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APPROXIMATION BY TRIGONOMETRIC POLYNOMIALS OF FUNCTIONS HAVING (α, ψ) - DERIVATIVES IN WEIGHTED VARIABLE EXPONENT LEBESGUE SPACES

R. Akgün

Balikesir University
10145, Balikesir, Turkey
rakgun@balikesir.edu.tr

V. Kokilashvili *

A. Razmadze Mathematical Institute
1, M. Aleksidze Str., Tbilisi 0193, Georgia
I. Javakhishvili Tbilisi State University
2, University Str., Tbilisi 0186, Georgia
kokil@rmi.ge

UDC 517.9

We prove direct simultaneous and converse approximation theorems by trigonometric polynomials for functions f and (α, ψ) -derivatives of f in weighted Lebesgue spaces with variable exponent. Bibliography: 11 titles.

1 Introduction

Let $\mathbf{T} := [0, 2\pi]$, and let $\mathcal{P}(\mathbf{T})$ be the class of Lebesgue measurable functions $p(x) : \mathbf{T} \rightarrow (1, \infty)$ such that

$$1 < p_*(\mathbf{T}) := \operatorname{ess\,inf}_{x \in \mathbf{T}} p(x) \leq p^* := \operatorname{ess\,sup}_{x \in \mathbf{T}} p(x) < \infty.$$

A function $\omega : \mathbf{T} \rightarrow [0, \infty]$ is called a *weight* on \mathbf{T} if it is a 2π -periodic, a.e. positive, and Lebesgue measurable function. We define the weighted variable exponent Lebesgue space $L_\omega^{p(\cdot)}$ as the collection of 2π -periodic Lebesgue measurable functions $f : \mathbf{T} \rightarrow \mathbb{R}$ with the finite norm

$$\|f\|_{p(\cdot), \omega} := \inf \left\{ \alpha > 0 : \int_{\mathbf{T}} |(f(x)/\alpha) \omega(x)|^{p(x)} dx \leq 1 \right\},$$

where $p \in \mathcal{P}(\mathbf{T})$. The space $L_{2\pi}^{p(\cdot)}$ is a Banach space.

For given $p \in \mathcal{P}(\mathbf{T})$ the class of weights ω satisfying the condition [1]

$$\|\omega \chi_Q\|_{p(\cdot), 1} \|\omega^{-1} \chi_Q\|_{p'(\cdot), 1} \leq C |Q|$$

* To whom the correspondence should be addressed.

for all balls Q in \mathbf{T} is denoted by $A_{p(\cdot)}(\mathbf{T})$. Here, $p'(x) := p(x)/(p(x) - 1)$ is the conjugate exponent of $p(x)$. The variable exponent $p(x)$ is said to be *log-Hölder continuous* on \mathbf{T} if there exists a constant $c \geq 0$ such that

$$|p(x_1) - p(x_2)| \leq \frac{c}{\log(e + 1/|x_1 - x_2|)} \quad \forall x_1, x_2 \in \mathbf{T}. \quad (1.1)$$

We denote by $\mathcal{P}^{\log}(\mathbf{T})$ the class of exponents $p \in \mathcal{P}(\mathbf{T})$ such that $1/p : \mathbf{T} \rightarrow [0, 1]$ is log-Hölder continuous on \mathbf{T} .

If $p \in \mathcal{P}^{\log}(\mathbf{T})$ and $f \in L_{\omega}^{p(\cdot)}$, then, as was proved in [1], the Hardy–Littlewood maximal function \mathcal{M} is bounded in $L_{\omega}^{p(\cdot)}$ if and only if $\omega \in A_{p(\cdot)}(\mathbf{T})$.

Let $f \in L_{\omega}^{p(\cdot)}$, and let

$$\mathcal{A}_h f(x) := \frac{1}{h} \int_{x-h/2}^{x+h/2} f(t) dt, \quad x \in \mathbf{T},$$

be the Steklov mean operator. If $p \in \mathcal{P}^{\log}(\mathbf{T})$ and $\omega \in A_{p(\cdot)}(\mathbf{T})$, then \mathcal{A}_h is bounded in $L_{\omega}^{p(\cdot)}$.

For $x, h \in \mathbf{T}$ and $0 \leq r$ we define

$$\sigma_h^r f(x) := (I - \mathcal{A}_h)^r f(x) = \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(r+1)}{\Gamma(k+1) \Gamma(r-k+1)} (\mathcal{A}_h)^k,$$

where $f \in L_{\omega}^{p(\cdot)}$, Γ is the Gamma function, and I is the identity operator.

If $p \in \mathcal{P}^{\log}(\mathbf{T})$, $\omega \in A_{p(\cdot)}(\mathbf{T})$, and $f \in L_{\omega}^{p(\cdot)}$, then

$$\|\sigma_h^r f\|_{p(\cdot), \omega} \leq c \|f\|_{p(\cdot), \omega}. \quad (1.2)$$

For $0 \leq r$ we can define the *fractional moduli of smoothness* for $p \in \mathcal{P}^{\log}(\mathbf{T})$, $\omega \in A_{p(\cdot)}(\mathbf{T})$, and $f \in L_{\omega}^{p(\cdot)}$ by the formula

$$\Omega_r(f, \delta)_{p(\cdot), \omega} := \sup_{0 < h_i, t \leq \delta} \left\| \prod_{i=1}^{[r]} (I - \mathcal{A}_{h_i}) \sigma_t^{\{r\}} f \right\|_{p(\cdot), \omega}, \quad r \geq 1, \quad \delta \geq 0,$$

where

$$\begin{aligned} \Omega_0(f, \delta)_{p(\cdot), \omega} &:= \|f\|_{p(\cdot), \omega}, \\ \Omega_r(f, \delta)_{p(\cdot), \omega} &:= \sup_{0 < t \leq \delta} \|\sigma_t^r f\|_{p(\cdot), \omega}, \quad 0 < r < 1, \end{aligned}$$

$[r]$ denotes the integer part of the nonnegative real number r and $\{r\} := r - [r]$.

