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Bernoulli numbers and certain convolution sums with divisor functions

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Abstract
In this paper, we investigate the convolution sums

$$\sum_{(a+b+c)x=n} a, \qquad \sum_{ax+by=n} ab, \qquad \sum_{ax+by+cz=n} abc, \qquad \sum_{ax+by+cz+du=n} abcd,$$

where $a,b,c,d,x,y,z,u,n \in \mathbb{N}$. Many new equalities and inequalities involving convolution sums, Bernoulli numbers and divisor functions have also been given.

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1 Introduction

Throughout this paper, $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{C}$ will denote the sets of positive integers, rational integers, and complex numbers, respectively. The Bernoulli polynomials $B_k(x)$, which are usually defined by the exponential generating function

$$\frac{te^{xt}}{e^t-1} = \sum_{k=0}^{\infty} B_k(x)\frac{t^k}{k!},$$

play an important role in different areas of mathematics, including number theory and the theory of finite differences. The Bernoulli polynomials satisfy the following well-known identity:

$$\sum_{j=0}^{N} j^k = \frac{B_{k+1}(N+1)-B_{k+1}(0)}{k+1}, \qquad k \geq 1. \tag{1}$$

It is well known that $B_k = B_k(0)$ are rational numbers. It can be shown that $B_{2k+1} = 0$ for $k \geq 1$, and is alternatively positive and negative for even k. The B_k are called Bernoulli numbers.

For $n,k \in \mathbb{N}$ with $s \in \mathbb{N} \cup \{0\}$, we define

$$\sigma_s(n) = \sum_{d|n} d^s, \qquad F_k(n) = \begin{cases} 1, & \text{if } k|n, \\ 0, & \text{if } k\nmid n. \end{cases}$$

The exact evaluation of the basic convolution sum

$$\sum_{m=1}^{n-1} \sigma_1(m)\sigma_1(n-m)$$

first appeared in a letter from Besge to Liouville in 1862. Ramanujan's work has been extended by many authors, *e.g.*, see [1]. For example, the following identity

$$\sum_{m=1}^{n-1} \sigma_1(m)\sigma_1(n-m) = \frac{1}{12}\bigl(5\sigma_3(n) + (1-6n)\sigma_1(n)\bigr) \tag{2}$$

is due to the works of Huard *et al.* [2]. In [1], Ramanujan also found nine identities, including (2), of the form

$$\sum_{m=0}^{n} \sigma_r(m)\sigma_s(n-m) = A\sigma_{r+s+1}(n) + Bn\sigma_{r+s-1}(n),$$

where A and B are certain rational numbers. We refer to [3] for a similar work. Lahiri [4] obtained the most general result by evaluating the sum

$$\sum_{m_1+\cdots+m_r=n} m_1^{a_1}\cdots m_r^{a_r}\sigma_{b_1}(m_1)\cdots\sigma_{b_r}(m_r) \quad (r \geq 3),$$

where the sum is over all positive integers $m_1,\ldots,m_r$ satisfying $m_1+\cdots+m_r = n$, $a_i \in \mathbb{N}\cup\{0\}$, and $b_i \in \mathbb{N}$.

The convolution identities have many beautiful applications in modern number theory, in particular in modular forms, since they appear in the coefficients of the Fourier expansions of classical Eisenstein series. For example, a very well-known work of Serre on p-adic modular forms (see [5]). For some of the history of the subject, and for a selection of these articles, we mention [4, 6] and [3], and especially [2] and [7]. We also refer to [8] and [9].

In this paper, we shall investigate the convolution sums

$$\sum_{(a+b+c)x=n} a, \qquad \sum_{ax+by=n} ab, \qquad \sum_{ax+by+cz=n} abc, \qquad \sum_{ax+by+cz+du=n} abcd.$$

In fact, we will prove the following results.

Theorem 1.1 *Let n be a positive integer. Then we have*

$$\sum_{(a+b+c)x=n} a = \frac{1}{6}\sigma_3(n) - \frac{1}{2}\sigma_2(n) + \frac{1}{3}\sigma_1(n) > \frac{1}{6}B_3(n-1) \tag{3}$$

with $n \geq 3$.

Remark 1.2 Let α be a fixed integer with $\alpha \geq 3$, and let

$$\mathrm{Pyr}_\alpha(x) = \frac{1}{6}\bigl\{(x)(x+1)\bigl((\alpha-2)x+5-\alpha\bigr)\bigr\}$$

be the αth order pyramid number. In fact, in (3), if $n = p$ is a prime number, then we obtain

$$\sum_{(a+b+c)x=p} a = \mathrm{Pyr}_3(p-2). \tag{4}$$

This result is similar to [10, (13)].

Theorem 1.3 *Let M be an odd positive integer. Let $R, r \in \mathbb{N} \cup \{0\}$ with $R \geq r$. Then we have*

$$A_1(R,r) := \sum_{\substack{ax+by=2^R M \\ ax=2^r m \\ m \text{ odd}}} ab = 8^{R-r-1}\left(2^{r+1}-1\right)^2 \sigma_3(M) > \frac{8^{R-r-1}}{3}\left(2^{r+1}-1\right)^2 B_3(q+1) \tag{5}$$

with $M = 2q + 1$.

Theorem 1.4 *Let m_1 be an odd positive integer. Let $r_1, r_2 \in \mathbb{N}$ and $r_3 \in \mathbb{N} \cup \{0\}$ with $r_1 > r_2 > r_3$. Then we have*

$$A_2(r_1,r_2,r_3) := \sum_{\substack{ax+by+cz=2^{r_1} m_1 \\ ax+by=2^{r_2} m_2 \\ ax=2^{r_3} m_3 \\ m_2 \text{ odd} \\ m_3 \text{ odd}}} abc = 2^{5r_1-2r_2-3r_3-8}\left(2^{r_2+1}-1\right)\left(2^{r_3+1}-1\right)^2 \sigma_5(m_1). \tag{6}$$

Theorem 1.5 *Let m_1 be an odd positive integer. Let $r_1, r_2, r_3 \in \mathbb{N}$ and $r_4 \in \mathbb{N} \cup \{0\}$ with $r_1 > r_2 > r_3 > r_4$. Then we have*

$$\begin{aligned}\sum_{\substack{ax+by+cz+du=2^{r_1} m_1 \\ ax+by+cz=2^{r_2} m_2 \\ ax+by=2^{r_3} m_3 \\ ax=2^{r_4} m_4 \\ m_2 \text{ odd} \\ m_3 \text{ odd} \\ m_4 \text{ odd}}} abcd &= \frac{1}{17} \cdot 2^{-2r_2-2r_3-3r_4-10}\left(2^{r_2+1}-1\right)\left(2^{r_3+1}-1\right)\left(2^{r_4+1}-1\right)^2 \\ &\quad \times \left\{2^{7r_1}\sigma_7(m_1) + 15(-1)^{r_1-r_2} 2^{4r_2+3r_1-1} b(m_1)\right\}\end{aligned}$$

when $\sum_{n=1}^{\infty} b(n)q^n = q\prod_{n=1}^{\infty}(1-q^n)^8(1-q^{2n})^8$.

