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Symmetry Reduction of Chazy Equation

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Abstract

In this study, invariant group of the Chazy Equation found by symmerty group analysis. First and second reduction made by the method of differential invariant. This equation is finally reduced to first-order ODE. Solution of first-order ODE made by Phase-Plane Tecniques.

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1 Introduction

Towards the end of the nineteenth century, Sophus Lie introduced the notion of Lie group in order to study the solutions of ordinary differential equations (ODEs). He showed the following main property: the order of an ordinary differential equation can be reduced by one if it is invariant under one-parameter Lie group of point transformations.This observation unified and extended the available integration techniques.

In the last century, the application of the Lie group method has been developed by a number of mathematicians. Ovsiannikov [13], Olver [11], [12], Ibragimov [9], Baumann [1] and Bluman, Anco [3] and Bluman, Kumei [2] and Bluman, Cole [4] are some of the mathematicians who have enormous amount of studies in this field.

Roughly speaking, a Lie point symmetry of a system is a local group of transformations that maps every solution of the system to another solution of the same system. In other words, it maps the solution set of the system to itself. Elementary examples of Lie groups are translations, rotations and scalings.

Lie groups and hence their infinitesimal generators can be naturally "extended" to act on the space of independent variables, state variables (dependent variables) and derivatives of the state variables up to any finite order. Lie symmetries were introduced by Lie in order to solve ordinary differential equations. Another application of symmetry methods is to reduce systems of differential equations, finding equivalent systems of differential equations of simpler form. This is called reduction. In the literature, one can find the classical reduction process.[11] Also symmetry groups can be used for classifying different symmetry classes of solutions. Lie's fundamental theorems underline that Lie groups can be characterized by their infinitesimal generators. These mathematical objects form a Lie algebra of infinitesimal generators. Deduced "infinitesimal symmetry conditions" (defining equations of the symmetry group) can be explicitly solved in order to find the closed form of symmetry groups, and thus the associated infinitesimal generators.

Lie algebras can be generated by a generating set of infinitesimal generators. To every Lie group, one can associate a Lie algebra. Lie algebra is an algebra constituted by a vector space equipped with Lie bracket as additional operation. The base field of a Lie algebra depends on the concept of invariant. Here only finite-dimensional Lie algebras are considered.

2 One-Parameter Lie Groups in the Plane

The definitions in this section are given by [6]. A one-parameter Lie group in two variables is a transformation of the form

$$\begin{aligned} \widetilde{x} &= F[x,y,s], \\ \widetilde{y} &= G[x,y,s]. \end{aligned} \tag{1}$$

where s is scalar parameter whose value defines a one-to-one invertible map from a source space $S:(x,y)$ to a target space $\widetilde{S}:(\widetilde{x},\widetilde{y})$. The functions F and G are smooth analytic functions of the group parameter s and therefore expandable in a Taylor series about any value on the open interval that contains s. At $s_0=0$ the transformation reduces to an identity. Thus

$$x = F[x,y,0],$$

$$y = G[x, y, 0].$$

Now expand (1) in a Taylor series about $s = 0$:

$$\widetilde{x} = x + s\left[\frac{\partial F}{\partial s}\right]_{s=0} + O(s^2) + ...,$$

$$\widetilde{y} = y + s\left[\frac{\partial G}{\partial s}\right]_{s=0} + O(s^2) + ...$$

The derivatives of the varios $F$and G with respect to the group parameter s evaluated at $s = 0$ are called the infinitesimal of the group and are traditionally denetod by ξ, η.

$$\xi[x, y] = \frac{\partial F[x, y, s]}{\partial s} \mid_{s=0}, \qquad \eta[x, y] = \frac{\partial G[x, y, s]}{\partial s} \mid_{s=0}$$

The vector (ξ, η) is also called vektor field of the group (1). The operator

$$X \equiv \xi[x, y]\frac{\partial}{\partial x} + \eta[x, y]\frac{\partial}{\partial y}$$

is called the group operator.

