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Hypersurfaces satisfying some curvature conditions in the semi-Euclidean space

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Abstract

We consider some conditions on conharmonic curvature tensor K, which has many applications in physics and mathematics, on a hypersurface in the semi-Euclidean space $\mathbb{E}_s^{n+1}$. We prove that every conharmonicaly Ricci-symmetric hypersurface M satisfying the condition $K \cdot R = 0$ is pseudosymmetric. We also consider the condition $K \cdot K = L_K Q(g, K)$ on hypersurfaces of the semi-Euclidean space $\mathbb{E}_s^{n+1}$.

1. Introduction

Let (M, g) be an n-dimensional, $n \geqslant 3$, differentiable manifold of class C^∞. The conharmonic curvature tensor K was defined by Ishii in [14]. K is invariant under the action of the conformal transformations of (M, g) which preserve, in a certain sense, real harmonic functions on (M, g), and which therefore are called *conharmonic transformations*. It satisfies all the symmetry properties of the Riemannian curvature tensor R. There are many physical applications of the tensor K. For example, in [2], Abdussattar showed that the sufficient condition for a space–time to be conharmonic to a flat space–time is that the tensor K vanishes identically. A conharmonically flat space–time is either empty in which case it is flat or is filled with a distribution represented by the energy momentum tensor T possessing the algebraic structure of an electromagnetic field and is conformal to a flat space–time [2]. Also he described the gravitational field due to a distribution of pure radiation in presence of disordered radiation by means of a spherically symmetric conharmonically flat space–time.

In the present study, our aim is to study hypersurfaces, of dimension $n \geqslant 4$, in $(n+1)$-dimensional semi-Euclidean space $\mathbb{E}_s^{n+1}$ whose shape operator $\mathscr{A}$ satisfies the condition

$$\mathscr{A}^3 = \mathrm{tr}(\mathscr{A})\mathscr{A}^2 + \beta\mathscr{A} + \gamma I_d \tag{1}$$

at every point $x \in M$ for some β and $\gamma \in \mathbb{R}$. We show that if a conharmonically Ricci-symmetric hypersurface M satisfies the condition $K \cdot R = 0$, where R denotes the curvature tensor of M, then M is pseudosymmetric. We also consider the condition $K \cdot K = L_K Q(g, K)$ and we obtain that if a hypersurface M, whose shape operator in $(n+1)$-dimensional semi-Euclidean space $\mathbb{E}_s^{n+1}$ is of the form $\mathscr{A}^3 = \mathrm{tr}(\mathscr{A})\mathscr{A}^2 + \beta\mathscr{A}$, satisfies the condition $K \cdot K = L_K Q(g, K)$ then M is pseudosymmetric. It can be easily seen that every Einstein pseudosymmetric manifold (M, g) satisfies the conditions $K \cdot R = 0$ and $K \cdot K = L_K Q(g, K)$.

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It is known that semi-symmetric manifolds are trivially pseudosymmetric. Semisymmetric space–times were classified by Petrov in [17]. The classification of pseudosymmetric space–times were given in [13], which are physically most relevant cases of vacuum, Einstein, perfect fluid, and electromagnetic (non)-null Maxwell fields: every vacuum Petrov type D space–time with real Ψ_2, (where $\Psi_2 = C_{1324}$ and C is the Weyl conformal curvature tensor) with respect to the principal null tetrad is pseudosymmetric (e.g. the Schwarzschild and Kantowski–Sachs metrics). Every Petrov type D non-null Maxwell field is pseudosymmetric and also every Petrov type D Einstein space is pseudosymmetric. A Petrov type N Einstein space–time is pseudosymmetric. Einstein and perfect fluid (e.g., Robertson–Walker) conformally flat space–times are pseudosymmetric. The metrics in the Kinnersley classes I (see [15]) (i.e. the NUT space–times), $II.F$, $III.A$, and $IV.A$ are all pseudosymmetric (see [13]). Hence we conclude that all semi-symmetric Petrov type space–times or all pseudosymmetric, Einstein Petrov space–times satisfy the conditions $K \cdot R = 0$ and $K \cdot K = L_K Q(g, K)$.