Theorem 1.6 *Let M be an odd positive integer. Let $r, R \in \mathbb{N} \cup \{0\}$. Then we have*

$$\begin{aligned}&\sum_{R \leq r < \log_2(\frac{2^R M}{m})} \sum_{\substack{ax+by=2^R M \\ ax=2^r m \\ m \text{ odd}}} ab \\ &= \frac{1}{12}\left\{\left(3 \cdot 2^{2R+2} - 2^{R+3} + 1\right)\sigma_3(M) - \left(3 \cdot 2^{R+1} M - 1\right)\left(2^{R+1}-1\right)\sigma_1(M)\right\}.\end{aligned}$$

Corollary 1.7 *For $R > r$, we have the following lower bound of $A_1(R,r)$ and the upper bound of $A_2(r_1,r_2,r_3)$,*

$$A_1(R,r) > \frac{7}{8}\sigma_3\left(2^{R-r-1}M\right)$$

and

$$A_2(r_1, r_2, r_3) < \frac{1}{18}\sigma_5\big(2^{r_1-1}m_1\big).$$

2 Bernoulli number derived from Diophantine equations $\sum_{(a+b+c)x=n} a$

Lemma 2.1 *Let $n \in \mathbb{N}$. Let $f : \mathbb{Z} \to \mathbb{C}$ be an odd function. Then*

$$\sum_{\substack{(a,b,c,x)\in\mathbb{N}^4\\(a+b+c)x=n}} \big(f(a+b) + f(b-c)\big) = \sum_{e|n}\sum_{k=1}^{e-2}(e-k-1)f(e-k).$$

Proof We can write the equality as

$$\begin{aligned}
&\sum_{(a+b+c)x=n} \big\{f(a+b) + f(b-c)\big\} \\
&\quad = \sum_{k\geq 1} f(k)\Bigg\{\sum_{\substack{(a+b+c)x=n\\a+b=k}} 1 + \sum_{\substack{(a+b+c)x=n\\b-c=k}} 1 - \sum_{\substack{(a+b+c)x=n\\b-c=-k}} 1\Bigg\} \\
&\quad = \sum_{k\geq 1} f(k)\sum_{\substack{(a+b+c)x=n\\a+b=k}} 1 \\
&\quad = \sum_{e|n}\big\{(e-2)f(e-1) + (e-3)f(e-2) + \cdots + \big(e-(e-1)\big)f(2)\big\} \\
&\quad = \sum_{e|n}\sum_{k=1}^{e-2}(e-k-1)f(e-k).
\end{aligned}$$

This completes the proof of the lemma. □

Proof of Theorem 1.1 Let $f(x) = x$. Then Lemma 2.1 becomes

$$\sum_{(a+b+c)x=n} \{a + 2b - c\} = \frac{1}{3}\sigma_3(n) - \sigma_2(n) + \frac{2}{3}\sigma_1(n) \tag{7}$$

and

$$\sum_{(a+b+c)x=n} a = \frac{1}{6}\sigma_3(n) - \frac{1}{2}\sigma_2(n) + \frac{1}{3}\sigma_1(n). \tag{8}$$

Using (1), we note that

$$\sum_{j=0}^{p-2} j^2 = \frac{1}{3}\big\{B_3(p-1) - B_3\big\} = \frac{B_3(p-1)}{3}$$

since $B_3 = 0$. It is easily checked that

$$\frac{p(p-1)(p-2)}{3} > \frac{(p-1)(p-2)(2p-3)}{6}.$$

We can write that

$$\sum_{(a+b+c)x=n} a > \frac{1}{6} B_3(n-1)$$

with $n \geq 3$. This completes the proof of the theorem. □

We list the first ten values of $\sum_{(a+b+c)x=n} a$ in Table 1.

Remark 2.2 Let

$$f(x) := \sum_{(a+b+c)t=x} a$$

and

$$g(x) := \frac{1}{6} x(x-1)(x-2) = \mathrm{Pyr}_3(x-2).$$

If x is a prime integer, by (4) and (8), then $f(x) = g(x)$.

The first nine values of $f(x)$ and $g(x)$ are given in Figure 1. In Figure 1, we plot the graphs for the values of the sums $f(x)$ and $g(x)$ in Remark 2.2 when $x = 3, 4, 5, 6, 7, 8, 9, 10, 11$.

Table 1 The first ten values of $\sum_{(a+b+c)x=n} a$

n	1	2	3	4	5	6	7	8	9	10
$\sum_{(a+b+c)x=n} a$	0	0	1	4	10	21	35	60	85	130

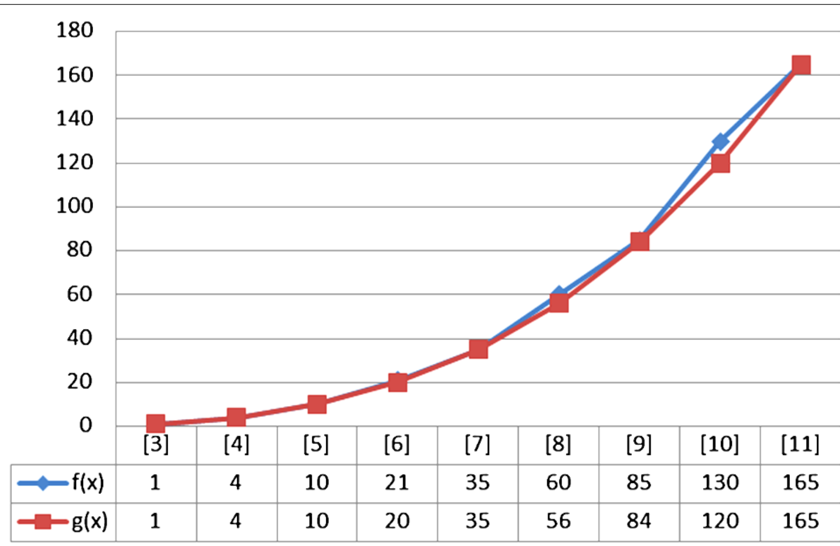

Figure 1 $x = 3, 4, 5, 6, 7, 8, 9, 10, 11$.

