Physics Letters A 373 (2009) 221–226

Contents lists available at ScienceDirect

Physics Letters A

www.elsevier.com/locate/pla

Fractional optimal control problem of a distributed system in cylindrical coordinates

Necati Özdemir *, Derya Karadeniz, Beyza B. İskender

Department of Mathematics, Faculty of Science and Arts, Balıkesir University, Cagis Campus, 10145 Balıkesir, Turkey

ARTICLE INFO

Article history:
Received 2 August 2008
Received in revised form 30 October 2008
Accepted 11 November 2008
Available online 21 November 2008
Communicated by R. Wu

Keywords:
Fractional optimal control
Axis-symmetry
Riemann–Liouville fractional derivative
Grünwald–Letnikov approach
Cylindrical coordinates

ABSTRACT

In this work, Fractional Optimal Control Problem (FOCP) of a Distributed system is investigated in cylindrical coordinates. Axis-symmetry naturally arises in the problem formulation. The fractional time derivative is described in the Riemann–Liouville (RL) sense. The performance index of a FOCP is considered as a function of state and control variables and system dynamics are given as a Partial Fractional Differential Equation (PFDE). The method of separation of variables is used to find the solution of the problem. Eigenfunctions are used to eliminate the terms containing space parameters and to define the problem in terms of a set of generalized state and control variables. For numerical computations, Grünwald–Letnikov (GL) approach is used. A time-invariant example is considered to demonstrate the effectiveness of the formulation. The comparison of analytical and numerical solutions is given using simulation results and also it can be seen that analytical and numerical results converge each other. In addition, simulation results for different values of order of derivative, time discretizations and eigenfunctions are analyzed.

1. Introduction

In the last years, it has been showed that the accurate modelling of dynamics of many physical systems can be obtained using Fractional Differential Equations (FDEs). Therefore, there has been a great deal of interest in the solution methods of FDEs in analytical and numerical sense. When FDEs describe the performance index and system dynamics, an optimal control problem reduces to a FOCP. The Fractional Optimal Control (FOC) of a distributed system is a FOC for which system dynamics are defined with PFDEs. There has been very little work in the area of FOCPs, especially FOC of a distributed system.

In the area of Fractional Order Controls and Systems, there are some papers which must be mentioned here. Oustaloup [1] investigated fractional order controls for dynamic systems and showed that the CRONE method has better performance than classical PID controller. Podlubny [2] demonstrated that PI, PD and PID controllers are particular cases of the fractional $PI^{\lambda}D^{\mu}$ controller. Podlubny, Dorcak and Kostial [3] compared RL–GL and Caputo fractional derivatives from the viewpoint of formulation and solution of engineering and physics problems, and they also presented the fractional $PI^{\lambda}D^{\mu}$ controller. Dorcak [4] analyzed dynamic properties and numerical methods of simulation of fractional-order systems. Petras, Dorcak and Kostial [5] dealt with fractional-order controlled systems and fractional-order controllers in discrete time domain. Machado [6,7] introduced algorithms for fractional-order discrete time controllers. Özdemir and İskender [8] applied fractional PI^{λ} controller for fractional order linear system subject to input hysteresis. Although, these cited papers show that the research area of fractional-order systems and controllers is popular, they do not mention FOCPs.

Recently, some papers related to the theories and solution methods of FOCPs have been presented. A general formulation and a numerical scheme for FOCPs in RL sense are investigated in Agrawal [9]. Agrawal [10] presents an eigenfunction expansion approach for a FOCP for a class of distributed system whose dynamics are defined in Caputo sense. Özdemir et al. [11] also use eigenfunction expansion approach to formulate a FOCP of a 2-dimensional distributed system. A general scheme for stochastic analysis of FOCPs is proposed in Agrawal [12]. A formulation for FOCPs whose dynamics are described in terms of Caputo fractional derivative is researched in Agrawal [13,14] and the same problem is investigated in terms of RL fractional derivatives in Agrawal and Baleanu [15].

In this Letter, we formulate a FOCP of a 3-dimensional distributed system defined in cylindrical coordinates. For this reason, the axis-symmetric case arises naturally in this problem. It is formulated in terms of RL fractional derivative and GL approach is used for numerical computation. Özdemir et al. [11] consider the problem in 2-dimensional case (Cartesian coordinates), whereas

* Corresponding author. Tel.: +90 266 6121000, ext: 215; fax: +90 266 6121215.
E-mail addresses: nozdemir@balikesir.edu.tr (N. Özdemir), fractional_life@hotmail.com (D. Karadeniz), beyzabillur@hotmail.com (B.B. İskender).

0375-9601/$ – see front matter
doi:10.1016/j.physleta.2008.11.019

in this Letter we formulate the problem in 3-dimensional case (cylindrical coordinates). The solution of problem is obtained for different number of eigenfunctions and time discretization. Also, the papers related to axis-symmetry can be given as follows: El-Shahed and Salem [16] find the solution of fractional generalization of Navier–Stokes equations described by polar coordinates. Fractional radial diffusion in a cylinder and in a sphere are proposed in Povstenko [17,18], respectively. Özdemir et al. [19], and Özdemir and Karadeniz [20] have recently formulated an axis-symmetric fractional diffusion-wave problem.