In this case, for $p \in \mathcal{P}^{\log}(\mathbf{T})$, $\omega \in A_{p(\cdot)}(\mathbf{T})$, and $f \in L_{\omega}^{p(\cdot)}$ we have

$$\Omega_r(f, \delta)_{p(\cdot), \omega} \leq c \|f\|_{p(\cdot), \omega},$$

where the constant $c > 0$ depends only on r and p .

Remark 1.1. The modulus of smoothness $\Omega_r(f, \delta)_{p(\cdot), \omega}$, $r \in \mathbb{R}^+$ has the following properties for $p \in \mathcal{P}^{\log}(\mathbf{T})$, $\omega \in A_{p(\cdot)}(\mathbf{T})$, and $f \in L_{\omega}^{p(\cdot)}$:

- (i) $\Omega_r(f, \delta)_{p(\cdot), \omega}$ is a nonnegative and nondecreasing function of $\delta \geq 0$,
- (ii) $\Omega_r(f_1 + f_2, \cdot)_{p(\cdot), \omega} \leq \Omega_r(f_1, \cdot)_{p(\cdot), \omega} + \Omega_r(f_2, \cdot)_{p(\cdot), \omega}$,
- (iii) $\lim_{\delta \rightarrow 0^+} \Omega_r(f, \delta)_{p(\cdot), \omega} = 0$.

If $p \in \mathcal{P}^{\log}(\mathbf{T})$ and $\omega \in A_{p(\cdot)}(\mathbf{T})$, then $\omega^{p(x)} \in L^1(\mathbf{T})$. This implies that the set of trigonometric polynomials is dense [2] in $L_\omega^{p(\cdot)}$. This allows us to consider approximation problems in $L_\omega^{p(\cdot)}$. Approximation by trigonometric polynomials in $L_\omega^{p(\cdot)}$ was considered in [3]–[8]. In [9, 10], on the basis of the transformed Fourier series, the so-called lambda derivatives were introduced and inequalities are obtained in a refined form like the Besov and Timan inequalities.

On the other hand, if $p \in \mathcal{P}^{\log}(\mathbf{T})$ and $\omega \in A_{p(\cdot)}$, then $L_\omega^{p(\cdot)} \subset L^1(\mathbf{T})$. For given $f \in L_\omega^{p(\cdot)}$ we introduce the *Fourier series* and the *conjugate Fourier series* of f by the formulas

$$f(x) \sim \frac{a_0(f)}{2} + \sum_{k=1}^{\infty} (a_k(f) \cos kx + b_k(f) \sin kx) \quad (1.3)$$

and

$$\tilde{f}(x) \sim \sum_{k=1}^{\infty} (a_k(f) \sin kx - b_k(f) \cos kx).$$

We say that a function $f \in L_\omega^{p(\cdot)}$, $p \in \mathcal{P}(\mathbf{T})$, $\omega \in A_{p(\cdot)}(\mathbf{T})$, has a (α, ψ) -derivative f_α^ψ if for a given sequence $\psi(k)$, $k = 1, 2, \dots$ and a number $\alpha \in \mathbb{R}$ the series

$$\sum_{k=1}^{\infty} \frac{1}{\psi(k)} \left(a_k(f) \cos k \left(x + \frac{\alpha\pi}{2k} \right) + b_k(f) \sin k \left(x + \frac{\alpha\pi}{2k} \right) \right) \quad (1.4)$$

is the Fourier series of the function f_α^ψ . Taking $\psi(k) = k^{-\alpha}$, $k = 1, 2, \dots$, $\alpha \in \mathbb{R}^+$, we have the fractional derivative $f^{(\alpha)}$ of f in the sense of Weyl. Taking $\psi(k) = k^{-\alpha} \ln^{-\beta} k$, $k = 1, 2, \dots$, $\alpha, \beta \in \mathbb{R}^+$, we have the power logarithmic-fractional derivative $f^{(\alpha, \beta)}$ of f .

Let \mathfrak{M} be the set of functions $\psi(v)$ that are convex downwards for any $v \geq 1$ and satisfy the condition $\lim_{v \rightarrow \infty} \psi(v) = 0$.

We associate every function $\psi \in \mathfrak{M}$ with a pair of functions $\eta(t) = \psi^{-1}(\psi(t)/2)$ and $\mu(t) = t/(\eta(t) - t)$.

We set

$$\mathfrak{M}_0 := \{\psi \in \mathfrak{M} : 0 < \mu(t) \leq K\}.$$

We define

$$E_n(f)_{p(\cdot), \omega} := \inf_{T \in \mathcal{T}_n} \|f - T\|_{p(\cdot), \omega}, \quad n = 0, 1, 2, \dots, \quad f \in L_\omega^{p(\cdot)},$$

where \mathcal{T}_n is the class of trigonometric polynomials of degree not greater than n .

Theorem 1.1. *Let $p \in \mathcal{P}^{\log}(\mathbf{T})$, $\omega^{-p_0} \in A_{(p(\cdot)/p_0)'}(\mathbf{T})$ for some $p_0 \in (1, p_*(\mathbf{T}))$, $\alpha \in \mathbb{R}$ and $f, f_\alpha^\psi \in L_\omega^{p(\cdot)}$. If $\psi(k)$, ($k \in \mathbb{N}$) is an arbitrary nonincreasing sequence of nonnegative numbers such that $\psi(k) \rightarrow 0$ as $k \rightarrow \infty$, then for every $n = 0, 1, 2, 3, \dots$ there exists a constant $c > 0$ independent of n such that*

$$E_n(f)_{p(\cdot), \omega} \leq c\psi(n+1) E_n \left(f_\alpha^\psi \right)_{p(\cdot), \omega}.$$

Corollary 1.1. *Under the assumptions of Theorem 1.1,*

$$E_n(f)_{p(\cdot),\omega} \leq c\psi(n+1) \left\| f_\alpha^\psi \right\|_{p(\cdot),\omega}$$

with a constant $c > 0$ independent of n .