3 Two lemmas

Lemma 3.1 *Let $n \in \mathbb{N}$ and $r, m \in \mathbb{N} \cup \{0\}$ with $r \geq m$. Let $f : \mathbb{Z} \to \mathbb{C}$ be a function. Then*

$$\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4 \\ ax+by=2^r n \\ y=x+2^m}} f(a+b) = \sum_{j=1}^{2^{r-m}n-1} \sum_{l=2^m}^{(2^r-2^{r-m})n+j-1} \sum_{k|2^m j} f(k)\delta_{k,2^r n-l},$$

where the Kronecker delta symbol is defined by

$$\delta_{i,j} = \begin{cases} 0, & i \neq j, \\ 1, & i = j. \end{cases}$$

Proof We note that

$$\begin{aligned} \sum_{\substack{(a,b,x,y)\in\mathbb{N}^4 \\ ax+by=2^r n \\ y=x+2^m}} f(a+b) &= \sum_{k\geq 1} f(k) \sum_{\substack{ax+by=2^r n \\ y=x+2^m \\ a+b=k}} 1 \\ &= \sum_{k\geq 1} f(k) \sum_{\substack{ax+b(x+2^m)=2^r n \\ a+b=k}} 1 = \sum_{k\geq 1} f(k) \sum_{\substack{(a+b)x+2^m b=2^r n \\ a+b=k}} 1 \end{aligned} \tag{9}$$

and

$$\begin{aligned} &\sum_{\substack{(a+b)x+2^m b=2^r n \\ a+b=k}} 1 \\ &= \sum_{\substack{(a+1)x=2^r n-2^m \\ a+1=k}} 1 + \sum_{\substack{(a+2)x=2^r n-2^m\cdot 2 \\ a+2=k}} 1 + \cdots + \sum_{\substack{(a+2^{r-m}n-1)x=2^r n-2^m(2^{r-m}n-1) \\ a+(2^{r-m}n-1)=k}} 1 \\ &= \sum_{\substack{k|(2^r n-2^m) \\ k\geq 2}} 1 + \sum_{\substack{k|(2^r n-2^m\cdot 2) \\ k\geq 3}} 1 + \cdots + \sum_{\substack{k|2^m\cdot 2 \\ k\geq 2^{r-m}n-1}} 1 + \sum_{\substack{k|2^m \\ k\geq 2^{r-m}n}} 1 \\ &= F_k\big(2^r n - 2^m\big)(\delta_{k,2} + \delta_{k,3} + \cdots + \delta_{k,2^r n-2^m}) \\ &\quad + F_k\big(2^r n - 2^m\cdot 2\big)(\delta_{k,3} + \delta_{k,4} + \cdots + \delta_{k,2^r n-2^m\cdot 2}) \\ &\quad + \cdots \\ &\quad + F_k\big(2^m\cdot 2\big)(\delta_{k,2^{r-m}n-1} + \delta_{k,2^{r-m}n} + \cdots + \delta_{k,2^m\cdot 2}) \\ &\quad + F_k\big(2^m\big)(\delta_{k,2^{r-m}n} + \delta_{k,2^{r-m}n+1} + \cdots + \delta_{k,2^m}) \\ &= \sum_{j=1}^{2^{r-m}n-1} F_k\big(2^m j\big)(\delta_{k,2^r n-2^m} + \delta_{k,2^r n-(2^m+1)} + \cdots + \delta_{k,2^r n-((2^r-2^{r-m})n+j-1)}) \\ &= \sum_{j=1}^{2^{r-m}n-1} F_k\big(2^m j\big) \sum_{l=2^m}^{(2^r-2^{r-m})n+j-1} \delta_{k,2^r n-l}. \end{aligned}$$

Therefore, (9) becomes

$$\begin{aligned}\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ ax+by=2^r n\\ y=x+2^m}} f(a+b) &= \sum_{k\geq 1} f(k) \sum_{j=1}^{2^{r-m}n-1} F_k\big(2^m j\big) \sum_{l=2^m}^{(2^r-2^{r-m})n+j-1} \delta_{k,2^r n-l}\\ &= \sum_{j=1}^{2^{r-m}n-1} \sum_{l=2^m}^{(2^r-2^{r-m})n+j-1} \sum_{k|2^m j} f(k)\delta_{k,2^r n-l}.\end{aligned}$$

This completes the proof of the lemma. □

Example 3.2

(a) Letting $m = r = 0$ in Lemma 3.1,

$$\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ ax+by=n\\ y=x+1}} f(a+b) = \sum_{j=1}^{n-1} \sum_{l=1}^{j-1} \sum_{k|j} f(k)\delta_{k,n-l}.$$

(b) If $m = r = 1$ in Lemma 3.1, then

$$\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ ax+by=2n\\ y=x+2}} f(a+b) = \sum_{j=1}^{n-1} \sum_{l=2}^{n+j-1} \sum_{k|2j} f(k)\delta_{k,2n-l}.$$

Corollary 3.3 *Let $n \in \mathbb{N}$ and $r, m \in \mathbb{N}\cup\{0\}$ with $r \geq m$. Let $f : \mathbb{Z} \to \mathbb{C}$ be a complex-valued function. Then*

$$\sum_{m=0}^{r} \sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ ax+by=2^r n\\ y=x+2^m}} f(a+b) = \sum_{m=0}^{r} \sum_{j=1}^{2^{r-m}n-1} \sum_{l=2^m}^{(2^r-2^{r-m})n+j-1} \sum_{k|2^m j} f(k)\delta_{k,2^r n-l}.$$

Proof It is obvious by Lemma 3.1. □

Example 3.4 Let $f(x) = x^2$. Then we have

$$\sum_{\substack{ax+by=n\\ y=x+1}} (a+b)^2 = \sum_{j=2}^{n-1} \sum_{l=1}^{j-1} \sum_{k|j} k^2\delta_{k,n-l}.$$

Lemma 3.5 *Let n be an odd positive integer, and let $f : \mathbb{Z} \to \mathbb{C}$ be a complex-valued function. Then*

$$\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ ax+by=n\\ y=x+2}} f(a+b) = \sum_{j=1}^{\frac{n-3}{2}} \sum_{l=1}^{j+\frac{n-1}{2}} \sum_{k|(2j+1)} f(k)\delta_{k,n-l}.$$

Proof It is similar to Lemma 3.1. □

4 A study of $\sum_{ax+by=n} ab$

Proof of Theorem 1.3 We observe that

$$\sum_{\substack{ax+by=2^R M \\ ax=2^r m \\ m \text{ odd}}} ab = \sum_{\substack{m<2^{R-r}M \\ 2\nmid m}} \biggl(\sum_{a|2^r m} a\biggr)\biggl(\sum_{b|2^r(2^{R-r}M-m)} b\biggr). \tag{10}$$

Thus, for odd m, we have

$$\sum_{a|2^r m} a = \sigma_1(2^r)\sigma_1(m) = (2^{r+1}-1)\sigma_1(m). \tag{11}$$