This Letter is organized as follows. In Section 2, the definitions of RL fractional derivative and FOCP are given. In Section 3, an axis-symmetric FOCP defined in cylindrical coordinates is presented. In Section 4, the GL approach is given and numerical results are analyzed. In Appendix A, the analytical solution of the problem is presented. Finally, Section 5 shows conclusions of this work.

2. Mathematical tools

Several definitions of a fractional derivative such as Riemann–Liouville, Caputo, Grünwald–Letnikov, Weyl, Marchaud and Riesz have been proposed. In this section, we formulate the problem in terms of the Riemann–Liouville fractional derivatives, which are defined as:

The left Riemann–Liouville fractional derivative

$$ {}_aD_t^\alpha f(t)=\frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n\int_a^t(t-\tau)^{n-\alpha-1}f(\tau)\,d\tau, \tag{1}$$

and the right Riemann–Liouville fractional derivative

$$ {}_tD_b^\alpha f(t)=\frac{1}{\Gamma(n-\alpha)}\left(-\frac{d}{dt}\right)^n\int_t^b(\tau-t)^{n-\alpha-1}f(\tau)\,d\tau, \tag{2}$$

where $f(.)$ is a time-dependent function, $\Gamma(.)$ is the Euler's gamma function, α is the order of derivative such that $n-1<\alpha<n$ and t represents time which belongs to $[a,b]$ $(a,b\in\mathbb{R})$. When α is an integer, these definitions reduce to ordinary differential operators, i.e.,

$$ {}_aD_t^\alpha f(t)=\left(\frac{d}{dt}\right)^n, \qquad {}_tD_b^\alpha f(t)=\left(-\frac{d}{dt}\right)^n, \quad \alpha=n, \quad n=1,2,\ldots. \tag{3}$$

Using the above definitions, the FOCP of interest is defined in Agrawal [9,15] as follows: Find the optimal control $u(t)$ that minimizes the performance index

$$ J(u)=\int_0^1F(x,u,t)\,dt \tag{4}$$

subject to the system dynamic constraints

$$ {}_0D_t^\alpha x=G(x,u,t) \tag{5}$$

and initial condition

$$ x(0)=x_0, \tag{6}$$

where $x(t)$ and $u(t)$ are the state and the control variables, respectively, F and G are two arbitrary functions. For $\alpha=1$, this problem reduces to a standard optimal control problem. Moreover, we take $0<\alpha<1$ and assume that $x(t)$, $u(t)$ and $G(x,u,t)$ are all scalar functions for simplicity. In the case of $\alpha>1$, additional initial conditions could be necessary. In the usual sense, the differential equations which describe the dynamics of the system are written in the state-space form, in that case, the order of the derivatives turns out to be less than 1. For this reason, we consider $0<\alpha<1$ in this work. We further consider the necessary terminal conditions which is determined by using Lagrange multiplier technique as follows:

$$ {}_0D_t^\alpha x=G(x,u,t), \tag{7}$$

$$ {}_tD_1^\alpha\lambda=\frac{\partial F}{\partial x}+\frac{\partial G}{\partial x}\lambda, \tag{8}$$

$$ \frac{\partial F}{\partial u}+\frac{\partial G}{\partial u}\lambda=0, \tag{9}$$

where λ is the Lagrange multiplier and

$$ x(0)=x_0 \quad\text{and}\quad \lambda(1)=0. \tag{10}$$

Eqs. (7)–(9) represent the Euler–Lagrange equations for the FOCP defined by Eqs. (4)–(6). This indicates that the solution of FOCPs requires not only right derivatives but also left derivatives.

3. The axis-symmetric FOCP formulation

Let us consider the following problem: Find the control $u(r,z,t)$ that minimizes the performance index

$$ J(u)=\frac{1}{2}\int_0^1\int_0^L\int_0^R r\big[Ax^2(r,z,t)+Bu^2(r,z,t)\big]\,dr\,dz\,dt \tag{11}$$

subject to the system dynamic constraints

$$ {}_0D_t^\alpha x(r,z,t)=\beta\left(\frac{\partial^2x(r,z,t)}{\partial r^2}+\frac{1}{r}\frac{\partial x(r,z,t)}{\partial r}+\frac{\partial^2x(r,z,t)}{\partial z^2}\right)+u(r,z,t), \tag{12}$$

initial condition

$$ x(r,z,0)=x_0(r,z) \quad (0<r<R,\ 0<z<L) \tag{13}$$

and the boundary conditions

$$ \frac{\partial x(0,z,t)}{\partial r}=\frac{\partial x(R,z,t)}{\partial r}=\frac{\partial x(r,0,t)}{\partial z}=\frac{\partial x(r,L,t)}{\partial z}=0, \tag{14}$$

where $x(r,z,t)$ and $u(r,z,t)$ are the state and the control functions depending on r, z, which represent cylindrical coordinates, and t. A and B are two arbitrary functions. R is the radius and L is the length of cylindrical domain on which problem is defined. The upper limit for time t is taken as 1 for convenience. This limit can be any positive number.