It is also known that a conformally flat quasi-Einstein manifold is pseudosymmetric and every three-dimensional pseudosymmetric manifold is a quasi-Einstein manifold and conversely [9]. The Robertson Walker space–times are quasi-Einstein manifolds. Quasi-Einstein manifolds arose during the study of exact solutions of the Einstein field equations.

There are many studies about Einstein field equations. For example, in [12], using some almost forgotten concepts developed by A. Einstein in his quest for a general field theory (see [10]), El Naschie derived the particles content of the standard model of high energy elementary particles. In [11], possible connections between Gö del's classical solution of Einstein's field equations and E-infinity were discussed.

The paper is organized as follows: In Section 2, we give a brief account of conharmonic curvature tensor, Weyl tensor, pseudosymmetric manifolds and Kulkarni–Nomizu product. In Section 3, we give some informations about hypersurfaces of semi-Euclidean space $\mathbb{E}_s^{n+1}$ and the main results of the study are presented.

2. Preliminaries

We denote by ∇, R, C, K, S and κ the Levi–Civita connection, the Riemannian–Christoffel curvature tensor, the Weyl conformal curvature tensor, the conharmonic curvature tensor, the Ricci tensor and the scalar curvature of (M, g), respectively. The Ricci operator $\mathscr{S}$ is defined by $g(\mathscr{S}X, Y) = S(X, Y)$, where $X, Y \in \chi(M)$, $\chi(M)$ being the Lie algebra of vector fields on M. Furthermore, the tensor S^2 is defined by

$$S^2(X, Y) = S(\mathscr{S}X, Y). \tag{2}$$

Next, we define the endomorphisms $\mathscr{R}(X, Y)$, $\mathscr{C}(X, Y)$ and $\mathscr{K}(X, Y)Z$ of $\chi(M)$ by

$$\mathscr{R}(X, Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z,$$
$$\mathscr{C}(X, Y)Z = \mathscr{R}(X, Y)Z - \frac{1}{n-2}\Big(X \wedge \mathscr{S}Y + \mathscr{S}X \wedge Y - \frac{\kappa}{n-1} X \wedge Y\Big)Z,$$

and

$$\mathscr{K}(X, Y)Z = \mathscr{R}(X, Y)Z - \frac{1}{n-2}(X \wedge \mathscr{S}Y + \mathscr{S}X \wedge Y)Z \tag{3}$$

respectively, where $(X \wedge Y)Z$ is the tensor, defined by

$$(X \wedge Y)Z = g(Y, Z)X - g(X, Z)Y,$$

and $Z \in \chi(M)$.

The Riemannian–Christoffel curvature tensor R, the Weyl conformal curvature tensor C and the conharmonic curvature tensor K are defined by

$$R(X, Y, Z, W) = g(\mathscr{R}(X, Y)Z, W),$$
$$C(X, Y, Z, W) = g(\mathscr{C}(X, Y)Z, W),$$
$$K(X, Y, Z, W) = g(\mathscr{K}(X, Y)Z, W),$$

respectively, where $W \in \chi(M)$. The $(0,4)$-tensor G is defined by $G(X, Y, Z, W) = g((X \wedge Y)Z, W)$.

For a $(0,k)$-tensor field T, $k \geqslant 1$, and $(0,2)$-tensor field A on (M, g) we define the tensors $R \cdot T$, $K \cdot T$, $C \cdot T$ and $Q(A, T)$ by

$$(R(X, Y) \cdot T)(X_1, \ldots, X_k) = -T(\mathscr{R}(X, Y)X_1, X_2, \ldots, X_k) - \cdots - T(X_1, \ldots, X_{k-1}, \mathscr{R}(X, Y)X_k), \tag{4}$$

$$(K(X,Y)\cdot T)(X_1,\ldots,X_k) = -T(\mathscr{K}(X,Y)X_1,X_2,\ldots,X_k) - \cdots - T(X_1,\ldots,X_{k-1},\mathscr{K}(X,Y)X_k), \tag{5}$$

$$(C(X,Y)\cdot T)(X_1,\ldots,X_k) = -T(\mathscr{C}(X,Y)X_1,X_2,\ldots,X_k) - \cdots - T(X_1,\ldots,X_{k-1},\mathscr{C}(X,Y)X_k), \tag{6}$$

$$Q(A,T)(X_1,\ldots,X_k;X,Y) = -T((X\wedge_A Y)X_1,X_2,\ldots,X_k) - \cdots - T(X_1,\ldots,X_{k-1},(X\wedge_A Y)X_k), \tag{7}$$

respectively, where the tensor $X\wedge_A Y$ is defined by

$$(X\wedge_A Y)Z = A(Y,Z)X - A(X,Z)Y.$$

If $A = g$ then we simply denote it by $X\wedge Y$.