Similarly, since $2^{R-r}M-m$ is odd, we have

$$\sum_{b|2^r(2^{R-r}M-m)} b = (2^{r+1}-1)\sigma_1(2^{R-r}M-m). \tag{12}$$

From (11) and (12), we can write (10) as

$$\begin{aligned}
\sum_{\substack{ax+by=2^R M \\ ax=2^r m \\ m \text{ odd}}} ab &= (2^{r+1}-1)^2 \sum_{\substack{m<2^{R-r}M \\ 2\nmid m}} \sigma_1(m)\sigma_1(2^{R-r}M-m) \\
&= (2^{r+1}-1)^2 \biggl\{ \sum_{m<2^{R-r}M} \sigma_1(m)\sigma_1(2^{R-r}M-m) \\
&\quad - \sum_{\substack{m<2^{R-r}M \\ 2|m}} \sigma_1(m)\sigma_1(2^{R-r}M-m) \biggr\} \\
&= (2^{r+1}-1)^2 \biggl\{ \sum_{m<2^{R-r}M} \sigma_1(m)\sigma_1(2^{R-r}M-m) \\
&\quad - \sum_{m<2^{R-r-1}M} \sigma_1(2m)\sigma_1(2^{R-r}M-2m) \biggr\}.
\end{aligned} \tag{13}$$

Let us consider the second term of (13). Since $\sigma_1(2m) = 3\sigma_1(m) - 2\sigma_1(\frac{m}{2})$, so we obtain

$$\begin{aligned}
&\sum_{m<2^{R-r-1}M} \sigma_1(2m)\sigma_1(2^{R-r}M-2m) \\
&\quad = \sum_{m<2^{R-r-1}M} \biggl\{3\sigma_1(m) - 2\sigma_1\biggl(\frac{m}{2}\biggr)\biggr\}\sigma_1(2^{R-r}M-2m) \\
&\quad = 3\sum_{m<2^{R-r-1}M} \sigma_1(m)\sigma_1(2^{R-r}M-2m) \\
&\qquad - 2\sum_{m<2^{R-r-2}M} \sigma_1(m)\sigma_1(2^{R-r}M-4m).
\end{aligned} \tag{14}$$

Therefore, (13) becomes

$$\begin{aligned}
\sum_{\substack{ax+by=2^R M\\ ax=2^r m\\ m \text{ odd}}} ab &= \left(2^{r+1}-1\right)^2 \sum_{\substack{m<2^{R-r}M\\ 2\nmid m}} \sigma_1(m)\sigma_1\left(2^{R-r}M-m\right)\\
&= \left(2^{r+1}-1\right)^2 \left\{ \sum_{m<2^{R-r}M} \sigma_1(m)\sigma_1\left(2^{R-r}M-m\right)\right.\\
&\quad - 3\sum_{m<2^{R-r-1}M} \sigma_1(m)\sigma_1\left(2^{R-r}M-2m\right)\\
&\quad \left. + 2\sum_{m<2^{R-r-2}M} \sigma_1(m)\sigma_1\left(2^{R-r}M-4m\right)\right\}\\
&= \left(2^{r+1}-1\right)^2 \cdot 8^{R-r-1}\sigma_3(M),
\end{aligned} \tag{15}$$

where we refer to (2),

$$\sum_{m<n/2} \sigma_1(m)\sigma_1(n-2m) = \frac{1}{24}\left\{2\sigma_3(n) + (1-3n)\sigma_1(n) + 8\sigma_3(n/2) + (1-6n)\sigma_1(n/2)\right\}$$

in [2, (4.4)] and

$$\sum_{m<n/4} \sigma_1(m)\sigma_1(n-4m) = \frac{1}{48}\left\{\sigma_3(n) + (2-3n)\sigma_1(n) + 3\sigma_3(n/2) + 16\sigma_3(n/4) + (2-12n)\sigma_1(n/4)\right\}$$

in [2, Theorem 4]. Thus, we obtain

$$\begin{aligned}
A_1(R,r) &= 8^{R-r-1}\left(2^{r+1}-1\right)^2\sigma_3(M)\\
&> 8^{R-r-1}\left(2^{r+1}-1\right)^2\left(\sigma_3(M)-\sigma_1(M)\right)\\
&> 8^{R-r-1}\left(2^{r+1}-1\right)^2\left(\frac{q(q+1)(2q+1)}{6}\right)\\
&\geq 8^{R-r-1}\left(2^{r+1}-1\right)^2\left(\frac{B_3(q+1)-B_3}{3}\right)
\end{aligned} \tag{16}$$

with $M = 2q+1$. This completes the proof of this theorem. □

Theorem 4.1 *Let M be an odd positive integer. Let $R \in \mathbb{N}$ and $r \in \mathbb{N}\cup\{0\}$ with $R > r$. Then we have*

(a)

$$\sum_{\substack{ax+by=2^R M\\ ax=2^r m\\ m \text{ odd}\\ x \text{ even}}} ab = 8^{R-r-1}\left(2^r-1\right)\left(2^{r+1}-1\right)\sigma_3(M),$$

(b)

$$\sum_{\substack{ax+by=2^R M\\ ax=2^r m\\ m \text{ odd}\\ x \text{ even}\\ y \text{ even}}} ab = 8^{R-r-1}\left(2^r-1\right)^2\sigma_3(M),$$

(c)

$$\sum_{\substack{ax+by=2^R M\\ ax=2^r m\\ m \text{ odd}\\ x \text{ even}\\ y \text{ odd}}} ab = \sum_{\substack{ax+by=2^R M\\ ax=2^r m\\ m \text{ odd}\\ x \text{ odd}\\ y \text{ even}}} ab = 2^{3R-2r-3}\left(2^r-1\right)\sigma_3(M),$$

(d)

$$\sum_{\substack{ax+by=2^R M\\ ax=2^r m\\ m \text{ odd}\\ x \text{ odd}\\ y \text{ odd}}} ab = 2^{3R-r-3}\sigma_3(M).$$

Proof

(a) First, we note that

$$\sum_{\substack{m<2^{R-r}M\\ 2\nmid m}} \sigma_1(m)\sigma_1\left(2^{R-r}M-m\right) = 8^{R-r-1}\sigma_3(M), \tag{17}$$

by (15). Therefore,

$$\begin{aligned}\sum_{\substack{ax+by=2^R M\\ ax=2^r m\\ m \text{ odd}\\ x \text{ even}}} ab &= \sum_{\substack{2ax+by=2^R M\\ 2ax=2^r m\\ m \text{ odd}}} ab\\ &= \sum_{\substack{2ax+by=2^R M\\ ax=2^{r-1} m\\ m \text{ odd}}} ab\\ &= \sum_{\substack{m<2^{R-r}M\\ 2\nmid m}} \left(\sum_{a|2^{r-1}m} a\right)\left(\sum_{b|2^r(2^{R-r}M-m)} b\right)\\ &= \left(2^r-1\right)\left(2^{r+1}-1\right)\sum_{\substack{m<2^{R-r}M\\ 2\nmid m}} \sigma_1(m)\sigma_1\left(2^{R-r}M-m\right)\\ &= 8^{R-r-1}\left(2^r-1\right)\left(2^{r+1}-1\right)\sigma_3(M),\end{aligned}$$

where we use (17) for the last line.