We assume that $x(r,z,t)$ and $u(r,z,t)$ can be written as

$$ x(r,z,t)=\sum_{i=1}^m\sum_{j=1}^m x_{ij}(t)J_0\left(\mu_j\frac{r}{R}\right)\sin\left(\frac{i\pi}{L}z\right), \tag{15}$$

$$ u(r,z,t)=\sum_{i=1}^m\sum_{j=1}^m u_{ij}(t)J_0\left(\mu_j\frac{r}{R}\right)\sin\left(\frac{i\pi}{L}z\right), \tag{16}$$

where $J_0(\mu_j\frac{r}{R})\sin(\frac{i\pi}{L}z)$, $i,j=1,2,\ldots,m$, are the eigenfunctions which are obtained by using the method of separation of variables. Here, J_0 is zero order Bessel function of first kind and μ_j are the roots of J_0. $x_{ij}(t)$ and $u_{ij}(t)$ are the state and the control eigencoordinates; m is a finite positive integer that theoretically should go to infinity. However, we take m as a finite number for computational purposes. By substituting Eqs. (15) and (16) into Eq. (11), we obtain

$$ J=\frac{R^2L}{8}\int_0^1\sum_{i=1}^m\sum_{j=1}^m J_1^2(\mu_j)\big[Ax_{ij}^2(t)+Bu_{ij}^2(t)\big]\,dt. \tag{17}$$

Substituting Eqs. (15) and (16) into Eq. (12), and equating the coefficients of

$$J_0\left(\mu_i \frac{r}{R}\right)\sin\left(\frac{i\pi}{L}z\right),$$

we get

$$ {}_0D_t^\alpha x_{ij}(t) = -\beta\left\{\left(\frac{\mu_j}{R}\right)^2 + \left(\frac{i\pi}{L}\right)^2\right\}x_{ij}(t) + u_{ij}(t), \quad i,j = 1,2,\ldots,m. \tag{18}$$

Substituting $x(r,z,t)$ from Eq. (15) into Eq. (13), multiplying both sides by $rJ_0(\mu_k \frac{r}{R})$ and integrating from 0 to R, we obtain

$$x_{ij}(0) = \frac{2}{R^2 J_1^2(\mu_j)\sin(\frac{i\pi}{L}z)}\int_0^R rJ_0\left(\mu_j\frac{r}{R}\right)x_0(r,z)\,dr, \quad i,j = 1,2,\ldots,m. \tag{19}$$

We determine the necessary condition by substituting Eqs. (17) and (18) into Eqs. (7)–(9) as follows:

$$ {}_tD_1^\alpha \lambda_{ij}(t) - \frac{R^2L}{4}AJ_1^2(\mu_j)x_{ij}(t) + \beta\left\{\left(\frac{\mu_j}{R}\right)^2 + \left(\frac{i\pi}{L}\right)^2\right\}\lambda_{ij}(t) = 0, \tag{20}$$

$$\frac{R^2L}{4}BJ_1^2(\mu_j)u_{ij}(t) + \lambda_{ij}(t) = 0, \tag{21}$$

$$ {}_0D_t^\alpha x_{ij}(t) + \beta\left\{\left(\frac{\mu_j}{R}\right)^2 + \left(\frac{i\pi}{L}\right)^2\right\}x_{ij}(t) - u_{ij}(t) = 0, \tag{22}$$

and

$$\lambda_{ij}(1) = 0, \quad i,j = 1,2,\ldots,m, \tag{23}$$

where $\lambda_{ij}(t)$, $i,j = 1,2,\ldots,m$, are the Lagrange multipliers. By arranging the terms of Eqs. (20)–(22), we obtain

$$ {}_tD_1^\alpha u_{ij}(t) = -\frac{A}{B}x_{ij}(t) - \beta\left\{\left(\frac{\mu_j}{R}\right)^2 + \left(\frac{i\pi}{L}\right)^2\right\}u_{ij}(t), \quad i,j = 1,2,\ldots,m. \tag{24}$$

Note that, for $\alpha = 1$ the fractional differential equations (18) and (24) reduce to the following form:

$$\dot{x}_{ij}(t) = -\beta\left\{\left(\frac{\mu_j}{R}\right)^2 + \left(\frac{i\pi}{L}\right)^2\right\}x_{ij}(t) + u_{ij}(t), \tag{25}$$

$$\dot{u}_{ij}(t) = \frac{A}{B}x_{ij}(t) + \beta\left\{\left(\frac{\mu_j}{R}\right)^2 + \left(\frac{i\pi}{L}\right)^2\right\}u_{ij}(t). \tag{26}$$

The general solution of these linear differential equations is given in Appendix A.