Using Theorem 1.1 and Theorem 1.4 in [4], we have the following Jackson type direct theorem.

Theorem 1.2. *Suppose that $p \in \mathcal{P}^{\log}(\mathbf{T})$, $\omega^{-p_0} \in A_{(p(\cdot)/p_0)'}(\mathbf{T})$ for some $p_0 \in (1, p_*(\mathbf{T}))$, $\alpha \in \mathbb{R}$ and $f, f_\alpha^\psi \in L_\omega^{p(\cdot)}$. If $\psi(k)$, ($k \in \mathbb{N}$) is an arbitrary nonincreasing sequence of nonnegative numbers such that $\psi(k) \rightarrow 0$ as $k \rightarrow \infty$, then for every $n = 1, 2, 3, \dots$ there exists a constant $c > 0$ independent of n such that*

$$E_n(f)_{p(\cdot),\omega} \leq c\psi(n+1) \Omega_r \left(f_\alpha^\psi, \frac{1}{n} \right)_{p(\cdot),\omega}.$$

Theorem 1.3. *If $p \in \mathcal{P}^{\log}(\mathbf{T})$, $\omega^{-p_0} \in A_{(p(\cdot)/p_0)'}(\mathbf{T})$ for some $p_0 \in (1, p_*(\mathbf{T}))$, $\alpha \in \mathbb{R}$, $\psi \in \mathfrak{M}_0$, and*

$$\sum_{\nu=1}^{\infty} (\nu\psi(\nu))^{-1} E_\nu(f)_{p(\cdot),\omega} < \infty,$$

then $f_\alpha^\psi \in L_\omega^{p(\cdot)}$ and

$$E_n \left(f_\alpha^\psi \right)_{p(\cdot),\omega} \leq c \left((\psi(n))^{-1} E_n(f)_{p(\cdot),\omega} + \sum_{\nu=n+1}^{\infty} (\nu\psi(\nu))^{-1} E_\nu(f)_{p(\cdot),\omega} \right),$$

where the constant $c > 0$ depends only on α and p .

Corollary 1.2. *Under the assumptions of Theorem 1.3, if $r \in (0, \infty)$ and*

$$\sum_{\nu=1}^{\infty} (\nu\psi(\nu))^{-1} E_\nu(f)_{p(\cdot),\omega} < \infty,$$

there exist constants $c, C > 0$ depending only on ψ , r , and p such that

$$\Omega_r \left(f_\alpha^\psi, \frac{1}{n} \right)_{p(\cdot),\omega} \leq \frac{c}{n^r} \sum_{\nu=0}^n \nu^{r-1} (\psi(\nu))^{-1} E_\nu(f)_{p(\cdot),\omega} + C \sum_{\nu=n+1}^{\infty} (\nu\psi(\nu))^{-1} E_\nu(f)_{p(\cdot),\omega}.$$

Theorem 1.4. *Suppose that $p \in \mathcal{P}^{\log}(\mathbf{T})$, $\omega^{-p_0} \in A_{(p(\cdot)/p_0)'}(\mathbf{T})$ for some $p_0 \in (1, p_*(\mathbf{T}))$, $\alpha \in [0, \infty)$, and $f, f_\alpha^\psi \in L_\omega^{p(\cdot)}$. If $\psi(k)$, ($k \in \mathbb{N}$) is an arbitrary nonincreasing sequence of nonnegative numbers such that $\psi(k) \rightarrow 0$ as $k \rightarrow \infty$, then there exists $T \in \mathcal{I}_n$, $n = 1, 2, 3, \dots$ and a constant $c > 0$ depending only on α and p such that*

$$\|f_\alpha^\psi - T_\alpha^\psi\|_{p(\cdot),\omega} \leq cE_n \left(f_\alpha^\psi \right)_{p(\cdot),\omega}.$$

In the particular case $\psi(k) = k^{-\alpha} \ln^{-\beta} k$, $k = 1, 2, \dots$, $\alpha, \beta \in \mathbb{R}^+$, we have the following new results for power logarithmic-fractional derivatives.

Theorem 1.5. If $p \in \mathcal{P}^{\log}(\mathbf{T})$, $\omega^{-p_0} \in A_{(p(\cdot)/p_0)'(\mathbf{T})}$ for some $p_0 \in (1, p_*(\mathbf{T}))$, $\alpha, \beta \in \mathbb{R}$, and $f, f^{(\alpha, \beta)} \in L_{\omega}^{p(\cdot)}$, then for every $n = 1, 2, 3, \dots$ there exists a constant $c > 0$ independent of n such that

$$E_n(f)_{p(\cdot), \omega} \leq \frac{c}{n^{\alpha} \ln^{\beta}(n+1)} E_n(f^{(\alpha, \beta)})_{p(\cdot), \omega}.$$

Corollary 1.3. Under the assumptions of Theorem 1.5,

$$E_n(f)_{p(\cdot), \omega} \leq \frac{c}{n^{\alpha} \ln^{\beta}(n+1)} \|f^{(\alpha, \beta)}\|_{p(\cdot), \omega}$$

with a constant $c > 0$ independent of n .