If the tensors $R\cdot R$ and $Q(g,R)$ are linearly dependent then M is called *pseudosymmetric*. This is equivalent to

$$R\cdot R = L_R Q(g,R) \tag{8}$$

holding on the set $U_R = \{x\in M^n | Q(g,R)\neq 0 \text{ at } x\}$, where L_R is some function on U_R (see [5, Section 3.1]). If $R\cdot R = 0$ then M is called *semi-symmetric* (see [18]).

If the tensors $C\cdot C$ and $Q(g,C)$ are linearly dependent then M is said to have pseudosymmetric Weyl tensor. This is equivalent to

$$C\cdot C = L_C Q(g,C)$$

holding on the set $U_C = \{x\in M | C\neq 0 \text{ at } x\}$, where L_C is some function on U_C (see [8]).

The Kulkarni–Nomizu product $A\barwedge B$ is given by

$$\begin{aligned}(A\barwedge B)(X_1,X_2,X_3,X_4) = {} & A(X_1,X_4)B(X_2,X_3) + A(X_2,X_3)B(X_1,X_4) - A(X_1,X_3)B(X_2,X_4)\\ & - A(X_2,X_4)B(X_1,X_3).\end{aligned} \tag{9}$$

We note that if $A = B$ then we have $\overline{A} = \frac{1}{2}A\barwedge A$, where the $(0,4)$-tensor $\overline{A}$ is defined by

$$\overline{A}(X_1,X_2,X_3,X_4) = A(X_1,X_4)A(X_2,X_3) - A(X_1,X_3)A(X_2,X_4). \tag{10}$$

Further, for a symmetric $(0,2)$-tensor A and a $(0,k)$-tensor T, $k\geqslant 2$, we define their Kulkarni–Nomizu product $A\barwedge T$ by

$$\begin{aligned}(A\barwedge T)(X_1,X_2,X_3,X_4;Y_3,\ldots,Y_k) = {} & A(X_1,X_4)T(X_2,X_3,Y_3,\ldots,Y_k) + A(X_2,X_3)T(X_1,X_4;Y_3,\ldots,Y_k)\\ & - A(X_1,X_3)T(X_2,X_4,Y_3,\ldots,Y_k) - A(X_2,X_4)T(X_1,X_3;Y_3,\ldots,Y_k)\end{aligned} \tag{11}$$

(see [6]). For symmetric $(0,2)$-tensor fields A and B we have the following identity ([6]):

$$A\barwedge Q(B,A) = Q(B,\overline{A}). \tag{12}$$

Note that

$$\overline{g} = G. \tag{13}$$

3. Hypersurfaces

Let $M, n = \dim M \geqslant 3$, be a connected hypersurface immersed isometrically in a semi-Riemannian manifold $(N,\widetilde{g})$. We denote by g the metric tensor of M induced from the metric tensor $\widetilde{g}$. Further, we denote by $\widetilde{\nabla}$ and ∇ the Levi–Civita connections corresponding to the metric tensors $\widetilde{g}$ and g, respectively. Let ξ be a local unit normal vector field on M in N and let $\varepsilon = \widetilde{g}(\xi,\xi) = \pm 1$. We can present the Gauss formula and the Weingarten formula of M in N in the following form:

$$\widetilde{\nabla}_X Y = \nabla_X Y + \varepsilon H(X,Y)\xi, \qquad \widetilde{\nabla}_X \xi = -\mathscr{A}(X)$$

respectively, where X, Y are vector fields tangent to M, H is the second fundamental tensor and $\mathscr{A}$ is the shape operator of M in N and $g(\mathscr{A}(X),Y) = H(X,Y)$. Furthermore, for $k>1$ we also have that $H^k(X,Y) = g(\mathscr{A}^k(X),Y)$, $\mathrm{tr}(H^k) = \mathrm{tr}(\mathscr{A}^k)$, $k\geqslant 1$, $H^1 = H$ and $\mathscr{A}^1 = \mathscr{A}$. We denote by R and $\widetilde{R}$ the Riemann–Christoffel curvature tensors of M and N, respectively.