4. GL approximation and numerical results

We use Grünwald–Letnikov approach to solve the FOCP numerically. In order to explain this algorithm, we first divide the entire time-domain into N subdomains with $h = \frac{1}{N}$ sizes and the times at grid points j are given as $t_j = jh$, $j = 0,1,\ldots,N$. Now, we consider the following fractional differential equations correspond to Eqs. (18) and (24) as

$$ {}_0D_t^\alpha x = ax + bu,$$

$$ {}_tD_b^\alpha u = cx + du,$$

where a, b, c and d are arbitrary coefficients. ${}_0D_t^\alpha x$ and ${}_tD_1^\alpha u$ fractional derivatives are approximated at node M using the GL formula in the following form

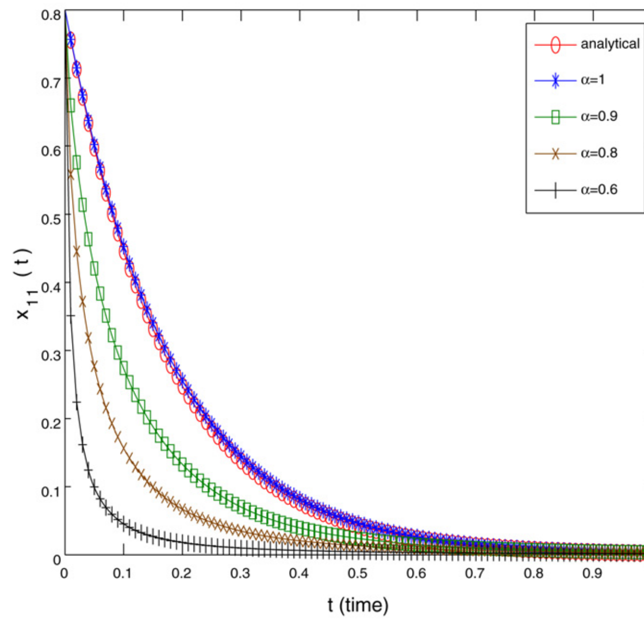

Fig. 1. Evolution of state $x_{11}(t)$ for different values of α, $r = z = 0.5$, $m = 5$ and $N = 100$.

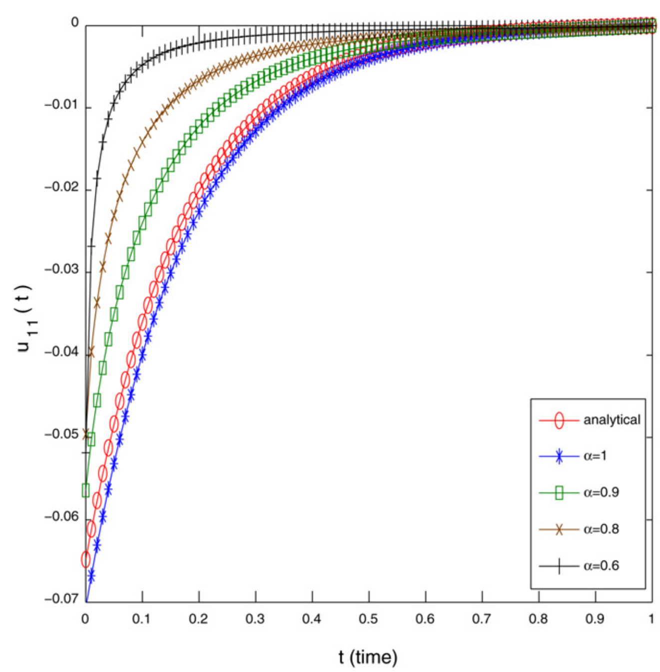

Fig. 2. Evolution of control $u_{11}(t)$ for different values of α, $r = z = 0.5$, $m = 5$ and $N = 100$.

$$ {}_0D_t^\alpha x = \frac{1}{h^\alpha}\sum_{j=0}^{M} w_j^{(\alpha)} x(hM - jh),$$

$$ {}_tD_1^\alpha u = \frac{1}{h^\alpha}\sum_{j=0}^{N-M} w_j^{(\alpha)} u(hM + jh),$$

where

$$w_0^\alpha = 1, \qquad w_j^\alpha = \binom{\alpha}{j} = \left(1 - \frac{\alpha+1}{j}\right)w_{(j-1)}^\alpha.$$

