{1-\gamma}{\alpha E T_0} \sigma_{yy}(\xi, \tau) \tag{17}$$

or

$$\overline{\sigma}_{yy}(\xi, \tau) = -\frac{\alpha E T_0}{1-\gamma} \left[\overline{T}(\xi, \tau) - \overline{T}_{average}(\xi, \tau)\right]. \tag{18}$$

Next, let us compute $\overline{\sigma}_{yy}(1, \tau)$, which represents the stress on the boundary, by using Eq. (13), and assume that

$$|\overline{\sigma}_{yy}(1, \tau)| = \overline{\sigma}_{crit}.$$

Taking into consideration that the maximal stresses will be at the boundary and keeping positive values of $\overline{\sigma}_{\max}(t) = \overline{\sigma}_{yy}(1, \tau) = \overline{\sigma}_{crit}$, we have

$$\overline{\sigma}_{crit} = -g(\tau) + 2\kappa^2 \int_0^{\tau} \sum_{n=1}^{\infty} \left[1 - (-1)^n\right]^2 (\tau - u)^{\alpha-1}$$

$$\times E_{\alpha,\alpha} \left[-\kappa^2 \xi_n^2 (\tau - u)^{\alpha}\right] g(u)\, du. \tag{19}$$

or

$$g(\tau) = \overline{\sigma}_{crit} + 2\kappa^2 \int_0^{\tau} \sum_{n=1}^{\infty} \left[1 - (-1)^n\right]^2 (\tau - u)^{\alpha-1}$$

$$\times E_{\alpha,\alpha} \left[-\kappa^2 \xi_n^2 (\tau - u)^{\alpha}\right] g(u)\, du. \tag{20}$$

Note that Eq. (20) is an integral equation for temperature control $g(\tau)$ for which we consider the numerical solution.

IV. Numerical Algorithm

Here, we first rearrange Eq. (20) by a successive changing of variables. In the first step, we take $y = \tau - u$ and so the integral in (20) reduces to

$$I = \int_0^{\tau} \sum_{n=1}^{\infty} c_n E_{\alpha,\alpha} \left[-\kappa^2 \xi_n^2 y^{\alpha}\right] y^{\alpha-1} g(\tau - y)\, dy \tag{21}$$

where $c_n = \left[1 - (-1)^n\right]^2$. By the second changing of variable $z = y^{\alpha}$, Eq. (21) leads to

$$I = \frac{1}{\alpha} \int_0^{\tau} \sum_{n=1}^{\infty} c_n E_{\alpha,\alpha} \left[-\kappa^2 \xi_n^2 z\right] g\left(\tau - z^{\frac{1}{\alpha}}\right) dz, \tag{22}$$

and so the integral equation (20) for $g(\tau)$ becomes

$$g(\tau) = \overline{\sigma}_{crit} + \frac{2\kappa^2}{\alpha} \int_0^{\tau} \sum_{n=1}^{\infty} c_n E_{\alpha,\alpha} \left[-\kappa^2 \xi_n^2 z\right] g\left(\tau - z^{\frac{1}{\alpha}}\right) dz. \tag{23}$$

Let us explain the numerical iteration applied to Eq. (23). The iterative form is the following

$$g_{m+1}(\tau) = \overline{\sigma}_{crit}$$

$$+\frac{2\kappa^2}{\alpha} \int_0^{\tau} \sum_{n=1}^{\infty} c_n E_{\alpha,\alpha} \left[-\kappa^2 \xi_n^2 z\right] g_m\left(\tau - z^{\frac{1}{\alpha}}\right) dz$$

$$(m = 0, 1, 2, ...)$$

where we assume the initial value of $g_0(\tau) = \overline{\sigma}_{crit} = 1$. Next, we calculate the iterative values $g_m(\tau)$ $(m = 1, 2, ...)$. Note that we have to know the values of $g_m(\tau)$ at the time nodes $\tau - z^{\frac{1}{\alpha}}$. They may not be calculated in the first iteration because of time discretization. If we take the time interval $[0, T]$ and divide it into N equal subintervals, we only know the values of $g_m(\tau)$ at $\tau = Nh$. To calculate other values of $g_m(\tau)$ for the values $lh < \tau - z^{\frac{1}{\alpha}} < (l+1)h$, $(l = 1, 2, ..., N)$, we use a linear approximation. The obtained results are illustrated in Figures 1 and 2 under some variation of problem parameters. We plot these figures with the assumption that $\kappa = 0.5$ and the upper limit of the sum for $g_m(\tau)$ equals to 20. The dependence of the fourth iteration value of control function $g_4(t)$ on changing of fractional order α is analyzed in Figure 1. We validate contribution of the iteration number to the solution in Figure 2. Note that, we only show the results up to the iteration number $m = 4$. The solutions coincide for $m > 5$.

V. Conclusion

In this work, the time-fractional heat subconduction equation with the temperature boundary control has been considered. It has been aimed to take the thermal stresses under constraint with an optimal boundary control function. For these purposes, a mathematical formulation and solution has been introduced. In the description of the problem, the Caputo fractional derivative has been used. Therefore, the problem studied in [20] has been generalized by the help of fractional fundamentals. To take the numerical results, the linear approximation has been applied for the iterations calculated with MATLAB. Finally, solutions under the variation of problem parameter have been shown and evaluated with the figures.

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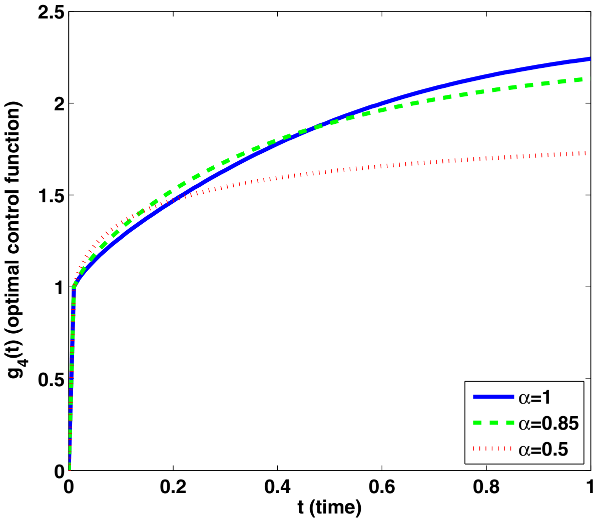

Fig. 1. Dependence of optimal control on the variation of α

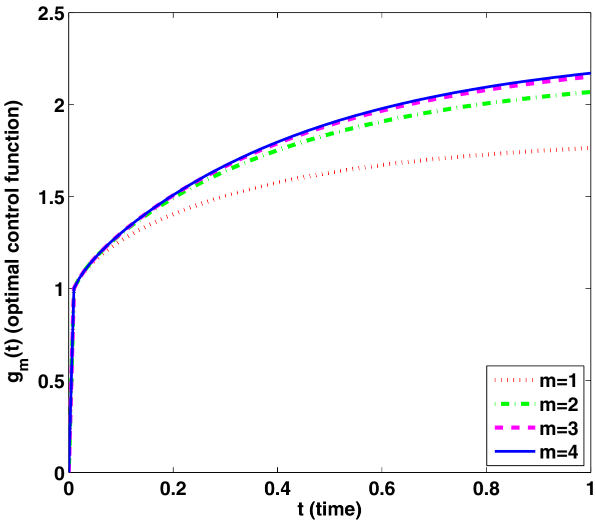

Fig. 2. Dependence of optimal control solution on iteration number