(b) We observe that

$$\sum_{\substack{ax+by=2^R M\\ ax=2^r m\\ m \text{ odd}\\ x \text{ even}\\ y \text{ even}}} ab = \sum_{\substack{2ax+2by=2^R M\\ 2ax=2^r m\\ m \text{ odd}}} ab = \sum_{\substack{ax+by=2^{R-1} M\\ ax=2^{r-1} m\\ m \text{ odd}}} ab$$

$$= 8^{R-r-1}\left(2^r-1\right)^2\sigma_3(M),$$

by replacing R with $R-1$ and r with $r-1$ in Theorem 1.3.

(c) We can write

$$\sum_{\substack{ax+by=2^R M\\ ax=2^r m\\ m \text{ odd}\\ x \text{ even}\\ y \text{ odd}}} ab = \sum_{\substack{ax+by=2^R M\\ ax=2^r m\\ m \text{ odd}\\ x \text{ even}}} ab - \sum_{\substack{ax+by=2^R M\\ ax=2^r m\\ m \text{ odd}\\ x \text{ even}\\ y \text{ even}}} ab.$$

So we use Theorem 4.1(a) and (b). We have that

$$\sum_{\substack{ax+by=2^R M\\ ax=2^r m\\ m \text{ odd}\\ x \text{ odd}\\ y \text{ even}}} ab = \sum_{\substack{ax+2by=2^R M\\ ax=2^r m\\ m \text{ odd}\\ x \text{ odd}}} ab$$

$$= \sum_{\substack{m<2^{R-r}M\\ 2\nmid m}} \Bigg(\sum_{\substack{a\mid 2^r m\\ \frac{2^r m}{a} \text{ odd}}} a\Bigg)\Bigg(\sum_{b\mid 2^{r-1}(2^{R-r}M-m)} b\Bigg). \tag{18}$$

Then, since

$$\sum_{\substack{a\mid 2^r m\\ \frac{2^r m}{a} \text{ odd}}} a = \sum_{a\mid m} 2^r a = 2^r\sum_{a\mid m} a = 2^r\sigma_1(m),$$

so (18) becomes

$$2^r\left(2^r-1\right)\sum_{\substack{m<2^{R-r}M\\ 2\nmid m}} \sigma_1(m)\sigma_1\left(2^{R-r}M-m\right).$$

Finally, we refer to (17).

(d) Since

$$\sum_{\substack{ax+by=2^R M\\ ax=2^r m\\ m \text{ odd}\\ x \text{ odd}\\ y \text{ odd}}} ab = \sum_{\substack{ax+by=2^R M\\ ax=2^r m\\ m \text{ odd}}} ab - \Bigg\{\sum_{\substack{ax+by=2^R M\\ ax=2^r m\\ m \text{ odd}\\ x \text{ even}\\ y \text{ even}}} ab + \sum_{\substack{ax+by=2^R M\\ ax=2^r m\\ m \text{ odd}\\ x \text{ even}\\ y \text{ odd}}} ab + \sum_{\substack{ax+by=2^R M\\ ax=2^r m\\ m \text{ odd}\\ x \text{ odd}\\ y \text{ even}}} ab\Bigg\},$$

we use Theorem 1.3 and Theorem 4.1(b) and (c). $\square$

Corollary 4.2 *Let M be an odd positive integer. Let $R \in \mathbb{N}$ and $r \in \mathbb{N} \cup \{0\}$ with $R > r$. Then we have*

$$\sum_{\substack{ax+by=2^R M \\ ax \neq 2^r m \\ m \text{ odd}}} ab = \frac{1}{12}\left[\left\{\frac{5}{7}\left(8^{R+1}-1\right) - 3 \cdot 2^{3R-3r-1}\left(2^{r+1}-1\right)^2\right\}\sigma_3(M) - \left(2^{R+1}-1\right)\left(3 \cdot 2^{R+1}M - 1\right)\sigma_1(M)\right].$$

Proof From (2), we deduce that

$$\sum_{m=1}^{n-1} \sigma_1(m)\sigma_1(n-m) = \sum_{ax+by=n} ab = \frac{1}{12}\left(5\sigma_3(n) + (1-6n)\sigma_1(n)\right). \tag{19}$$

So for $n = 2^R M$ with an odd M, we have

$$\begin{aligned}
\sum_{ax+by=2^R M} ab &= \frac{1}{12}\left(5\sigma_3\left(2^R M\right) + \left(1 - 6 \cdot 2^R M\right)\sigma_1\left(2^R M\right)\right) \\
&= \frac{1}{12}\left(\frac{5}{7}\left(2^{3(R+1)}-1\right)\sigma_3(M) + \left(1 - 3 \cdot 2^{R+1}M\right)\left(2^{R+1}-1\right)\sigma_1(M)\right) \\
&= \sum_{\substack{ax+by=2^R M \\ ax = 2^r m \\ m \text{ odd}}} ab + \sum_{\substack{ax+by=2^R M \\ ax \neq 2^r m \\ m \text{ odd}}} ab.
\end{aligned}$$

Thus, we refer to Theorem 1.3. □

Corollary 4.3 *Let M be an odd positive integer. Let $R \in \mathbb{N}$ and $r \in \mathbb{N} \cup \{0\}$ with $R > r$. Then we have*

$$\sum_{r=0}^{R-1} \sum_{\substack{ax+by=2^R M \\ ax = 2^r m \\ m \text{ odd}}} ab = \frac{1}{21}\left\{5 \cdot 2^{3R+1} - 21 \cdot 2^{2R} + 7 \cdot 2^{R+1} - 3\right\}\sigma_3(M).$$

Proof By Theorem 1.3, we have

$$\begin{aligned}
\sum_{r=0}^{R-1} \sum_{\substack{ax+by=2^R M \\ ax = 2^r m \\ m \text{ odd}}} ab &= \sum_{r=0}^{R-1} 8^{R-r-1}\left(2^{r+1}-1\right)^2 \sigma_3(M) \\
&= 8^R \sigma_3(M) \sum_{r=0}^{R-1}\left(2^{-r-1} + 2^{-3r-3} - 2^{-2r-1}\right).
\end{aligned} \tag{20}$$