Therefore, the above equations are rewritten as

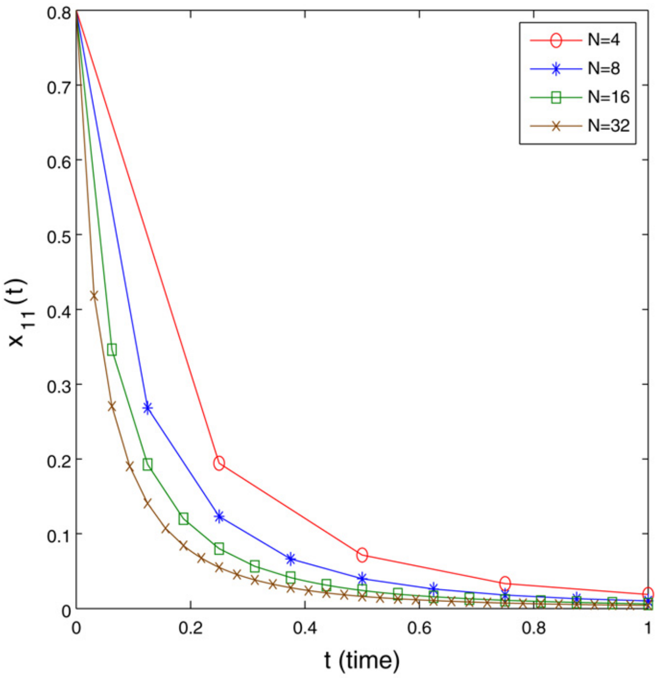

Fig. 3. Evolution of state $x_{11}(t)$ for different values of N, $\alpha = 0.75$, $r = z = 0.5$ and $m = 5$.

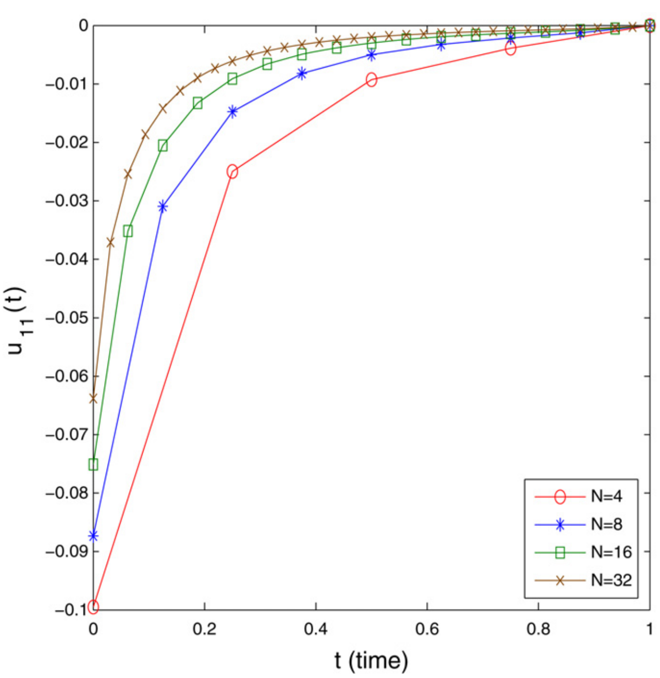

Fig. 4. Evolution of control $u_{11}(t)$ for different values of N, $\alpha = 0.75$, $r = z = 0.5$ and $m = 5$.

$$\frac{1}{h^\alpha}\sum_{j=0}^{M} w_j^{(\alpha)} x(hM - jh) = ax(Mh) + bu(Mh),$$

$$\frac{1}{h^\alpha}\sum_{j=0}^{N-M} w_j^{(\alpha)} u(hM + jh) = cx(Mh) + du(Mh),$$

and

$$x(0) = x_0, \qquad u(1) = 0.$$

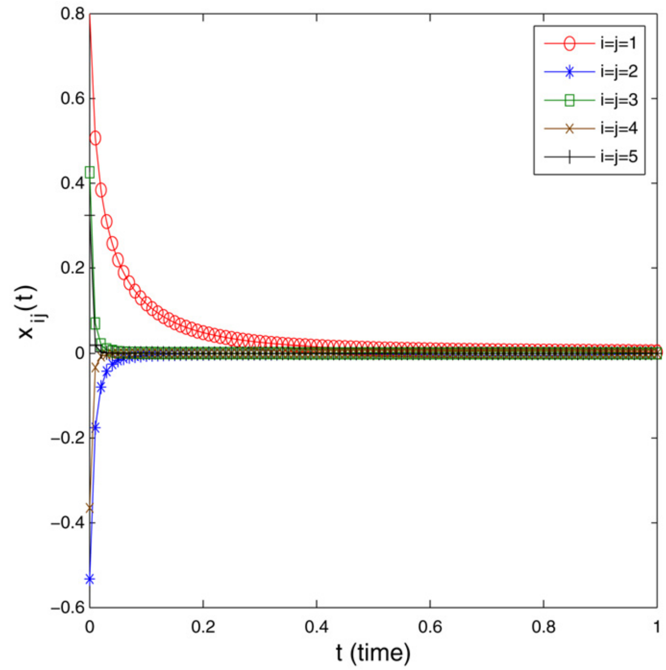

Fig. 5. Evolution of states $x_{ij}(t)$ for different values of m, $\alpha = 0.75$, $r = z = 0.5$ and $N = 100$.

