Glasgow Math. J. **51** (2009) 331–339. © 2009 Glasgow Mathematical Journal Trust.
doi:10.1017/S001708950900500X. Printed in the United Kingdom

PINCHING THEOREMS FOR TOTALLY REAL MINIMAL SUBMANIFOLDS IN CP^n

CENGİZHAN MURATHAN
Department of Mathematics, Uludağ University, 16059, Bursa, Turkey
e-mail: cengiz@uludag.edu.tr

and CİHAN ÖZGÜR
Department of Mathematics, Balıkesir University, 10145, Balıkesir, Turkey
e-mail: cozgur@balikesir.edu.tr

(Received 14 August 2007; revised 9 October 2008; accepted 15 October 2008)

Abstract. Let M be an n-dimensional totally real minimal submanifold in CP^n. We prove that if M is semi-parallel and the scalar curvature τ, $\frac{-(n-1)(n-2)(n+1)}{2} \leq \tau \leq 0$, then M is an open part of the Clifford torus $T^n \subset CP^n$. If M is semi-parallel and the scalar curvature $\tau, n(n-1) \leq \tau \leq \frac{n^3-3n+2}{2}$, then M is an open part of the real projective space RP^n.

2000 *Mathematics Subject Classification.* 53C42, 53C40, 53C20.

1. Introduction. Among all submanifolds of an almost Hermitian manifold, there are two typical classes: one is the class of holomorphic submanifolds and, the other is the class of totally real submanifolds. A submanifold M of an almost Hermitian manifold $\tilde{M}$ is called holomorphic (resp. totally real) if each tangent space of M is mapped into itself (resp. the normal space) by the almost complex structure of $\tilde{M}$.

Given an isometric immersion $f : M \longrightarrow \tilde{M}$, let h be the second fundamental form and $\overline{\nabla}$ the van der Waerden–Bortolotti connection of M. If $\overline{\nabla} h = 0$, then M is said to have *parallel second fundamental form*. The class of isometric immersions in a Riemannian manifold with parallel second fundamental form is very wide, as it is shown, for instance, in the classical paper of D. Ferus [**8**]. Certain generalisations of these immersions have been studied, obtaining classification theorems in some cases.

H. Naitoh [**11**] and M. Takeuchi [**13**] classified submanifolds in a real and complex space form with parallel second fundamental form. Among such examples, there exist three n-dimensional conformally flat totally real minimal submanifolds in a complex projective space CP^n of constant holomorphic sectional curvature 4:

(i) a totally geodesic submanifold;
(ii) a flat torus;

(iii) a Riemannian product

$$S_{1,n-1} : S^1(\sin a \cos a) \times S^{n-1}(\sin a),$$

where $S^n(r)$ is an n-dimensional sphere with radius r and $\tan a = \sqrt{n}$.

The purpose of this paper is to give the characterisation of (i) and (ii) of n dimension.

On the other hand, in [**7**], N. Ejiri studied four-dimensional compact orientable conformally flat totally real minimal submanifold in CP^4. Precisely, he proved the following theorem:

THEOREM A. *If M is four-dimensional compact, orientable and conformally flat and has non-negative Euler number and the scalar curvature τ, $0 \leq \tau \leq \frac{15}{2}$, then M is flat or locally isometric to $S_{1,3}$.*

In [**12**], D. Perrone considered six-dimensional case. Under the same conditions in Ejiri's result, he obtained that if the scalar curvature τ, $0 \leq \tau \leq \frac{70}{3}$, then M is locally isometric to $S_{1,5}$.