Then the first term of (20) is

$$\sum_{r=0}^{R-1} 2^{-r-1} = 1 - 2^{-R}. \tag{21}$$

Similarly, the other terms of (20) are

$$\sum_{r=0}^{R-1} 2^{-3r-3} = \frac{1}{7}\left(1 - 8^{-R}\right) \tag{22}$$

and

$$\sum_{r=0}^{R-1} 2^{-2r-1} = \frac{2}{3}\left(1 - 2^{-2R}\right). \tag{23}$$

From (21), (22) and (23), we get the result. □

Proof of Theorem 1.4 The proof starts as follows:

$$\begin{aligned}
&\sum_{\substack{ax+by+cz=2^{r_1}m_1\\ ax+by=2^{r_2}m_2\\ ax=2^{r_3}m_3\\ m_2 \text{ odd}\\ m_3 \text{ odd}}} abc \\
&= \sum_{\substack{m_2<2^{r_1-r_2}m_1\\ 2\nmid m_2}} \left(\sum_{\substack{m_3<2^{r_2-r_3}m_2\\ 2\nmid m_3}} \sum_{a|2^{r_3}m_3} a \cdot \sum_{b|2^{r_3}(2^{r_2-r_3}m_2-m_3)} b \right) \sum_{c|2^{r_2}(2^{r_1-r_2}m_1-m_2)} c \\
&= \sum_{\substack{m_2<2^{r_1-r_2}m_1\\ 2\nmid m_2}} \left\{ 8^{r_2-r_3-1}\left(2^{r_3+1}-1\right)^2 \sigma_3(m_2) \right\} \left(2^{r_2+1}-1\right) \sigma_1\left(2^{r_1-r_2}m_1 - m_2\right)
\end{aligned} \tag{24}$$

by Theorem 1.3. So Eq. (24) is equal to

$$\begin{aligned}
&8^{r_2-r_3-1}\left(2^{r_3+1}-1\right)^2\left(2^{r_2+1}-1\right) \sum_{\substack{m_2<2^{r_1-r_2}m_1\\ 2\nmid m_2}} \sigma_3(m_2)\sigma_1\left(2^{r_1-r_2}m_1 - m_2\right) \\
&\quad = 8^{r_2-r_3-1}\left(2^{r_3+1}-1\right)^2\left(2^{r_2+1}-1\right) \left\{ \sum_{m_2<2^{r_1-r_2}m_1} \sigma_3(m_2)\sigma_1\left(2^{r_1-r_2}m_1 - m_2\right) \right. \\
&\qquad \left. - \sum_{\substack{m_2<2^{r_1-r_2}m_1\\ 2|m_2}} \sigma_3(m_2)\sigma_1\left(2^{r_1-r_2}m_1 - m_2\right) \right\}.
\end{aligned} \tag{25}$$

Then the second term of (25) is

$$\begin{aligned}
&\sum_{\substack{m_2<2^{r_1-r_2}m_1\\ 2|m_2}} \sigma_3(m_2)\sigma_1\left(2^{r_1-r_2}m_1 - m_2\right) \\
&= \sum_{m_2<2^{r_1-r_2-1}m_1} \sigma_3(2m_2)\sigma_1\left(2^{r_1-r_2}m_1 - 2m_2\right) \\
&= \sum_{m_2<2^{r_1-r_2-1}m_1} \left\{ 9\sigma_3(m_2) - 8\sigma_3\left(\frac{m_2}{2}\right) \right\} \sigma_1\left(2^{r_1-r_2}m_1 - 2m_2\right)
\end{aligned}$$

$$
\begin{aligned}
&= 9 \sum_{m_2 < 2^{r_1-r_2-1} m_1} \sigma_3(m_2)\sigma_1\bigl(2^{r_1-r_2} m_1 - 2m_2\bigr) \\
&\quad - 8 \sum_{m_2 < 2^{r_1-r_2-2} m_1} \sigma_3(m_2)\sigma_1\bigl(2^{r_1-r_2} m_1 - 4m_2\bigr).
\end{aligned}
$$

So we refer to

$$
\sum_{m=1}^{n-1} \sigma_1(m)\sigma_3(n-m) = \frac{1}{240}\bigl[21\sigma_5(n) + (10-30n)\sigma_3(n) - \sigma_1(n)\bigr]
$$

in [2, (3.12)],

$$
\sum_{m<n/2} \sigma_3(m)\sigma_1(n-2m) = \frac{1}{240}\sigma_5(n) + \frac{1}{12}\sigma_5\left(\frac{n}{2}\right) + \frac{(1-3n)}{24}\sigma_3\left(\frac{n}{2}\right) - \frac{1}{240}\sigma_1(n)
$$

in [2, Theorem 6], and

$$
\begin{aligned}
\sum_{m<n/4} \sigma_3(m)\sigma_1(n-4m) &= \frac{1}{3{,}840}\sigma_5(n) + \frac{1}{256}\sigma_5\left(\frac{n}{2}\right) + \frac{1}{12}\sigma_5\left(\frac{n}{4}\right) \\
&\quad + \frac{(1-3n)}{24}\sigma_3\left(\frac{n}{4}\right) - \frac{1}{240}\sigma_1(n) + \frac{1}{256}a(n)
\end{aligned}
$$

with $\sum_{n=1}^{\infty} a(n)q^n = q\prod_{n=1}^{\infty}(1-q^{2n})^{12}$ in [6, Theorem 4.2]. Therefore, (24) becomes

$$
\begin{aligned}
\sum_{\substack{ax+by+cz=2^{r_1}m_1 \\ ax+by=2^{r_2}m_2 \\ ax=2^{r_3}m_3 \\ m_2 \text{ odd} \\ m_3 \text{ odd}}} abc &= 2^{-2r_2-3r_3-8}\bigl(2^{r_2+1}-1\bigr)\bigl(2^{r_3+1}-1\bigr)^2\bigl\{32^{r_1}\sigma_5(m_1) + 32^{r_2}a\bigl(2^{r_1-r_2}m_1\bigr)\bigr\} \\
&= 2^{5r_1-2r_2-3r_3-8}\bigl(2^{r_2+1}-1\bigr)\bigl(2^{r_3+1}-1\bigr)^2\sigma_5(m_1),
\end{aligned}
$$

where we use the fact that $r_1 > r_2$ and $a(2n) = 0$ for $n \in \mathbb{N}$. This completes the proof this theorem. □

Proof of Theorem 1.5 From Theorem 1.4, we observe that

$$
\begin{aligned}
\sum_{\substack{ax+by+cz+du=2^{r_1}m_1 \\ ax+by+cz=2^{r_2}m_2 \\ ax+by=2^{r_3}m_3 \\ ax=2^{r_4}m_4 \\ m_2 \text{ odd} \\ m_3 \text{ odd} \\ m_4 \text{ odd}}} abcd &= 2^{5r_2-2r_3-3r_4-8}\bigl(2^{r_2+1}-1\bigr)\bigl(2^{r_3+1}-1\bigr)\bigl(2^{r_4+1}-1\bigr)^2 \\
&\quad \times \sum_{\substack{m_2 < 2^{r_1-r_2}m_1 \\ 2 \nmid m_2}} \sigma_5(m_2)\sigma_1\bigl(2^{r_1-r_2}m_1 - m_2\bigr).
\end{aligned}
$$