The Gauss equation of M in N has the following form:

$$R(X_1,X_2,X_3,X_4) = \widetilde{R}(X_1,X_2,X_3,X_4) + \varepsilon\overline{H}(X_1,X_2,X_3,X_4). \tag{14}$$

From now on we will assume that M is a hypersurface in a semi-Euclidean space $\mathbb{E}^{n+1}_s$. So Eq. (14) turns into

$$R(X_1,X_2,X_3,X_4) = \varepsilon\overline{H}(X_1,X_2,X_3,X_4), \tag{15}$$

where X_1, X_2, X_3, X_4 are vector fields tangent to M and $\overline{H} = \frac{1}{2} H \wedge H$. From (15), by contraction we get easily

$$S(X_1, X_4) = \varepsilon(\mathrm{tr}(H)H(X_1, X_4) - H^2(X_1, X_4)). \tag{16}$$

Moreover, contracting (16) we obtain

$$\kappa = \varepsilon(\mathrm{tr}(H)^2 - \mathrm{tr}(H^2)). \tag{17}$$

Now we give the following lemmas which will be used in the main results.

Lemma 3.1 [7]. *Let A and D be two symmetric $(0,2)$-tensors at point x of a semi-Riemannian manifold (M, g). If the condition*

$$\alpha Q(g, A) + \gamma Q(A, D) + \beta Q(g, D) = 0; \quad \alpha, \beta, \gamma \in \mathbb{R}, \ \ \gamma \neq 0$$

is satisfied at x, then the tensors $A - \frac{1}{n}\mathrm{tr}(A)g$ and $D - \frac{1}{n}\mathrm{tr}(D)g$ are linearly dependent.

Lemma 3.2 [7]. *Any hypersurface M immersed isometrically in an $(n+1)$-dimensional semi-Euclidean space $\mathbb{E}_s^{n+1}$, $n \geqslant 4$, satisfies the condition*

$$R \cdot R = Q(S, R). \tag{18}$$

Theorem 3.3. *Let M be a hypersurface in a semi-Euclidean space $\mathbb{E}_s^{n+1}$, $n \geqslant 4$. If the shape operator $\mathscr{A}$ of M satisfies (1) and the condition $K \cdot R = 0$ holds on M then M is pseudosymmetric.*

Proof. Using the definition of the second fundamental tensor, Eq. (1) can be written as

$$H^3 = \mathrm{tr}(H)H^2 + \beta H + \gamma g. \tag{19}$$

Let $X_h, X_i, X_j, X_k, X_l, X_m \in \chi(M)$. So using (5) we have

$$\begin{aligned}(K(X_h, X_i) \cdot R)(X_j, X_k, X_l, X_m) = &-R(\mathscr{K}(X_h, X_i)X_j, X_k, X_l, X_m) - R(X_j, \mathscr{K}(X_h, X_i)X_k, X_l, X_m) \\ &- R(X_j, X_k, \mathscr{K}(X_h, X_i)X_l, X_m)) - R(X_j, X_k, X_l, \mathscr{K}(X_h, X_i)X_m).\end{aligned} \tag{20}$$