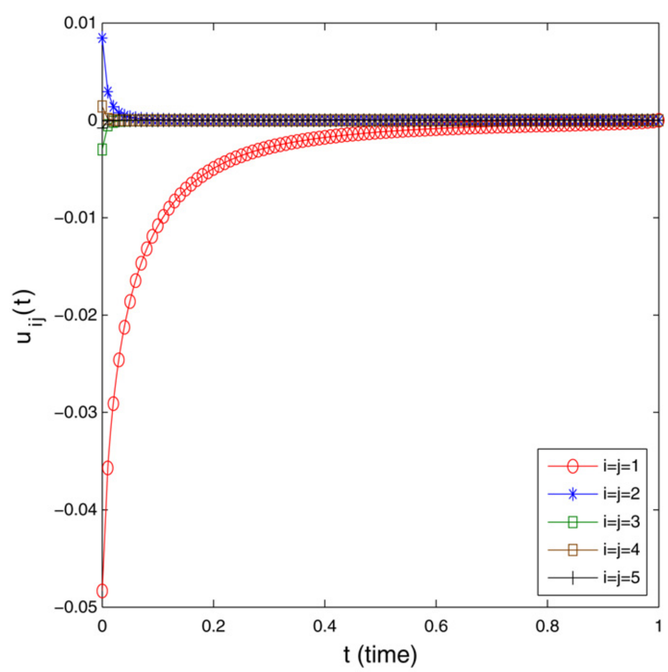

Fig. 6. Evolution of controls $u_{ij}(t)$ for different values of m, $\alpha = 0.75$, $r = z = 0.5$ and $N = 100$.

Consequently, we apply these steps to axis-symmetric FOCP and then obtain some simulation results by choosing the coefficients of Eq. (11); $A = B = 1$ and boundaries of the cylinder; $R = L = 1$. We consider the following initial conditions

$$x_0(r, z) = \sin\left(\frac{i\pi}{L} z\right). \tag{27}$$

Substituting Eq. (27) into Eq. (19), we get

$$x_{ij}(0) = \frac{1}{\mu_j \; J_1(\mu_j)}, \quad i, j = 1, 2, \ldots, m. \tag{28}$$

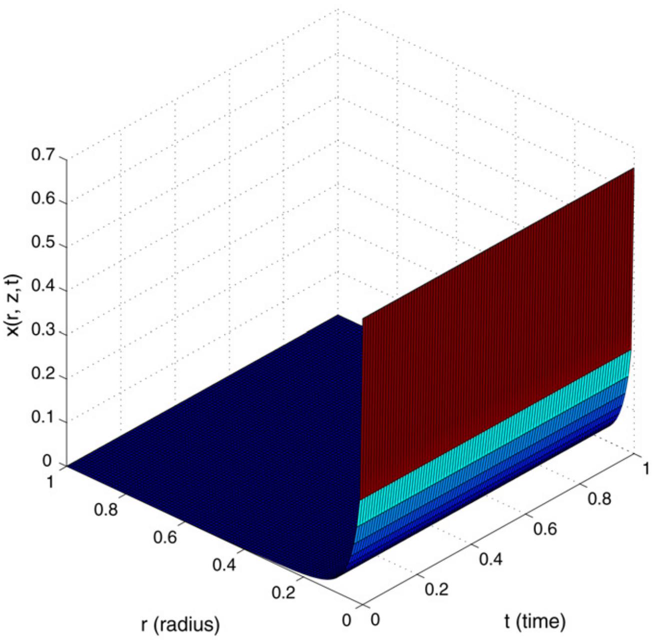

Fig. 7. Evolution of state function $x(r, z, t)$ as a function of r and t for $z = 0.5$, $\alpha = 0.75$, $m = 5$ and $N = 100$.

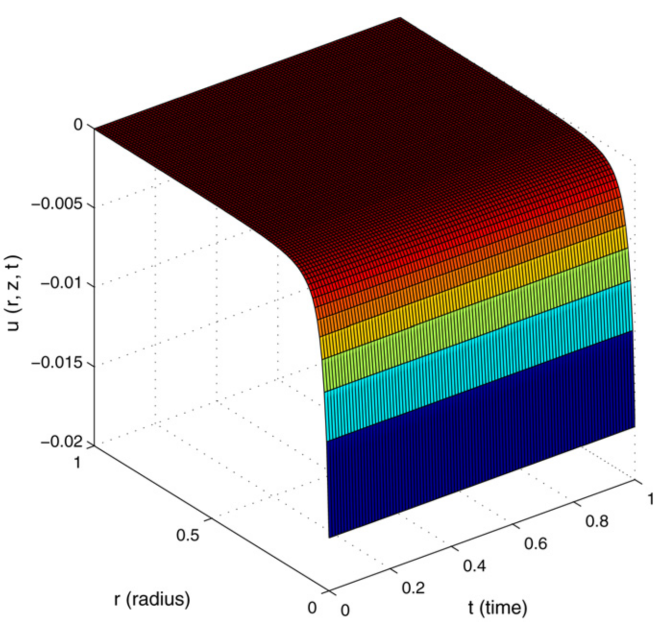

Fig. 8. Evolution of control function $u(r, z, t)$ as a function of r and t for $z = 0.5$, $\alpha = 0.75$, $m = 5$ and $N = 100$.

Figs. 1 and 2 show the state and the control response of the system, respectively, for fixed r and z coordinates with respect to different values of α. These figures also compare the analytical and the numerical solutions for $\alpha = 1$, and indicate that these are very close. Therefore, we can conclude that the numerical algorithm is effective.