2.1 Infinitesimal Transformation of the Third Derivatives

We consider the finite Lie point group in two variables (1). The third extended finite transformation group is

$$\begin{aligned}
\widetilde{x} &= F[x, y, s], \\
\widetilde{y} &= G[x, y, s], \\
\widetilde{y}_{\widetilde{x}} &= G_{\{1\}}[x, y, y_x, s], \\
\widetilde{y}_{\widetilde{x}\widetilde{x}} &= G_{\{2\}}[x, y, y_x, y_{xx}, s], \\
\widetilde{y}_{\widetilde{x}\widetilde{x}\widetilde{x}} &= G_{\{3\}}[x, y, y_x, y_{xx}, y_{xxx}, s],
\end{aligned} \tag{2}$$

where

$$G_{\{1\}}[x, y, y_x, s] = DG(DF)^{-1}, \qquad G_{\{2\}}[x, y, y_x, y_{xx}, s] = DG_{\{1\}}(DF)^{-1},$$

$$G_{\{3\}}[x, y, y_x, y_{xx}, y_{xxx}, s] = DG_{\{2\}}(DF)^{-1}.$$

The extend transformation (2) is a Lie group. The infinitesimal form of (1) is

$$\widetilde{x} = x + s\xi[x, y],$$

$$\widetilde{y} = y + s\eta[x, y],$$

where

$$\xi[x, y] = \frac{\partial F}{\partial s} \mid_{s=0}, \qquad \eta[x, y] = \frac{\partial G}{\partial s} \mid_{s=0}$$

and s is assumed to be small. The third extended infinitesimal group in the plane is

$$\begin{aligned}
\widetilde{x} &= x + s\xi[x, y], \\
\widetilde{y} &= y + s\eta[x, y], \\
\widetilde{y}_{\widetilde{x}} &= y_x + s\eta_{\{1\}}[x, y, y_x], \\
\widetilde{y}_{\widetilde{x}\widetilde{x}} &= y_{xx} + s\eta_{\{2\}}[x, y, y_x, y_{xx}], \\
\widetilde{y}_{\widetilde{x}\widetilde{x}\widetilde{x}} &= y_{xxx} + s\eta_{\{3\}}[x, y, y_x, y_{xx}, y_{xxx}],
\end{aligned}$$

where

$$\begin{aligned}
\eta_{\{1\}}[x, y, y_x] &= D\eta - y_x D\xi, \\
\eta_{\{2\}}[x, y, y_x, y_{xx}] &= D\eta_{\{1\}} - y_{xx} D\xi, \\
\eta_{\{2\}}[x, y, y_x, y_{xx}, y_{xxx}] &= D\eta_{\{2\}} - y_{xxx} D\xi.
\end{aligned}$$

The total diferantiation opetators are

$$D\xi = \frac{\partial \xi}{\partial x} + y_x \frac{\partial \xi}{\partial y},$$

$$D\eta_{\{1\}} = \frac{\partial \eta_{\{1\}}}{\partial x} + y_x \frac{\partial \eta_{\{1\}}}{\partial y} + y_{xx} \frac{\partial \eta_{\{1\}}}{\partial y_x},$$

$$D\eta_{\{2\}} = \frac{\partial \eta_{\{2\}}}{\partial x} + y_x \frac{\partial \eta_{\{2\}}}{\partial y} + y_{xx} \frac{\partial \eta_{\{2\}}}{\partial y_x} + y_{xxx} \frac{\partial \eta_{\{2\}}}{\partial y_{xx}}.$$

Note the quadratic dependence of the infinitesimal $\eta_{\{1\}}$ on y_x and $\eta_{\{2\}}$ is linear in y_{xx} and $\eta_{\{3\}}$ is linear in y_{xxx}. Thus

$$\eta_{\{1\}}[x, y, y_x] = \eta_x + (\eta_y - \xi_x) y_x - \xi_y (y_x)^2,$$

$$\eta_{\{2\}}[x, y, y_x, y_{xx}] = \eta_{xx} + (2\eta_{xy} - \xi_{xx})y_x + (\eta_{yy} - 2\xi_{xy})(y_x)^2 - \xi_{yy}(y_x)^3 + (\eta_y - 2\xi_x)y_{xx} - 3\xi_y y_x y_{xx},$$

$$\begin{aligned}\eta_{\{3\}}[x, y, y_x, y_{xx}, y_{xxx}] \; = \; & \eta_{xxx} + (3\eta_{xxy} - \xi_{xxx})y_x + (3\eta_{xyy} - 3\xi_{xxy})(y_x)^2 + (\eta_{yyy} - 3\xi_{xyy})(y_x)^3 \\ & -\xi_{yyy}(y_x)^4 + (3\eta_{xy} - \xi_{xx})y_{xx} + (3\eta_{yy} - 9\xi_{xy})y_x y_{xx} - 6\xi_{yy}(y_x)^2 y_{xx} \\ & -3\xi_y(y_{xx})^2 + (\eta_y - 3\xi_x)y_{xxx} - 4\xi_y y_x y_{xxx}.\end{aligned}$$