Theorem 1.6. If $p \in \mathcal{P}^{\log}(\mathbf{T})$, $\omega^{-p_0} \in A_{(p(\cdot)/p_0)'(\mathbf{T})}$ for some $p_0 \in (1, p_*(\mathbf{T}))$, $\alpha, \beta \in \mathbb{R}$, and $f, f^{(\alpha, \beta)} \in L_{\omega}^{p(\cdot)}$, then for every $n = 1, 2, 3, \dots$ there exists a constant $c > 0$ independent of n such that

$$E_n(f)_{p(\cdot), \omega} \leq \frac{c}{n^{\alpha} \ln^{\beta}(n+1)} \Omega_r \left(f^{(\alpha, \beta)}, \frac{1}{n} \right)_{p(\cdot), \omega}.$$

Theorem 1.7. If $p \in \mathcal{P}^{\log}(\mathbf{T})$, $\omega^{-p_0} \in A_{(p(\cdot)/p_0)'(\mathbf{T})}$ for some $p_0 \in (1, p_*(\mathbf{T}))$, $\alpha \in \mathbb{R}$, and

$$\sum_{\nu=1}^{\infty} \nu^{\alpha-1} \ln^{\beta} \nu E_{\nu}(f)_{p(\cdot), \omega} < \infty,$$

then $f^{(\alpha, \beta)} \in L_{\omega}^{p(\cdot)}$ and

$$E_n(f^{(\alpha, \beta)})_{p(\cdot), \omega} \leq c \left(n^{\alpha} \ln^{\beta} n E_n(f)_{p(\cdot), \omega} + \sum_{\nu=n+1}^{\infty} \nu^{\alpha-1} \ln^{\beta} \nu E_{\nu}(f)_{p(\cdot), \omega} \right),$$

where the constant $c > 0$ depends only on α, β , and p .

Corollary 1.4. Under the assumptions of Theorem 1.7, if $r \in (0, \infty)$ and

$$\sum_{\nu=1}^{\infty} \nu^{\alpha-1} \ln^{\beta} \nu E_{\nu}(f)_{p(\cdot), \omega} < \infty,$$

there exist constants $c, C > 0$ depending only on α, β, r , and p such that

$$\Omega_r \left(f^{(\alpha, \beta)}, \frac{1}{n} \right)_{p(\cdot), \omega} \leq \frac{c}{n^r} \sum_{\nu=1}^n \nu^{r+\alpha-1} \ln^{\beta} \nu E_{\nu}(f)_{p(\cdot), \omega} + C \sum_{\nu=n+1}^{\infty} \nu^{\alpha-1} \ln^{\beta} \nu E_{\nu}(f)_{p(\cdot), \omega}.$$

Theorem 1.8. If $p \in \mathcal{P}^{\log}(\mathbf{T})$, $\omega^{-p_0} \in A_{(p(\cdot)/p_0)'(\mathbf{T})}$ for some $p_0 \in (1, p_*(\mathbf{T}))$, $\alpha \in [0, \infty)$, and $f, f^{(\alpha, \beta)} \in L_{\omega}^{p(\cdot)}$, then there exists $T \in \mathcal{T}_n$, $n = 1, 2, 3, \dots$ and a constant $c > 0$ depending only on α and p such that

$$\|f^{(\alpha, \beta)} - T^{(\alpha, \beta)}\|_{p(\cdot), \omega} \leq c E_n(f^{(\alpha, \beta)})_{p(\cdot), \omega}.$$

Theorem 1.7 and Corollary 1.4 were proved in L^p ($\omega \equiv 1$, constant $p \in (1, \infty)$) in [11].

2 Auxiliary Results

We define the n th partial sum of (1.3)

$$S_n(f) := S_n(x, f) := \frac{a_0(f)}{2} + \sum_{k=1}^n (a_k(f) \cos kx + b_k(f) \sin kx), \quad n = 0, 1, 2, \dots$$

Lemma 2.1. [7] *If $p \in \mathcal{P}^{\log}(\mathbf{T})$, $\omega^{-p_0} \in A_{(p(\cdot)/p_0)'}(\mathbf{T})$ for some $p_0 \in (1, p_*(\mathbf{T}))$, and $f \in L_{\omega}^{p(\cdot)}$, then there are constants $c, C > 0$ such that*

$$\left\| \tilde{f} \right\|_{p(\cdot), \omega} \leq c \|f\|_{p(\cdot), \omega} \tag{2.1}$$

and

$$\|S_n(\cdot, f)\|_{p(\cdot), \omega} \leq C \|f\|_{p(\cdot), \omega}, \quad n = 1, 2, \dots \tag{2.2}$$

Remark 2.1. [4] Under the assumptions of Lemma 2.1, there exists a constant $c > 0$ such that

$$\|f - S_n(\cdot, f)\|_{p(\cdot), \omega} \leq c E_n(f)_{p(\cdot), \omega} \asymp E_n(\tilde{f})_{p(\cdot), \omega}.$$

Definition 2.1. Suppose that $p \in \mathcal{P}^{\log}(\mathbf{T})$, $\omega^{-p_0} \in A_{(p(\cdot)/p_0)'}(\mathbf{T})$ for some $p_0 \in (1, p_*(\mathbf{T}))$, $\psi(k)$, ($k \in \mathbb{N}$) is an arbitrary sequence, and $\alpha \in \mathbb{R}$. We write $(\alpha, \psi) \in B$ if

$$\left\| (T_n)_{\alpha}^{\psi} \right\|_{p(\cdot), \omega} \leq c (\psi(n))^{-1} \|T_n\|_{p(\cdot), \omega}$$

for any $T_n \in \mathcal{T}_n$, where the constant c is independent of n .

Proposition 2.1. *Suppose that $p \in \mathcal{P}^{\log}(\mathbf{T})$, $\omega^{-p_0} \in A_{(p(\cdot)/p_0)'}(\mathbf{T})$ for some $p_0 \in (1, p_*(\mathbf{T}))$, and ψ satisfies*

$$\sup_q \sum_{k=2^q}^{2^{q+1}} |(\psi_n(k+1))^{-1} - (\psi_n(k))^{-1}| \leq C \lambda_n, \tag{2.3}$$

where

$$(\psi_n(k))^{-1} = \begin{cases} (\psi(k))^{-1}, & 1 \leq k \leq n, \\ 0, & k > n, \end{cases}$$

and

$$\lambda_n = \max_k |(\psi_n(k))^{-1}| = \max_{k \leq n} |\psi(k)|^{-1}. \tag{2.4}$$

Then

$$\left\| (T_n)_{\alpha}^{\psi} \right\|_{p(\cdot), \omega} \leq c \lambda_n \|T_n\|_{p(\cdot), \omega},$$

where the constant c depends only on ψ and p .