Figs. 3 and 4 show dependence of the state and the control functions on the number of time discretization N, respectively. It can be seen from the figures that the algorithm is more stable when the number N is increased.

Figs. 5 and 6 show contribution of the number of eigenvalues (m) to the system response. It can be conclude that after $m = 5$ the eigenvalues approach to zero. Therefore, it is sufficient to truncate the calculations at $m = 5$.

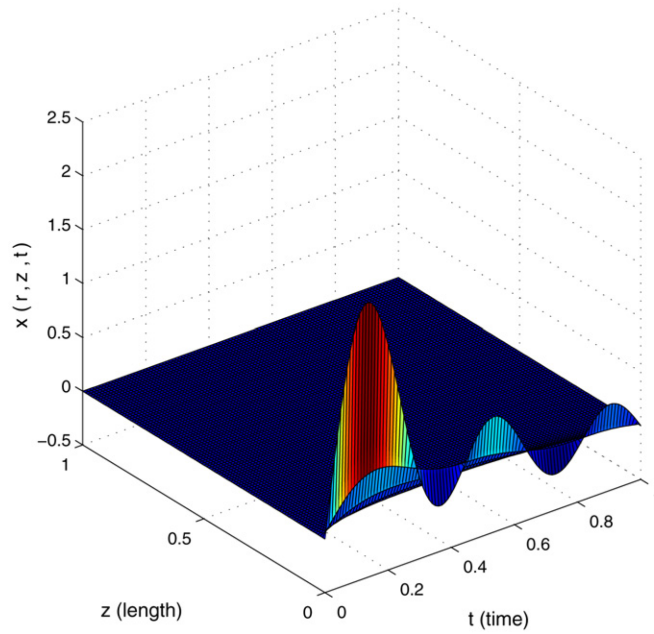

Fig. 9. Evolution of state function $x(r, z, t)$ as a function of z and t for $r = 0.5$, $\alpha = 0.75$, $m = 5$ and $N = 100$.

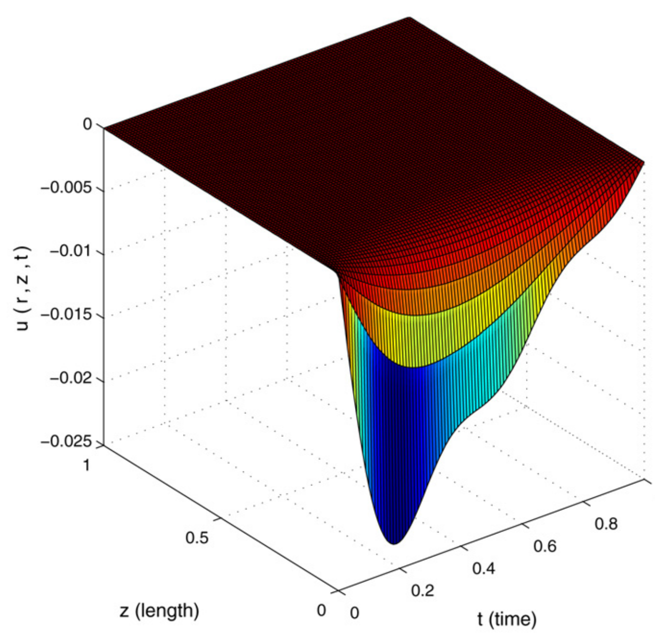

Fig. 10. Evolution of control function $u(r, z, t)$ as a function of z and t for $r = 0.5$, $\alpha = 0.75$, $m = 5$ and $N = 100$.

Figs. 7 and 8 show three-dimensional response of the state and the control of the system as a function of r and t by choosing $z = 0.5$ and $\alpha = 0.75$.

Figs. 9 and 10 also show three-dimensional response of the state and the control of the system as a function of z and t by choosing $r = 0.5$ and $\alpha = 0.75$.

5. Conclusions

FOCP of a distributed system was investigated in cylindrical coordinates in which the fractional time derivative was defined in the RL sense. Because of the cylindrical coordinates, axis-symmetry naturally arose in the problem formulation. The quadratic performance index of a FOCP was considered as a function of state and the control variables and system dynamic constraints were given as a PFDE. The method of separation of variables was used to find the solution of the problem. Therefore, the PFDE was decomposed into fractional, ordinary and Bessel differential equations. Solutions of the ordinary and the Bessel differential equations were called as eigenfunctions which were used to eliminate the terms containing space parameters and to define the problems in terms of a set of generalized state and control variables. For numerical computation, the GL approach was used. A time-invariant example was considered to demonstrate the effectiveness of the formulation. The simulation results showed that only a few eigenfunctions were sufficient to obtained the results, the solution converged when the time discretization was increased and as order of fractional derivative α approached to 1, the numerical results converged to the analytical ones.