2.2 The Invariance Condition for Third-Order ODE

The third-order ordinary diferential equation

$$\Psi[x, y, y_x, y_{xx}, y_{xxx}] = 0$$

is invariant under the third times extended group with infinitesimals $(\xi, \eta, \eta_{\{1\}}, \eta_{\{2\}}, \eta_{\{3\}})$ if and only if

$$X_{\{3\}}\Psi = 0$$

where the group operator of the third time extended group is

$$X_{\{3\}} = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \eta_{\{1\}}\frac{\partial}{\partial y_x} + \eta_{\{2\}}\frac{\partial}{\partial y_{xx}} + \eta_{\{3\}}\frac{\partial}{\partial y_{xxx}}.$$

The characteristic equations associated with the group operator are

$$\frac{dx}{\xi} = \frac{dy}{\eta} = \frac{dy_x}{\eta_{\{1\}}} = \frac{dy_{xx}}{\eta_{\{2\}}} = \frac{dy_{xxx}}{\eta_{\{3\}}}.$$

3 Invariant Group of the Chazy Equation

We consider the Chazy Equation in [7]

$$y_{xxx} - 2yy_{xx} + 3y_x^2 = 0. \qquad (3)$$

Let's look at third -order nonlineer equation with a solvable Lie algebra. We find invariant group of (3). First we determine the group that leaves

$$\Psi[x, y, y_x, y_{xx}, y_{xxx}] = y_{xxx} - 2yy_{xx} + 3y_x^2 = 0$$

invariant. The invariance contion is

$$X_{\{3\}}\Psi = 0.$$

This equation is obtained

$$X_{\{3\}}\Psi = \xi\frac{\partial\Psi}{\partial x} + \eta\frac{\partial\Psi}{\partial y} + \eta_{(1)}\frac{\partial\Psi}{\partial y_x} + \eta_{(2)}\frac{\partial\Psi}{\partial y_{xx}} + \eta_{(3)}\frac{\partial\Psi}{\partial y_{xxx}} = -2\eta y_{xx} + 6\eta_{(1)}y_x - 2\eta_{(2)}y + \eta_{(3)} = 0.$$

Where $\eta_{(1)}, \eta_{(2)}, \eta_{(3)}$ are infinitesimal transformations for the first derivative, second derivative, third derivatives given by section 2.1. Thus the determinin equations can be obtained as:

$$\eta_{xxx} = 0, \eta_{xx} = 0, \eta_{yy} = 0, \xi_y = 0, \quad \xi_{yy} = 0, \quad \xi_{yyy} = 0, 6\eta_x - \xi_{xxx} = 0, 6\eta_y - 6\xi_x = 0,$$

$$-4\eta_{xy} + 2\xi_{xx} = 0, -2\eta + 3\eta_{xy} - 3\xi_{xx} = 0, -2\eta_y + 4\xi_x = 0, \eta_y - 3\xi_x = 0$$

Finally, the infinitesimals of (3) satisfy the set of determinin equations. The resulting system of equations easily be solved to give the infinitesimals

$$\xi = 1, \qquad \eta = 0.$$

The infinitesimal generator of (3) is

$$X = \frac{\partial}{\partial x}.$$

3.1 First Reduction

In this section we use the method of diferential invariants to determine the invariants and reduced order of (3). The characteristic equations of the thrice extended operator $X_{\{3\}}$ a

$$\frac{dx}{1} = \frac{dy}{0} = \frac{dy_x}{0} = \frac{dy_{xx}}{0} = \frac{dy_{xxx}}{0},$$

and the first two invariants are

$$\phi = y, \quad G = y_x.$$

By the methhod of diferantial invariants [kitap] , the equation

$$\frac{DG}{D\phi} = \frac{\frac{\partial G}{\partial x}dx + \frac{\partial G}{\partial y}dy + \frac{\partial G}{\partial y_x}dy_x}{\frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy} = \frac{y_{xx}}{y_x}$$

is invariant, as is

$$\frac{D^2G}{D\phi^2} = \left(\frac{y_{xx}y_{xxx} - y_{xx}^2}{y_x^2}\right)\frac{1}{y_x} = \frac{y_x(2yy_{xx} - 3y_x^2) - y_{xx}^2}{y_x^3}$$

where (3) has been used to replace the third derivative. This equation can be rearranged to read

$$G\frac{D^2G}{D\phi^2} - 2\phi\frac{DG}{D\phi} + \left(\frac{DG}{D\phi}\right)^2 + 3G = 0 \tag{4}$$

This is the one reduced (3).