Proof. We can write

$$\begin{aligned} (T_n)_{\alpha}^{\psi} &= \sum_{k=1}^n \frac{1}{\psi(k)} \left(a_k(f) \cos k \left(x + \frac{\alpha\pi}{2k} \right) + b_k(f) \sin k \left(x + \frac{\alpha\pi}{2k} \right) \right) \\ &= \sum_{k=1}^n \frac{1}{\psi(k)} A_k \left(T_n, x + \frac{\alpha\pi}{2k} \right) = \sum_{k=1}^n \frac{1}{\psi(k)} \left\{ \cos \frac{\alpha\pi}{2} A_k(T_n, x) - \sin \frac{\alpha\pi}{2} A_k(\tilde{T}_n, x) \right\}. \end{aligned}$$

We define the multipliers

$$\mu_k = \begin{cases} (\psi(k))^{-1} \cos \frac{\alpha\pi}{2}, & 1 \leq k \leq n, \\ 0, & k > n, \quad k = 0, \end{cases}$$

$$\tilde{\mu}_k = \begin{cases} (\psi(k))^{-1} \sin \frac{\alpha\pi}{2}, & 1 \leq k \leq n, \\ 0, & k > n, \quad k = 0, \end{cases}$$

and operators

$$(AT_n)(x) = \sum_{k=1}^n \frac{1}{\psi(k)} \cos \frac{\alpha\pi}{2} A_k(T_n, x),$$

$$\left(\widetilde{AT}_n\right)(x) = \sum_{k=1}^n \frac{1}{\psi(k)} \sin \frac{\alpha\pi}{2} A_k\left(\widetilde{T}_n, x\right).$$

Therefore,

$$(T_n)_\alpha^\psi(\cdot) = (AT_n)(\cdot) - \left(\widetilde{AT}_n\right)(\cdot).$$

Using (2.3) and (2.4) we get

$$\sup_k |\mu_k| \leq \lambda_n, \quad \sup_k |\bar{\mu}_k| \leq \lambda_n,$$

$$\sup_q \sum_{k=2^q}^{2^{q+1}} |\mu(k+1) - \mu(k)| \leq C\lambda_n,$$

$$\sup_q \sum_{k=2^q}^{2^{q+1}} |\bar{\mu}(k+1) - \bar{\mu}(k)| \leq C\lambda_n.$$

Applying the Marcinkiewicz multiplier theorem for weighted variable exponent Lebesgue spaces [7], we find

$$\begin{aligned} \|(T_n)_\alpha^\psi\|_{p(\cdot), \omega} &= \|(AT_n) - \left(\widetilde{AT}_n\right)\|_{p(\cdot), \omega} \leq \|AT_n\|_{p(\cdot), \omega} + \left\|\widetilde{AT}_n\right\|_{p(\cdot), \omega} \\ &\leq C\lambda_n \left(\left\| \sum_{k=1}^n A_k(T_n, x) \right\|_{p(\cdot), \omega} + \left\| \sum_{k=1}^n A_k\left(\widetilde{T}_n, x\right) \right\|_{p(\cdot), \omega} \right). \end{aligned}$$

By the boundedness (2.1) of the conjugate operator, we obtain the desired inequality

$$\|(T_n)_\alpha^\psi\|_{p(\cdot), \omega} \leq C\lambda_n \left\| \sum_{k=1}^n A_k(T_n, x) \right\|_{p(\cdot), \omega} = C\lambda_n \|T_n\|_{p(\cdot), \omega}. \quad \square$$

Proposition 2.1 yields the following corollary which, in fact, is a generalized Bernstein inequality.

Corollary 2.1. *If $p \in \mathcal{P}^{\log}(\mathbf{T})$, $\omega^{-p_0} \in A_{(p(\cdot)/p_0)'}(\mathbf{T})$ for some $p_0 \in (1, p_*(\mathbf{T}))$, $\alpha \in \mathbb{R}$, $\psi(k)$, $(k \in \mathbb{N})$ is an arbitrary nonincreasing sequence of nonnegative numbers, and $T_n \in \mathcal{T}_n$, then $(\alpha, \psi) \in B$.*

The proof follows from $\lambda_n = (\psi(n))^{-1}$.

3 Proofs

Proof of Theorem 1.1. We set

$$A_k(x, f) := a_k \cos kx + b_k \sin kx.$$

Since the set of trigonometric polynomials is dense in $L_\omega^{p(\cdot)}$, for given $f \in L_\omega^{p(\cdot)}$ we have

$$E_n(f)_{p(\cdot), \omega} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

From the first inequality in Remark 2.1 we have

$$f(x) = \sum_{k=0}^{\infty} A_k(x, f)$$

in the norm $\|\cdot\|_{p(\cdot), \omega}$. For $k = 1, 2, 3, \dots$ we know that

$$\begin{aligned} A_k(x, f) &= a_k \cos k \left(x + \frac{\alpha\pi}{2} - \frac{\alpha\pi}{2} \right) + b_k \sin k \left(x + \frac{\alpha\pi}{2} - \frac{\alpha\pi}{2} \right) \\ &= A_k \left(x + \frac{\alpha\pi}{2k}, f \right) \cos \frac{\alpha\pi}{2} + A_k \left(x + \frac{\alpha\pi}{2k}, \tilde{f} \right) \sin \frac{\alpha\pi}{2} \end{aligned}$$

and

$$A_k(x, f_\alpha^\psi) = \frac{1}{\psi(k)} A_k \left(x + \frac{\alpha\pi}{2k}, f \right).$$