Thus, we refer to

$$\sum_{m<\frac{n}{2}} \sigma_5(m)\sigma_1(n-2m) = \frac{1}{2{,}142}\sigma_7(n) + \frac{2}{51}\sigma_7\left(\frac{n}{2}\right) + \frac{(1-2n)}{24}\sigma_5\left(\frac{n}{2}\right) + \frac{1}{504}\sigma_1(n) - \frac{1}{408}b(n)$$

and

$$\sum_{m<\frac{n}{4}} \sigma_5(m)\sigma_1(n-4m) = \frac{1}{137{,}088}\sigma_7(n) + \frac{1}{2{,}176}\sigma_7\left(\frac{n}{2}\right) + \frac{2}{51}\sigma_7\left(\frac{n}{4}\right) + \frac{(1-2n)}{24}\sigma_5\left(\frac{n}{4}\right) + \frac{1}{504}\sigma_1(n) - \frac{13}{6{,}528}b(n) - \frac{19}{816}b\left(\frac{n}{2}\right)$$

in [6, Theorem 5.2]. Also, to obtain the formula, we use the fact that $b(n) = -8b(\frac{n}{2})$ in [11, Remark 4.3]. □

Proof of Theorem 1.6 If $2^r m < 2^R M$, then $r < \log_2(\frac{2^R M}{m})$. We note that

$$\sum_{ax+by=2^R M} ab = \sum_{r=0}^{R-1} \sum_{\substack{ax+by=2^R M \\ ax=2^r m \\ m \text{ odd}}} ab + \sum_{R\le r<\log_2(\frac{2^R M}{m})} \left(\sum_{\substack{ax+by=2^R M \\ ax=2^r m \\ m \text{ odd}}} ab \right).$$

Thus, by (19) and Corollary 4.3, we get our result. □

Theorem 4.4 *Let M be an odd positive integer. Let $R \in \mathbb{N}$ and $r \in \mathbb{N} \cup \{0\}$ with $R > r$. We have*

(a)

$$\sum_{\substack{ax+by=2^R M \\ ax=2^r m \\ m \text{ odd}}} (-1)^a ab = 8^{R-r-1}\left(2^{r+1}-3\right)\left(2^{r+1}-1\right)\sigma_3(M),$$

(b)

$$\sum_{\substack{ax+by=2^R M \\ ax=2^r m \\ m \text{ odd}}} (-1)^{a+b} ab = 8^{R-r-1}\left(2^{r+1}-3\right)^2\sigma_3(M).$$

Proof

(a) The proof is similar to Theorem 1.3. Let us consider that

$$\sum_{\substack{ax+by=2^R M \\ ax=2^r m \\ m \text{ odd}}} (-1)^a ab = \sum_{\substack{m<2^{R-r}M \\ 2\nmid m}} \left(\sum_{a|2^r m}(-1)^a a\right)\left(\sum_{b|2^r(2^{R-r}M-m)} b\right). \tag{26}$$

Then

$$\begin{aligned}\sum_{a|2^r m}(-1)^a a &= -\sum_{a|m} a + \sum_{a|m} 2a + \sum_{a|m} 2^2 a + \cdots + \sum_{a|m} 2^r a \\ &= \left(-1 + 2 + 2^2 + \cdots + 2^r\right)\sum_{a|m} a \\ &= \left(2^{r+1} - 3\right)\sigma_1(m).\end{aligned}$$

Thus, (26) becomes

$$\begin{aligned}\sum_{\substack{ax+by=2^R M \\ ax=2^r m \\ m \text{ odd}}} (-1)^a ab &= \sum_{\substack{m<2^{R-r}M \\ 2\nmid m}} \left(2^{r+1} - 3\right)\sigma_1(m)\cdot\left(2^{r+1} - 1\right)\sigma_1\left(2^{R-r}M - m\right) \\ &= \left(2^{r+1} - 3\right)\left(2^{r+1} - 1\right) \sum_{\substack{m<2^{R-r}M \\ 2\nmid m}} \sigma_1(m)\sigma_1\left(2^{R-r}M - m\right).\end{aligned}$$

Then by (17), we get our result.

(b) We sketch the proof as follows:

$$\begin{aligned}\sum_{\substack{ax+by=2^R M \\ ax=2^r m \\ m \text{ odd}}} (-1)^{a+b} ab &= \sum_{\substack{ax+by=2^R M \\ ax=2^r m \\ m \text{ odd}}} (-1)^a a \cdot (-1)^b b \\ &= \sum_{\substack{m<2^{R-r}M \\ 2\nmid m}} \left(\sum_{a|2^r m} (-1)^a a\right)\left(\sum_{b|2^r(2^{R-r}M-m)} (-1)^b b\right).\end{aligned}$$

□

Proof of Corollary 1.7 Firstly, from (5), we note that

$$A_1(R,r) = \frac{1}{8}\left(8^{R-r}\right)\left(2^{r+1} - 1\right)^2 \sigma_3(M).$$

If $r \geq 0$, then

$$\begin{aligned}A_1(R,r) &\geq \frac{1}{8}\left(8^{R-r}\right)\sigma_3(M) \\ &> \frac{7}{8}\left(\frac{8^{R-r} - 1}{8 - 1}\right)\sigma_3(M).\end{aligned}$$

It is easily checked that $\sigma_3(2^{R-r-1}) = \frac{8^{R-r}-1}{8-1}$. So we obtain

$$A_1(R,r) > \frac{1}{8}\sigma_3\left(2^{R-r-1}M\right)$$

with $(2,M) = 1$. Secondly, by (6), we deduce that

$$A_2(r_1,r_2,r_3) = 2^{5r_1-8}\left(\frac{2^{r_2+1} - 1}{4^{r_2}}\right)\left(\frac{1}{2^{r_3}}\right)\left(\frac{2^{r_3+1} - 1}{2^{r_3}}\right)^2 \sigma_5(m_1).$$