Then using (3) and (15) we get

$$(K(X_h, X_i) \cdot R)(X_j, X_k, X_l, X_m) = \alpha + \alpha_2 + \alpha_3 + \alpha_4,$$

where

$$\begin{aligned}\alpha_1 = {} & H_{kl}(H_{ij}H^2_{hm} - H_{hj}H^2_{im} + H_{im}H^2_{jh} - H_{hm}H^2_{ij}) \\ & + H_{km}(-H_{ij}H^2_{hl} + H_{hj}H^2_{il} - H_{il}H^2_{jh} + H_{hl}H^2_{ij}) \\ & + H_{jm}(H_{ik}H^2_{hl} - H_{hk}H^2_{il} + H_{il}H^2_{hk} - H_{hl}H^2_{ik}) \\ & + H_{jl}(-H_{ik}H^2_{hm} + H_{hk}H^2_{im} - H_{im}H^2_{hk} + H_{hm}H^2_{ik}),\end{aligned} \tag{21}$$

$$\begin{aligned}\alpha_2 = {} & \frac{1}{n-2}[S_{ij}R_{hklm} - S_{hj}R_{iklm} + S_{ik}R_{jhlm} - S_{hk}R_{jilm} \\ & + S_{il}R_{jkhm} - S_{hl}R_{jkim} + S_{im}R_{jklh} - S_{hm}R_{jkli}],\end{aligned} \tag{22}$$

$$\begin{aligned}\alpha_3 = {} & \frac{1}{n-2}\mathrm{tr}(H)[H_{kl}(g_{ij}H^2_{hm} - g_{hj}H^2_{im} + g_{im}H^2_{jh} - g_{hm}H^2_{ij}) \\ & + H_{km}(-g_{ij}H^2_{hl} + g_{hj}H^2_{il} - g_{il}H^2_{jh} + g_{hl}H^2_{ij}) \\ & + H_{jm}(g_{ik}H^2_{hl} - g_{hk}H^2_{il} + g_{il}H^2_{hk} - g_{hl}H^2_{ik}) \\ & + H_{jl}(-g_{ik}H^2_{hm} + g_{hk}H^2_{im} - g_{im}H^2_{hk} + g_{hm}H^2_{ik})],\end{aligned} \tag{23}$$

$$\begin{aligned}\alpha_4 = {} & -\frac{1}{n-2}[H_{kl}(g_{ij}H^3_{hm} - g_{hj}H^3_{im} + g_{im}H^3_{jh} - g_{hm}H^3_{ij}) \\ & + H_{km}(-g_{ij}H^3_{hl} + g_{hj}H^3_{il} - g_{il}H^3_{jh} + g_{hl}H^3_{ij}) \\ & + H_{jm}(g_{ik}H^3_{hl} - g_{hk}H^3_{il} + g_{il}H^3_{hk} - g_{hl}H^3_{ik}) \\ & + H_{jl}(-g_{ik}H^3_{hm} + g_{hk}H^3_{im} - g_{im}H^3_{hk} + g_{hm}H^3_{ik}).\end{aligned} \tag{24}$$

Since M satisfies the condition (1), so combining (7), (11), (20)–(24) we have

$$K \cdot R = H \widetilde{\wedge} Q(H^2, H) - \frac{1}{n-2} Q(S, R) + \frac{1}{n-2} \lambda H \widetilde{\wedge} Q(g, H). \tag{25}$$

Thus by (12), Eq. (25) turns into

$$K \cdot R = Q(H^2, \overline{H}) - \frac{1}{n-2} Q(S, R) + \frac{1}{n-2} \lambda Q(g, \overline{H}). \tag{26}$$

From (16), since

$$H^2 = \mathrm{tr}(H)H - \varepsilon S,$$

using (15) and Lemma 3.2, Eq. (26) can be rewritten as

$$K \cdot R = -(\varepsilon + \frac{1}{n-2}) R \cdot R + \frac{\varepsilon}{n-2} Q(g, R). \tag{27}$$

Since the condition $K \cdot R = 0$ holds on M, by (27), the tensors $R \cdot R$ and $Q(g, R)$ are linearly dependent.

This completes the proof of the theorem. □

Definition 3.4 [16]. Let M be a hypersurface in a semi-Euclidean space $\mathbb{E}_s^{n+1}$, $n \geqslant 4$. If $K \cdot S = 0$ then M is called conharmonically Ricci-symmetric.