Therefore,

$$\begin{aligned} \sum_{k=0}^{\infty} A_k(x, f) &= A_0(x, f) + \cos \frac{\alpha\pi}{2} \sum_{k=1}^{\infty} A_k \left(x + \frac{\alpha\pi}{2k}, f \right) + \sin \frac{\alpha\pi}{2} \sum_{k=1}^{\infty} A_k \left(x + \frac{\alpha\pi}{2k}, \tilde{f} \right) \\ &= A_0(x, f) + \cos \frac{\alpha\pi}{2} \sum_{k=1}^{\infty} \psi(k) A_k(x, f_\alpha^\psi) + \sin \frac{\alpha\pi}{2} \sum_{k=1}^{\infty} \psi(k) A_k(x, (\tilde{f})_\alpha^\psi). \end{aligned}$$

Hence

$$f(\cdot) - S_n(\cdot, f) = \cos \frac{\alpha\pi}{2} \sum_{k=n+1}^{\infty} \psi(k) A_k(\cdot, f_\alpha^\psi) + \sin \frac{\alpha\pi}{2} \sum_{k=n+1}^{\infty} \psi(k) A_k(\cdot, (\tilde{f})_\alpha^\psi).$$

Since

$$\begin{aligned} \sum_{k=n+1}^{\infty} \psi(k) A_k(\cdot, f_\alpha^\psi) &= \sum_{k=n+1}^{\infty} \psi(k) [(S_k(\cdot, f_\alpha^\psi) - f_\alpha^\psi(\cdot)) - (S_{k-1}(\cdot, f_\alpha^\psi) - f_\alpha^\psi(\cdot))] \\ &= \sum_{k=n+1}^{\infty} (\psi(k) - \psi(k+1)) (S_k(\cdot, f_\alpha^\psi) - f_\alpha^\psi(\cdot)) - \psi(n+1) (S_n(\cdot, f_\alpha^\psi) - f_\alpha^\psi(\cdot)) \end{aligned}$$

and

$$\begin{aligned} \sum_{k=n+1}^{\infty} \psi(k) A_k(\cdot, (\tilde{f})_\alpha^\psi) &= \sum_{k=n+1}^{\infty} (\psi(k) - \psi(k+1)) (S_k(\cdot, (\tilde{f})_\alpha^\psi) - (\tilde{f})_\alpha^\psi(\cdot)) \\ &\quad - \psi(n+1) (S_n(\cdot, (\tilde{f})_\alpha^\psi) - (\tilde{f})_\alpha^\psi(\cdot)) \end{aligned}$$

we obtain

$$\begin{aligned}
\|f(\cdot) - S_n(\cdot, f)\|_{p(\cdot), \omega} &\leq \sum_{k=n+1}^{\infty} (\psi(k) - \psi(k+1)) \|S_k(\cdot, f_\alpha^\psi) - f_\alpha^\psi(\cdot)\|_{p(\cdot), \omega} \\
&+ \psi(n+1) \|S_n(\cdot, f_\alpha^\psi) - f_\alpha^\psi(\cdot)\|_{p(\cdot), \omega} + \sum_{k=n+1}^{\infty} (\psi(k) - \psi(k+1)) \\
&\times \|S_k(\cdot, (\tilde{f})_\alpha^\psi) - (\tilde{f})_\alpha^\psi(\cdot)\|_{p(\cdot), \omega} + \psi(n+1) \|S_n(\cdot, (\tilde{f})_\alpha^\psi) - (\tilde{f})_\alpha^\psi(\cdot)\|_{p(\cdot), \omega} \\
&\leq c \left[\sum_{k=n+1}^{\infty} (\psi(k) - \psi(k+1)) E_k(f_\alpha^\psi)_{p(\cdot), \omega} + \psi(n+1) E_n(f_\alpha^\psi)_{p(\cdot), \omega} \right] \\
&+ c \left[\sum_{k=n+1}^{\infty} (\psi(k) - \psi(k+1)) E_k((\tilde{f})_\alpha^\psi)_{p(\cdot), \omega} + \psi(n+1) E_n((\tilde{f})_\alpha^\psi)_{p(\cdot), \omega} \right].
\end{aligned}$$

Consequently, from the equivalence in Remark 2.1 we have

$$\begin{aligned}
&\|f(\cdot) - S_n(\cdot, f)\|_{p(\cdot), \omega} \\
&\leq c \left[\sum_{k=n+1}^{\infty} (\psi(k) - \psi(k+1)) + \psi(n+1) \right] \{E_k(f_\alpha^\psi)_{p(\cdot), \omega} + E_n((\tilde{f})_\alpha^\psi)_{p(\cdot), \omega}\} \\
&\leq c E_n(f_\alpha^\psi)_{p(\cdot), \omega} \left[\sum_{k=n+1}^{\infty} (\psi(k) - \psi(k+1)) + \psi(n+1) \right] \leq c \psi(n+1) E_n(f_\alpha^\psi)_{p(\cdot), \omega}. \quad \square
\end{aligned}$$

Proof of Theorem 1.3. Let T_n be the best approximating polynomial for $f \in L_\omega^{p(\cdot)}$. We set $n_0 = n, n_1 := [\eta(n)] + 1, \dots, n_k := [\eta(n_{k-1})] + 1, \dots$. In this case, the series

$$T_{n_0}(\cdot) + \sum_{k=1}^{\infty} (T_{n_k}(\cdot) - T_{n_{k-1}}(\cdot))$$

converges to f in the $L_\omega^{p(\cdot)}$ -norm. We consider the series

$$(T_{n_0}(\cdot))_\alpha^\psi + \sum_{k=1}^{\infty} (T_{n_k}(\cdot) - T_{n_{k-1}}(\cdot))_\alpha^\psi. \quad (3.1)$$

Applying the generalized Bernstein inequality (Corollary 2.1) to the difference $u_k(\cdot) := T_{n_k}(\cdot) - T_{n_{k-1}}(\cdot)$, we get

$$\begin{aligned}
\|(u_k)_\alpha^\psi\|_{p(\cdot), \omega} &\leq c \frac{\|u_k\|_{p(\cdot), \omega}}{\psi(n_k)} \leq c \frac{(\|T_{n_k} - f\|_{p(\cdot), \omega} + \|T_{n_{k-1}} - f\|_{p(\cdot), \omega})}{\psi(n_k)} \\
&\leq c E_{n_{k-1}+1}(f)_{p(\cdot), \omega} (\psi(n_k))^{-1}.
\end{aligned}$$