3.2 Second Reduction

We determine the action of the group

$$\widetilde{x} = e^b x, \quad \widetilde{y} = e^{-b}y$$

on the new variables (ϕ, G),

$$\widetilde{\phi} = e^{-b}\ y, \qquad \widetilde{G} = e^{-2b}G \tag{5}$$

and on equation (4), which we see invariant.

$$\widetilde{G}\frac{D^2\widetilde{G}}{D\widetilde{\phi}^2} - 2\widetilde{\phi}\frac{D\widetilde{G}}{D\widetilde{\phi}} + \left(\frac{D\widetilde{G}}{D\widetilde{\phi}}\right)^2 + 3\widetilde{G} = e^{-2b}\left(G\frac{D^2G}{D\phi^2} - 2\phi\frac{DG}{D\phi} + \left(\frac{DG}{D\phi}\right)^2 + 3G\right) = 0$$

Now solve the characteristic equations of (5)

$$\frac{d\phi}{-\phi} = \frac{dG}{-2G} = \frac{dG_\phi}{-G_\phi}.$$

The invariants at the second stage are

$$\gamma = \frac{G}{\phi^2}, \quad H = \frac{G_\phi}{\phi}. \tag{6}$$

Using the method of differential invariants to generate the second reduction of order:

$$\frac{DH}{D\gamma} = \frac{H_\phi + H_G \frac{dG}{d\phi} + H_{G_\phi} \frac{dG_\phi}{d\phi}}{\gamma_\phi + \gamma_G \frac{dG}{d\phi}} = \frac{-\frac{G_\phi}{\phi^2} + \frac{1}{\phi} G_{\phi\phi}}{-2\frac{G}{\phi^3} + \frac{1}{\phi^2} G_\phi}. \tag{7}$$

Using the once reduced equation to eliminate the second-derivative term, the right-hand side of (7) can be rearranged to read as follows:

$$\frac{DH}{D\gamma} = \frac{-\frac{1}{\phi^2}\left(\frac{dG}{d\phi}\right) + \frac{1}{\phi}\left(\frac{2\phi}{G}\left(\frac{dG}{d\phi}\right) - \frac{1}{G}\left(\frac{dG}{d\phi}\right)^2 - 3\right)}{-2\frac{G}{\phi^3} + \frac{1}{\phi^2}\left(\frac{dG}{d\phi}\right)} = \frac{-\frac{1}{\phi}\left(\frac{dG}{d\phi}\right) + \frac{2\phi}{G}\left(\frac{dG}{d\phi}\right) - \frac{1}{G}\left(\frac{dG}{d\phi}\right)^2 - 3}{-2\frac{G}{\phi^2} + \frac{1}{\phi}\left(\frac{dG}{d\phi}\right)}.$$

Using(6) in above equation, the Chazy equation is fnally reduced to the following first-order ODE:

$$\frac{dH}{d\gamma} = \frac{\gamma H - 2H + H^2 + 3\gamma}{2\gamma^2 - \gamma H} \tag{8}$$

3.3 The Solution

Solve the equation (8) by phase -plane techniques in [5], [10], [8]. The equation (8) take as a plane autonomous system which is a pair of simultaneous first-order diferential equations,

$$\dot{x} = f(x,y) = 2x^2 - xy, \qquad \dot{y} = g(x,y) = xy - 2y + y^2 + 3x.$$

This system has an equilibrium point (or fixed point or critical point or singular point $(x_0, y_0) = (0,0), (0,2)$ and $(\frac{1}{6}, \frac{1}{3})$ when $f(x_0, y_0) = g(x_0, y_0) = 0$.