Using the above definition we have the following theorem:

Lemma 3.5. *Let M be a hypersurface in a semi-Euclidean space $\mathbb{E}_s^{n+1}$, $n \geqslant 4$. If M is conharmonically Ricci-symmetric then there is a real valued function λ on M such that*

$$H^3 = \mathrm{tr}(H)H^2 + \lambda H + \frac{1}{n}[-\lambda \mathrm{tr}(H) - \mathrm{tr}(H)\mathrm{tr}(H^2) + \mathrm{tr}(H^3)]g. \tag{28}$$

Proof. Let $X_h, X_i, X_j, X_k \in \chi(M)$. So using (5) we have

$$(K \cdot H)(X_h, X_i; X_j, X_k) = -H(\mathscr{K}(X_j, X_k)X_h, X_i) - H(X_h, \mathscr{K}(X_j, X_k)X_i) \tag{29}$$

and similarly

$$(K \cdot H^2)(X_h, X_i; X_j, X_k) = -H^2(\mathscr{K}(X_j, X_k)X_h, X_i) - H^2(X_h, \mathscr{K}(X_j, X_k)X_i). \tag{30}$$

Then by making use of (3), (7) and (15) we get

$$K \cdot H = \frac{\varepsilon}{n-2}[(n-3)Q(H, H^2) - \mathrm{tr}(H)Q(g, H^2) + Q(g, H^3)] \tag{31}$$

and

$$K \cdot H^2 = \varepsilon Q(H, H^3) + \frac{1}{n-2}\varepsilon[-\mathrm{tr}(H)Q(H, H^2) - \mathrm{tr}(H)Q(g, H^3) + Q(g, H^4)]. \tag{32}$$

Since M is conharmonically Ricci-symmetric by the use of (16) we have

$$K \cdot S = \varepsilon(\mathrm{tr}(H)K \cdot H - K \cdot H^2) = 0. \tag{33}$$

Thus substituting (31) and (32) into (33) we obtain

$$-Q(H, H^3) + \mathrm{tr}(H)Q(H, H^2) + \frac{1}{n-2}[-\mathrm{tr}(H)^2 Q(g, H^2) + 2\mathrm{tr}(H)Q(g, H^3) - Q(g, H^4)] = 0. \tag{34}$$

Hence from (34), by a contraction we have

$$\begin{aligned} -\frac{1}{n-2}H^4 = {} & \frac{1}{n(n-2)}[-(n+2)\mathrm{tr}(H)H^3 + 2\mathrm{tr}(H)^2]H^2 + \frac{1}{n}[-\mathrm{tr}(H^3) + \mathrm{tr}(H)\mathrm{tr}(H^2)]H + [-\mathrm{tr}(H)^2\mathrm{tr}(H^2) \\ & + 2\mathrm{tr}(H)\mathrm{tr}(H^3) - \mathrm{tr}(H^4)]g. \end{aligned} \tag{35}$$

So substituting (35) into (34) we get

$$-\frac{1}{n}\mathrm{tr}(H)Q(g, \mathrm{tr}(H)H^2 - H^3) + Q(H, \mathrm{tr}(H)H^2 - H^3) + \frac{1}{n}[-\mathrm{tr}(H^3) + \mathrm{tr}(H)\mathrm{tr}(H^2)]Q(g, H) = 0.$$

Then by Lemma 3.1, the tensors

$$\mathrm{tr}(H)H^2 - H^3 - \frac{\mathrm{tr}(H)\mathrm{tr}(H^2) - \mathrm{tr}(H^3)}{n}g$$

and

$$H - \frac{1}{n}\mathrm{tr}(H)g$$

are linearly dependent, which proves the lemma. □

Hence by combining Theorem 3.3 and Lemma 3.5 we have the following theorem:

Theorem 3.6. *Let M be a hypersurface in a semi-Euclidean space $\mathbb{E}_s^{n+1}$, $n \geqslant 4$. If M is conharmonically Ricci-symmetric and the condition $K \cdot R = 0$ holds on M then M is pseudosymmetric.*

Example 3.7. Let $\mathbb{S}^2 = \{p \in \mathbb{R}^3 \text{ such that } |p| = 1\}$ be the standard unit sphere. First consider

$$M^4 = \mathbb{S}_1^2 \times \mathbb{S}_2^2 = \{(p,q) \in \mathbb{R}^6 = \mathbb{R}^3 \times \mathbb{R}^3 \text{ such that } |p| = |q| = 1\}.$$

Next we take the cone

$$C^5 = \{(tp, tq) \in \mathbb{R}^6 \text{ such that } |p| = |q| = 1, \quad t \in \mathbb{R}\}.$$

In [1], the authors show that the principal curvatures of C^5 are $\left(0, \frac{1}{\sqrt{2}t}, \frac{1}{\sqrt{2}t}, -\frac{1}{\sqrt{2}t}, -\frac{1}{\sqrt{2}t}\right)$ and the cone C^5 is Ricci-semi-symmetric, but not semi-symmetric.