Hence

$$\sum_{k=1}^{\infty} \left\| (u_k)_\alpha^\psi \right\|_{p(\cdot), \omega} \leq c \left(E_{n+1}(f)_{p(\cdot), \omega} (\psi(n))^{-1} + \sum_{k=1}^{\infty} E_{n_k+1}(f)_{p(\cdot), \omega} (\psi(n_k))^{-1} \right).$$

Let $z = \bar{\eta}(\tau) := \psi^{-1}(2\psi(t))$ for $\tau \geq \eta(1)$. Since $\psi \in \mathfrak{M}_0$ we get

$$\frac{\tau}{\tau - \bar{\eta}(\tau)} = \frac{\eta(z)}{\eta(z) - z} = 1 + \frac{z}{\eta(z) - z} = 1 + \mu(\psi, z) \leq c,$$

and for $\tau \in [t, \eta(t)]$, $\tau \geq \eta(1)$

$$\begin{aligned} \frac{\tau - \bar{\eta}(\tau)}{\eta(t) - t} &\leq \frac{\tau\eta(t)}{\eta(t)(\eta(t) - t)} = \frac{\tau}{\eta(t)} \left(1 + \frac{t}{\eta(t) - t}\right) \\ &\leq \frac{\tau}{\eta(t)} (1 + \mu(\psi, t)) \leq c \frac{\tau}{\eta(t)} \leq c. \end{aligned}$$

Then $\psi(\tau) \geq \psi(\eta(t)) > \psi(\tau)/2$ for any $\tau \in [t, \eta(t)]$, $\tau \geq \eta(1)$. Without loss of generality one can assume that $\eta(t) - t > 1$. In this case, we get

$$\begin{aligned} \frac{E_{n_k+1}(f)_{p(\cdot),\omega}}{\psi(n_k)} &\leq C \sum_{v=n_{k-1}}^{n_k-1} \frac{E_{v+1}(f)_{p(\cdot),\omega}}{\psi(v)} \frac{1}{\eta(n_{k-1}) - n_{k-1}} \\ &\leq c \sum_{v=n_{k-1}}^{n_k-1} \frac{E_{v+1}(f)_{p(\cdot),\omega}}{v\psi(v)} \frac{v}{(v - \bar{\eta}(v))} \frac{v - \bar{\eta}(v)}{\eta(n_{k-1}) - n_{k-1}} \leq \sum_{v=n_{k-1}}^{n_k-1} \frac{E_{v+1}(f)_{p(\cdot),\omega}}{v\psi(v)}. \end{aligned}$$

Therefore,

$$\sum_{k=1}^{\infty} \|(u_k)_\alpha^\psi\|_{p(\cdot),\omega} \leq c \left(E_{n+1}(f)_{p(\cdot),\omega} (\psi(n))^{-1} + \sum_{v=n+1}^{\infty} E_v(f)_{p(\cdot),\omega} (v\psi(v))^{-1} \right).$$

The right-hand side of the last inequality converges and, consequently, the series (3.1) converges in the norm to some function $S(\cdot)$ from $L_\omega^{p(\cdot)}$. Let $a_k^{(n)} := a_k(T_n)$ and $b_k^{(n)} := b_k(T_n)$, $k = 0, 1, 2, \dots$, be coefficients of polynomials T_n . The corresponding coefficients $\alpha_k^{(n)}$, $\beta_k^{(n)}$ of the polynomials $(T_n)_\alpha^\psi$ have the form

$$\begin{aligned} \alpha_k^{(n)} &= \frac{1}{\psi(k)} \left\{ \cos \frac{\alpha\pi}{2} a_k^{(n)} + \sin \frac{\alpha\pi}{2} b_k^{(n)} \right\}, \\ \beta_k^{(n)} &= \frac{1}{\psi(k)} \left\{ \cos \frac{\alpha\pi}{2} b_k^{(n)} - \sin \frac{\alpha\pi}{2} a_k^{(n)} \right\}. \end{aligned}$$

Since $(T_n(\cdot))_\alpha^\psi \rightarrow S(\cdot)$ as $n \rightarrow \infty$, we have $\alpha_k^{(n)} \rightarrow a_k(S)$ and $\beta_k^{(n)} \rightarrow b_k(S)$ as $n \rightarrow \infty$ for $k = 0, 1, 2, \dots$. Since $\alpha_k^{(n)} \rightarrow a_k(f)$ and $\beta_k^{(n)} \rightarrow b_k(f)$ as $n \rightarrow \infty$ for $k = 0, 1, 2, \dots$, we have

$$\begin{aligned} a_k(S) &= \frac{1}{\psi(k)} \left\{ \cos \frac{\alpha\pi}{2} a_k^{(n)} + \sin \frac{\alpha\pi}{2} b_k^{(n)} \right\}, \\ b_k(S) &= \frac{1}{\psi(k)} \left\{ \cos \frac{\alpha\pi}{2} b_k^{(n)} - \sin \frac{\alpha\pi}{2} a_k^{(n)} \right\}. \end{aligned}$$

We conclude that the Fourier series of S has the form (1.4). This means that the function f has a (ψ, α) -derivative f_α^ψ of class $L_\omega^{p(\cdot)}$ and

$$f_\alpha^\psi = (T_n)_\alpha^\psi + \sum_{k=1}^{\infty} (u_k)_\alpha^\psi \tag{3.2}$$

in the $L_\omega^{p(\cdot)}$ -norm. Therefore, from (3.2) it follows that

$$E_n(f_\alpha^\psi)_{p(\cdot),\omega} \leq c \left((\psi(n))^{-1} E_n(f)_{p(\cdot),\omega} + \sum_{\nu=n+1}^{\infty} (\nu\psi(\nu))^{-1} E_\nu(f)_{p(\cdot),\omega} \right). \quad \square$$