We can illustrate the behavior of system by drawing trajectories (i.e.,solution curve) in the(x,y)-plane, know in this context as the phase plane. The trajectories in such a phase portrait are marked with arrows to show the direction of increasing time. Note that trajectories can never cross, because the solution starting from any point in the plane is uniquely determined: so thete cannot be two such solution curves starting at the any given point. The only exception is at an equilibrium point (because the solution starting at an equilibrium point is just that single point, so it is no contradiction for two curves to meet there).

We can examine the stability of an equilibrium point by setting $x = x_0 + \xi$, $y = y_0 + \eta$ and using Taylor Seriesmin 2D for small ξ and η :

$$\dot{\xi} = \xi f_x + \eta f_y$$

$$\dot{\eta} = \xi g_x + \eta g_y$$

In matrix natation

$$\begin{pmatrix} \dot{\xi} \\ \dot{\eta} \end{pmatrix} = J \begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

$$\dot{\boldsymbol{\xi}} = J\boldsymbol{\xi} \tag{9}$$

where

$$J = \left(\begin{array}{cc} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{array} \right).$$

Let the eigenvalues of this stability matrix J be λ_1, λ_2 with corresponding eigenvectors e_1, e_2.The general solution of (9)

$$\boldsymbol{\xi} = Ae^{\lambda_1 t}\mathbf{e}_1 + Be^{\lambda_2 t}\mathbf{e}_2$$

where A, B are arbitrary constant. The behavior of the solution therefore depends on the eigenvalues. The critical point at $(x_0, y_0) = (0, 0)$ is nonlinear.

Equilibrium $(x_0, y_0) = (0, 2)$:

$$J(x_0, y_0) = \left(\begin{array}{cc} -2 & 0 \\ 5 & -3 \end{array} \right) \;, \quad \lambda_1 = -2 \;\;,\;\; \lambda_2 = 3 \;,\; \mathbf{e}_1 = [1 \quad 5]^T \;,\; \mathbf{e}_2 = [0 \quad 1]^T.$$

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = Ae^{-2t} \begin{pmatrix} 1 \\ 5 \end{pmatrix} + Be^{3t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\xi = Ae^{2t}, \quad \eta = 5Ae^{2t} + Be^{3t}$$

$$x = x_0 + \xi = Ae^{2t}, \qquad y = y_0 + \eta = 2 + 5Ae^{2t} + Be^{3t}$$

$$y = 2 + 5x + Be^{3\ln\left(\frac{A^2}{x^2}\right)}, \qquad y = \Psi(x, A, B)$$

Two real eigenvalues of opposite sign $(\lambda_1 < 0, \lambda_2 > 0)$ .Trajectories move invards along $\mathbf{e}_1$ but outwards along $\mathbf{e}_2$. Unless the initial value of $\boldsymbol{\xi}$ lies exactly paralel to $\mathbf{e}_1$,the solution will eventually move away from equilibrium point, so it is unstable. This is a saddle point.

Equilibrium $(x_0, y_0) = (\frac{1}{6}, \frac{1}{3})$:

$$J(x_0, y_0) = \begin{pmatrix} \frac{1}{3} & -\frac{1}{6} \\ \frac{10}{3} & -\frac{7}{6} \end{pmatrix} , \quad \lambda_1 = -\frac{1}{2} , \; \lambda_2 = -\frac{1}{3} , \mathbf{e}_1 = [1 \quad 5]^T , \mathbf{e}_2 = [1 \quad 4]^T.$$

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = Ae^{-\frac{1}{2}t}\begin{pmatrix} 1 \\ 5 \end{pmatrix} + Be^{-\frac{1}{3}t}\begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

$$\xi = Ae^{-\frac{1}{2}t} + Be^{-\frac{1}{3}t}, \quad \eta = 5Ae^{-\frac{1}{2}t} + 4Be^{-\frac{1}{3}t}$$

$$x = x_0 + \xi = \frac{1}{6} + Ae^{-\frac{1}{2}t} + Be^{-\frac{1}{3}t}, \qquad y = y_0 + \eta = \frac{1}{3} + 5Ae^{-\frac{1}{2}t} + 4Be^{-\frac{1}{3}t}$$

Two real, negative eigenvalues $(\lambda_1, \lambda_2 < 0)$ .In this case $|\boldsymbol{\xi}|$ decreases exponentially and trajectories move towardas the equilibrium point. This is stable node. The phase portrait is identical to that of an unstable node with the arrows reserved.

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