Appendix A

For $\alpha = 1$, Eqs. (25) and (26) and the terminal conditions given by Eqs. (19) and (23) represent the necessary conditions for the problem defined by Eqs. (11)–(14). These equations are rewritten as

$$\begin{cases} \dot{x}_{ij}(t) = -a_{ij}x_{ij}(t) + u_{ij}(t), \\ \dot{u}_{ij}(t) = e_0 x_{ij}(t) + a_{ij}u_{ij}(t), \end{cases} \quad i, j = 1, 2, \ldots, m, \tag{A.1}$$

and

$$\begin{cases} x_{ij}(0) = x_{ij0}, \\ u_{ij}(1) = 0, \end{cases} \quad i, j = 1, 2, \ldots, m, \tag{A.2}$$

where

$$a_{ij} = \beta\left\{\left(\frac{\mu_j}{R}\right)^2 + \left(\frac{i\pi}{L}\right)^2\right\}, \quad i, j = 1, 2, \ldots, m, \tag{A.3}$$

$$e_0 = \frac{A}{B}. \tag{A.4}$$

After some manipulations, Eq. (A.1) leads to

$$\ddot{x}_{ij}(t) - b_{ij}^2 x_{ij}(t) = 0, \tag{A.5}$$

where b_{ij} is given by

$$b_{ij} = \sqrt{e_0 + a_{ij}^2}. \tag{A.6}$$

The solution of Eq. (A.5) is obtained as

$$x_{ij}(t) = x_{ij0}\left[\frac{b_{ij}\cosh(b_{ij}(t-1)) - a_{ij}\sinh(b_{ij}(t-1))}{b_{ij}\cosh(b_{ij}) + a_{ij}\sinh(b_{ij})}\right]. \tag{A.7}$$

Using Eqs. (A.1) and (A.7), we get

$$u_{ij}(t) = x_{ij0}\left[\frac{(b_{ij}^2 - a_{ij}^2)\sinh(b_{ij}(t-1))}{b_{ij}\cosh(b_{ij}) + a_{ij}\sinh(b_{ij})}\right]. \tag{A.8}$$

Consequently, $x(r, z, t)$ and $u(r, z, t)$ which respect to $x_{ij}(t)$ and $u_{ij}(t)$ can be obtained.

References

[1] A. Outsaloup, La Commande CRONE: Commande Robuste d'Ordre Non Enteire, Editions Hermes, Paris, 1991.
[2] I. Podlubny, IEEE Trans. Automat. Control 44 (1999) 208.
[3] I. Podlubny, L. Dorcak, I. Kostial, in: Proceeding of the 36th IEEE Conference on Decision and Control, Part 5, San Diego, CA, USA, 10–12 December 1997, pp. 4985–4990.
[4] L. Dorcak, Numerical models for simulation the fractional-order control systems, UEF SAV, The Academy of Sciences Institute of Experimental Physics, Kosice, Slovak Republic, 1994.
[5] I. Petras, L. Dorcak, I. Kostial, in: ICCC' 2000 Conference, High Tatras, 23–26 May 2000, pp. 257–260.
[6] J.A.T. Machado, Syst. Anal. Model. Simul. 27 (1997) 107.
[7] J.A.T. Machado, Syst. Anal. Model. Simul. 34 (1999) 419.
[8] N. Özdemir, B.B. İskender, Fractional PI^{λ} controller for fractional order linear system with input hysteresis, in: ENOC-2008, Saint Petersburg, Russia, 4 June–30 July 2008.
[9] O.P. Agrawal, Nonlinear Dynam. 38 (2004) 323.
[10] O.P. Agrawal, in: Proceedings of the 2007 ASME International Design Engineering Technical Conferences, Las Vegas, Nevada, 4–7 September 2007.
[11] N. Özdemir, O.P. Agrawal, B.B. İskender, D. Karadeniz, Nonlinear Dynam. (2008), doi:10.1007/s11071-008-9360-4.
[12] O.P. Agrawal, A general scheme for stochastic analysis of fractional optimal control problems, in: A.L. Mahaute, J.A.T. Machado, J.C. Trigeassou, J. Sabatier (Eds.), Fractional Differentiation and Its Applications, 2005, pp. 615–624.
[13] O.P. Agrawal, in: Proceedings of the 2nd IFAC Conference on Fractional Differentiations and Its Applications, IFAC, Porto, Portugal, 19–21 July 2006.
[14] O.P. Agrawal, J. Dyn. Syst. Meas. Control 130 (1) (2008) 11010.
[15] O.P. Agrawal, D. Baleanu, J. Vib. Control 13 (2007) 1269.
[16] M. El-Shahed, A. Salem, Appl. Math. Comput. 156 (2004) 287.
[17] Y.Z. Povstenko, J. Mol. Liq. 137 (2008) 46.
[18] Y.Z. Povstenko, Nonlinear Dynam. 53 (2008) 55.
[19] N. Özdemir, O.P. Agrawal, D. Karadeniz, B.B. İskender, Axis-symmetric fractional diffusion-wave problem: Part I – Analysis, in: ENOC-2008, Saint Petersburg, Russia, 4 June–30 July 2008.
[20] N. Özdemir, D. Karadeniz, Phys. Lett. A 372 (2008) 5968.