Proof of Corollary 1.2. Since [4]

$$\Omega_r \left(f, \frac{1}{n} \right)_{p(\cdot),\omega} \leq \frac{c}{n^r} \sum_{\nu=1}^n \nu^{r-1} E_\nu(f)_{p(\cdot),\omega},$$

using Theorem 1.3, we have

$$\begin{aligned} \Omega_r \left(f_\alpha^\psi, \frac{1}{n} \right)_{p(\cdot),\omega} &\leq \frac{c}{n^r} \sum_{\nu=1}^n \nu^{r-1} E_\nu(f_\alpha^\psi)_{p(\cdot),\omega} \leq \frac{c}{n^r} \left\{ \sum_{\nu=1}^n \nu^{r-1} (\psi(\nu))^{-1} E_\nu(f)_{p(\cdot),\omega} \right. \\ &\quad \left. + \sum_{\nu=1}^n \nu^{r-1} \sum_{m=\nu+1}^{\infty} (m\psi(m))^{-1} E_m(f)_{p(\cdot),\omega} \right\}. \end{aligned}$$

Using the equality

$$\sum_{\nu=1}^n b_\nu \sum_{m=\nu}^n a_m = \sum_{m=1}^n a_m \sum_{\nu=1}^m b_\nu,$$

we get

$$\Omega_r \left(f_\alpha^\psi, \frac{1}{n} \right)_{p(\cdot),\omega} \leq \frac{c}{n^r} \sum_{\nu=0}^n \nu^{r-1} (\psi(\nu))^{-1} E_\nu(f)_{p(\cdot),\omega} + C \sum_{\nu=n+1}^{\infty} (\nu\psi(\nu))^{-1} E_\nu(f)_{p(\cdot),\omega}. \quad \square$$

Proof of Theorem 1.4. We set

$$W_n(f) := W_n(\cdot, f) := \frac{1}{n+1} \sum_{\nu=n}^{2n} S_\nu(\cdot, f)$$

for $n = 0, 1, 2, \dots$. Since $W_n(\cdot, f_\alpha^\psi) = (W_n(\cdot, f))_\alpha^\psi$, we have

$$\begin{aligned} \|f_\alpha^\psi(\cdot) - (S_n(\cdot, f))_\alpha^\psi\|_{p(\cdot),\omega} &\leq \|f_\alpha^\psi(\cdot) - W_n(\cdot, f_\alpha^\psi)\|_{p(\cdot),\omega} + \|(S_n(\cdot, W_n(f)))_\alpha^\psi - (S_n(\cdot, f))_\alpha^\psi\|_{p(\cdot),\omega} \\ &\quad + \|(W_n(\cdot, f))_\alpha^\psi - (S_n(\cdot, W_n(f)))_\alpha^\psi\|_{p(\cdot),\omega} := I_1 + I_2 + I_3. \end{aligned}$$

In this case, from the boundedness of S_n in $L_\omega^{p(\cdot)}$ we obtain the boundedness of W_n in $L_\omega^{p(\cdot)}$ and

$$\begin{aligned} I_1 &\leq \|f_\alpha^\psi(\cdot) - S_n(\cdot, f_\alpha^\psi)\|_{p(\cdot),\omega} + \|S_n(\cdot, f_\alpha^\psi) - W_n(\cdot, f_\alpha^\psi)\|_{p(\cdot),\omega} \\ &\leq cE_n(f_\alpha^\psi)_{p(\cdot),\omega} + \|W_n(\cdot, S_n(f_\alpha^\psi) - f_\alpha^\psi)\|_{p(\cdot),\omega} \leq cE_n(f_\alpha^\psi)_{p(\cdot),\omega}. \end{aligned}$$

From Lemma 2.1 we get

$$I_2 \leq c(\psi(n))^{-1} \|S_n(\cdot, W_n(f)) - S_n(\cdot, f)\|_{p(\cdot),\omega},$$

$$I_3 \leq c(\psi(n))^{-1} \|W_n(\cdot, f) - S_n(\cdot, W_n(f))\|_{p(\cdot),\omega} \leq c(\psi(n))^{-1} E_n(W_n(f))_{p(\cdot),\omega}.$$

Now, we have

$$\|S_n(\cdot, W_n(f)) - S_n(\cdot, f)\|_{p(\cdot),\omega} \leq \|S_n(\cdot, W_n(f)) - W_n(\cdot, f)\|_{p(\cdot),\omega} + \|W_n(\cdot, f) - f(\cdot)\|_{p(\cdot),\omega}$$

$$+\|f(\cdot) - S_n(\cdot, f)\|_{p(\cdot),\omega} \leq cE_n(W_n(f))_{p(\cdot),\omega} + cE_n(f)_{p(\cdot),\omega} + cE_n(f)_{p(\cdot),\omega}.$$

Since $E_n(W_n(f))_{p(\cdot),\omega} \leq cE_n(f)_{p(\cdot),\omega}$, we get

$$\begin{aligned} \|f_\alpha^\psi(\cdot) - (S_n(\cdot, f))_\alpha^\psi\|_{p(\cdot),\omega} &\leq cE_n(f_\alpha^\psi)_{p(\cdot),\omega} + c(\psi(n))^{-1}E_n(W_n(f))_{p(\cdot),\omega} \\ &\quad + cE_n(f)_{p(\cdot),\omega} \leq cE_n(f_\alpha^\psi)_{p(\cdot),\omega} + c(\psi(n))^{-1}E_n(f)_{p(\cdot),\omega}. \end{aligned}$$

Since $E_n(f)_{p(\cdot),\omega} \leq c\psi(n+1)E_n(f_\alpha^\psi)_{p(\cdot),\omega}$ in view of Theorem 1.1, we obtain

$$\|f_\alpha^\psi(\cdot) - (S_n(\cdot, f))_\alpha^\psi\|_{p(\cdot),\omega} \leq cE_n(f_\alpha^\psi)_{p(\cdot),\omega}. \quad \square$$

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