It can be easily seen that the cone C^5 satisfies the conditions $K \cdot S = 0$, $K \cdot R = 0$ and it is pseudosymmetric.

Now we consider hypersurfaces, of dimension $\geqslant 4$, in $(n+1)$-dimensional semi-Euclidean space $\mathbb{E}_s^{n+1}$ whose shape operator $\mathscr{A}$ satisfies the condition

$$\mathscr{A}^3 = \mathrm{tr}(\mathscr{A})\mathscr{A}^2 + \beta\mathscr{A}. \tag{36}$$

at every point $x \in M$.

Proposition 3.8. *Let M be a hypersurface in a semi-Euclidean space $\mathbb{E}_s^{n+1}$, $n \geqslant 3$, satisfying the condition (36). Then the Ricci tensor S of M has the following property:*

$$S^2 = -\varepsilon\beta S. \tag{37}$$

Proof. Using the definition of the second fundamental tensor, Eq. (36) can be written as

$$H^3 = \mathrm{tr}(H)H^2 + \beta H. \tag{38}$$

So by the use of (2) we get

$$S^2 = H^4 - 2\mathrm{tr}(H)H^3 + \mathrm{tr}(H)^2H^2. \tag{39}$$

Hence applying (38) and (16) into (39), we obtain (37). □

Using (3) and (7) we easily obtain the following proposition.

Proposition 3.9. *Let M be a hypersurface in a semi-Euclidean space $\mathbb{E}_s^{n+1}$, $n \geqslant 4$. Then we have the following identity:*

$$Q(g,K) = Q(g,R) + \frac{1}{n-2}Q(S,G). \tag{40}$$

Lemma 3.10. *Let M be a hypersurface in a semi-Euclidean space $\mathbb{E}_s^{n+1}$, $n \geqslant 4$, satisfying the condition (36). Then the following relation is fulfilled on M*

$$K \cdot K = \frac{n-3}{n-2}R \cdot R + \frac{\beta\varepsilon}{n-2}Q(g,R) - \frac{\varepsilon\beta}{(n-2)^2}Q(S,G). \tag{41}$$

Proof. Let $X_h, X_i, X_j, X_k, X_l, X_m \in \chi(M)$. So using (5) we have

$$(K(X_h,X_i)\cdot K)(X_j,X_k,X_l,X_m) = -K(\mathscr{K}(X_h,X_i)X_j,X_k,X_l,X_m) - K(X_j,\mathscr{K}(X_h,X_i)X_k,X_l,X_m) - K(X_j,X_k,\mathscr{K}(X_h,X_i)X_l,X_m)) - K(X_j,X_k,X_l,\mathscr{K}(X_h,X_i)X_m). \quad (42)$$

Then by making use of (3), (11) and Proposition 3.8, similar to the proof of Theorem 3.3 we get

$$K\cdot K = R\cdot R - \frac{1}{n-2}Q(S,R) + \frac{\beta}{n-2}H\widetilde{\wedge}Q(g,H) - \frac{\varepsilon\beta}{(n-2)^2}g\widetilde{\wedge}Q(S,g).$$

But from (18) and (12) we have

$$K\cdot K = R\cdot R - \frac{1}{n-2}R\cdot R + \frac{\beta}{n-2}Q(g,\overline{H}) - \frac{\varepsilon\beta}{(n-2)^2}Q(S,\bar{g}).$$

So using (13) and (15) we obtain (41). □

Now we investigate hypersurfaces in a semi-Euclidean space $\mathbb{E}_s^{n+1}$, $n \geqslant 4$, satisfying the condition $K\cdot K = L_K Q(g,K)$, where L_K is some function on $U_K = \{x \in M |\ K \neq 0 \text{ at } x\}$.