$2t - t^2 = -(t-1)^2 + 1$ and $0 < 2t - t^2 \leq \frac{3}{4}$ with $0 < t \leq \frac{t}{2}$. Put $t = (\frac{1}{2})^{r_2}$ then

$$0 < \frac{2^{r_2+1} - 1}{4^{r_2}} \leq \frac{3}{4}. \tag{27}$$

Thirdly, we consider $f(t) = t(2-t)^2$ with $0 < t < 1$. Then, we easily check that $0 < f(t) \leq \frac{16}{27}$ so

$$0 < \frac{2^{r_3+1} - 1}{8^{r_3}} \leq \frac{16}{27}. \tag{28}$$

Consider

$$\frac{2^{5r_1}}{32} - \frac{2^{5r_1} - 1}{32 - 1} = \frac{1 - (32)^{r_1 - 1}}{31} < 0 \tag{29}$$

with $r_1 > 1$. From (29), we deduce that

$$2^{5r_1 - 5} < \sigma_5\bigl(2^{r_1 - 1}\bigr)$$

and

$$2^{5r_1 - 8} < \frac{1}{8}\sigma_5\bigl(2^{r_1 - 1}\bigr). \tag{30}$$

From (27), (28) and (30), we compute that $A_2(r_1, r_2, r_3) < \frac{1}{18}\sigma_5(2^{r_1-1}m_1)$. □

5 A study of $\sum_{ax+by+cz+du=n} abcd$

Corollary 5.1 *Let m_1 be an odd positive integer. Let $r_1, r_2, r_3 \in \mathbb{N}$ and $r_4 \in \mathbb{N} \cup \{0\}$ with $r_1 > r_2 > r_3 > r_4$. If $r_2, r_3, r_4 \not\equiv -1 \pmod 8$, then we have*

$$2^{7r_1+1}\sigma_7(m_1) \equiv 15(-1)^{r_1 - r_2 + 1} \cdot 2^{4r_2 + 3r_1} b(m_1) \pmod{17 \cdot 2^{2r_2 + 2r_3 + 3r_4 + 11}}.$$

Proof From Theorem 1.5, we have

$$\sum_{\substack{ax+by+cz+du=2^{r_1}m_1 \\ ax+by+cz=2^{r_2}m_2 \\ ax+by=2^{r_3}m_3 \\ ax=2^{r_4}m_4 \\ m_2 \text{ odd} \\ m_3 \text{ odd} \\ m_4 \text{ odd}}} abcd = \frac{1}{17} \cdot 2^{-2r_2 - 2r_3 - 3r_4 - 10}\bigl(2^{r_2+1} - 1\bigr)\bigl(2^{r_3+1} - 1\bigr)\bigl(2^{r_4+1} - 1\bigr)^2$$
$$\times \bigl\{2^{7r_1}\sigma_7(m_1) + 15(-1)^{r_1 - r_2} 2^{4r_2 + 3r_1 - 1} b(m_1)\bigr\}. \tag{31}$$

Since $r_2, r_3, r_4 \not\equiv -1 \pmod 8$ by the assumption, therefore, $2^{r_2+1} - 1 \not\equiv 0 \pmod{17}$. So from (31), we have

$$2^{-2r_2 - 2r_3 - 3r_4 - 10}\bigl\{2^{7r_1}\sigma_7(m_1) + 15(-1)^{r_1 - r_2} 2^{4r_2 + 3r_1 - 1} b(m_1)\bigr\} \equiv 0 \pmod{17}. \tag{32}$$

By multiplying (32) by $2^{2r_2 + 2r_3 + 3r_4 + 11}$, we obtain the proof. □

Remark 5.2 This is a similar result to that in [12, Theorem 5.3].

6 Another convolution sums

Theorem 6.1 *Let $M \in \mathbb{N}$ with $3 \nmid M$. Let $R \in \mathbb{N}$ and $r \in \mathbb{N} \cup \{0\}$ with $R \geq r$. Then we have*

$$A_1^*(R,r) := \sum_{\substack{ax+by=3^R M \\ ax=3^r m \\ 3\nmid m}} ab = \frac{1}{2} \cdot 3^{3R-3r-2}\left(3^{r+1}-1\right)^2 \sigma_3(M) \tag{33}$$

and if $R > r$, then

$$A_1^*(R,r) > \frac{52}{9} \cdot \sigma_3\left(3^{R-r-1}M\right).$$

Proof It is similar to Theorem 1.3. So we obtain that

$$\begin{aligned}
\sum_{\substack{ax+by=3^R M \\ ax=3^r m \\ 3\nmid m}} ab &= \frac{1}{4}\left(3^{r+1}-1\right)^2 \sum_{\substack{m<3^{R-r}M \\ 3\nmid m}} \sigma_1(m)\sigma_1\left(3^{R-r}M-m\right) \\
&= \frac{1}{4}\left(3^{r+1}-1\right)^2 \left\{ \sum_{m<3^{R-r}M} \sigma_1(m)\sigma_1\left(3^{R-r}M-m\right) \right. \\
&\quad \left. - \sum_{\substack{m<3^{R-r}M \\ 3\mid m}} \sigma_1(m)\sigma_1\left(3^{R-r}M-m\right) \right\} \\
&= \frac{1}{4}\left(3^{r+1}-1\right)^2 \left\{ \sum_{m<3^{R-r}M} \sigma_1(m)\sigma_1\left(3^{R-r}M-m\right) \right. \\
&\quad \left. - \sum_{m<3^{R-r-1}M} \sigma_1(3m)\sigma_1\left(3^{R-r}M-3m\right) \right\}.
\end{aligned}$$

Then we refer to

$$\sum_{m<\frac{n}{3}} \sigma_1(3m)\sigma_1(n-3m) = \frac{1}{36}\left\{7\sigma_3(n) + (3-18n)\sigma_1(n) + 8\sigma_3\left(\frac{n}{3}\right)\right\},$$

if $n \equiv 0 \pmod 3$ in [2, Theorem 7]. Therefore, we get (33). By (33), we note that

$$\begin{aligned}
A_1^*(R,r) &= \frac{1}{2} 3^{3R-3r-2}\left(3^{r+1}-1\right)^2 \sigma_3(M) \\
&= \frac{1}{2} \cdot \frac{1}{9} 3^{3(R-r)}\left(3^{r+1}-1\right)^2 \sigma_3(M).
\end{aligned} \tag{34}$$

It is well known that

$$\left(3^{r+1}-1\right)^2 \geq 4 \tag{35}$$

Table 2 Values of $b(n)$ $(1 \leq n \leq 12)$

n	$b(n)$	n	$b(n)$
1	1	7	1,016
2	–8	8	–512
3	12	9	–2,043
4	64	10	1,680
5	–210	11	1,092
6	–96	12	768

with $r \geq 0$. Combine (34) and (35),

$$\begin{aligned} A_1^*(R,r) &> \frac{2}{9} \cdot 3^{3(R-r)} \sigma_3(M) \\ &> \frac{2}{9} \cdot 26 \cdot \frac{(3^{3(R-r)} - 1)}{27 - 1} \sigma_3(M) \\ &= \frac{52}{9} \cdot \sigma_3\bigl(3^{R-r-1}M\bigr) \end{aligned}$$

with $(3, M) = 1$. This completes the proof of this theorem. □

Appendix

The first twelve values of $b(n)$ for $n \in \mathbb{N}$ are given in Table 2.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in this article. All authors read and approved the final manuscript.

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