Hence by the use of Lemma 3.10 we have the following characterization:

Theorem 3.11. *Let M be a hypersurface in a semi-Euclidean space $\mathbb{E}_s^{n+1}$, $n \geqslant 4$, satisfying the condition (36). If the condition $K\cdot K = L_K Q(g,K)$ holds on M then M is pseudosymmetric.*

Proof. By the use of (40), Eq. (41) can be written as

$$K\cdot K + \frac{\beta\varepsilon}{n-2}Q(g,K) = \frac{n-3}{n-2}R\cdot R + \frac{2\beta\varepsilon}{n-2}Q(g,R).$$

But since the condition $K\cdot K = L_K Q(g,K)$ holds on M the last equation implies $\frac{n-3}{n-2}R\cdot R + \frac{2\beta\varepsilon}{n-2}Q(g,R) = 0$, which gives us M is pseudosymmetric. □

4. Conclusions

As a generalization of semi-symmetric spaces, pseudosymmetric spaces have been studied by many mathematicians. It has also some physical applications. In this study, we give new conditions of pseudosymmetry type spaces which have some examples as some of Petrov space–times and quasi-Einstein manifolds.

References

[1] Abdalla BE, Dillen F. A Ricci-semi-symmetric hypersurface of Euclidean space which is not semi-symmetric. Proc Amer Math Soc 2002;130(6):1805.
[2] Abdussattar DB. On conharmonic transformations in general relativity. Bull Calcutta Math Soc 1996;6:465–70.
[5] Deszcz R. On pseudosymmetric spaces. Bull Soc Belg Math Ser A 1992;44:1–34.
[6] Deszcz R, Glogowska M. Some nonsemisymmetric Ricci-semisymmetric warped product hypersurfaces. Publ Inst Math Beograd (N.S.) 2002;72(86):81–93.
[7] Deszcz R, Verstraelen L. Hypersurfaces of semi-Riemannian conformally flat manifolds. In: Geometry and topology of submanifolds, III. Leeds: World Scientific Publishing; 1990. p. 131–47.
[8] Deszcz R, Verstraelen L, Yaprak Ş. Hypersurfaces with pseudosymmetric Weyl tensor in conformally flat manifolds. In: Geometry and topology of submanifolds, IX. Valenciennes/Lyon/Leuven: World Scientific Publishing; 1997. p. 131–77.
[9] Deszcz R, Verstraelen L, Yaprak Ş. Warped products realizing a certain condition of pseudosymmetry type imposed on the Weyl curvature tensor. Chinese J Math 1994;22:39–157.
[10] Einstein A. Grundzuge der Relativitats theory. Berlin: Springer; 2002.
[11] El Naschie MS. Gödel universe, dualities and high energy particles in *E*-infinity. Chaos, Solitons & Fractals 2005;25(3):759–64.
[12] El Naschie MS. Is Einstein's general field equation more fundamental than quantum field theory and particle physics? Chaos, Solitons & Fractals 2006;30(3):525–31.
[13] Haesen S, Verstraelen L. Classification of the pseudosymmetric space–times. J Math Phys 2004;45(6):2343–6.
[14] Ishii Y. On conharmonic transformations. Tensor (N.S.) 1957;7:73–80.
[15] Kinnersley W. Type D vacuum metrics. J Math Phys 1969;10:1195–203.

[16] Özgür C, Arslan K, Murathan C. Conharmonically semi-parallel hypersurfaces, Proceedings of the Centennial "G. Vranceanu", Part II (Bucharest, 2000). An Univ Bucureşti Mat Inform 2001;50(2):121–32.
[17] Petrov AZ. Einstein spaces, Translated from the Russian by R.F. Kelleher. In: Woodrow J. (Translated editor). Pergamon Press: Oxford–Edinburgh–New York; 1969.
[18] Szabo ZI. Structure theorems on Riemannian spaces satisfying $R(X, Y) \cdot R = 0$. I. The local version. J Diff Geom 1982;17:531–82.