Recently, A. M. Li and G. Zhao [**9**] proved the following theorems:

THEOREM B. *Let M be an n-dimensional totally real minimal submanifold with constant sectional curvature c in CP^n. Then M is either totally geodesic or flat.*

THEOREM C. *Let M be an n-dimensional totally real minimal embedding submanifold in CP^n with constant sectional curvature. Then M is either an open part of the real projective space $RP^n \subset CP^n$ or an open part of the Clifford torus $T^n \subset CP^n$.*

THEOREM D. *Let M be an n-dimensional totally real minimal submanifold with parallel second fundamental form in CP^n. If $\tau \leq 0$ (namely $\|h\|^2 \geq n(n-1)$), then M is an open part of the Clifford torus $T^n \subset CP^n$.*

Furthermore, in [**4**], J. Deprez defined the immersion to be *semi-parallel* if

$$\bar{R}(X, Y) \cdot h = (\overline{\nabla}_X \overline{\nabla}_Y - \overline{\nabla}_Y \overline{\nabla}_X - \overline{\nabla}_{[X,Y]})h = 0 \tag{1}$$

holds for any vectors X, Y tangent to M. The semi-parallelity condition is a local holonomy condition on the second fundamental form with respect to the connection $\overline{\nabla}$, which is the induced connection on the tensor product of the Levi-Civita connection on the tangent bundle and the normal connection in the normal bundle of the submanifold M. It is well known that if second fundamental form of M is parallel, then it is semi-parallel. But the converse is not necessary to be parallel. J. Deprez studied semi-parallel immersions in real space forms [**4**, **5**] . In [**10**], Ü. Lumiste showed that a semi-parallel submanifold is the second-order envelope of the family of submanifolds with parallel second fundamental form. Later, studying hypersurfaces in the sphere and the hyperbolic space, F. Dillen showed that they are flat surfaces, hypersurfaces with parallel Weingarten endomorphism or rotation hypersurfaces of certain helices [**6**].

In the present study, taking semi-parallelity condition instead of the parallelity of the second fundamental form of M in Li and Zhao's [**9**] result we obtain the following results:

THEOREM 1. *Let M be an n-dimensional totally real minimal submanifold in CP^n. If M is semi-parallel and the scalar curvature τ, $n(n-1) \leq \tau \leq \frac{n^3-3n+2}{2}$, then M is an open part of real projective space RP^n.*

THEOREM 2. *Let M be an n-dimensional totally real minimal submanifold in CP^n. If M is semi-parallel and the scalar curvature τ, $\frac{-(n-1)(n-2)(n+1)}{2} \leq \tau \leq 0$, then M is an open part of the Clifford torus $T^n \subset CP^n$.*

2. Preliminaries. Let M be an n-dimensional totally real submanifold of complex projective space CP^n; that is M is immersed in CP^n and $J(T_xM)$ is orthogonal to T_xM for all $x \in M$, where J denotes the almost complex structure of CP^n (see [**14**] and [**15**]). We denote by $\widetilde{g}$ and g the Riemannian metric of CP^n and M, respectively. The Gauss and Weingarten formulas are given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X, Y)$$

and

$$\widetilde{\nabla}_X \xi = -A_\xi X + D_X \xi,$$

respectively, where ξ is a normal vector field and X, Y are tangent vector fields on M; h is called the *second fundamental form* of M. If $h = 0$, then M is said to be *totally geodesic*. The *mean curvature vector* H of M is defined to be

$$H = \frac{1}{n} tr(h).$$

A submanifold M is said to be *minimal* if $H = 0$ identically.

The *covariant derivative* $\overline{\nabla}h$ of h is defined by

$$(\overline{\nabla}_X h)(Y, Z) = \nabla^{\perp}_X (h(Y, Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z), \tag{2}$$

where, $\overline{\nabla}h$ is a normal bundle valued tensor of type (0, 3) and is called *the third fundamental form* of M. Here, $\overline{\nabla}$ is called the *van der Waerden–Bortolotti connection* of M. If $\overline{\nabla}h = 0$, then f is called *parallel* [**8**]. The *second covariant derivative* $\overline{\nabla}^2 h$ of h is defined by

$$\begin{aligned}(\overline{\nabla}^2 h)(Z, W, X, Y) &= (\overline{\nabla}_X \overline{\nabla}_Y h)(Z, W)\\ &= \nabla^{\perp}_X((\overline{\nabla}_Y h)(Z, W)) - (\overline{\nabla}_Y h)(\nabla_X Z, W)\\ &\quad -(\overline{\nabla}_X h)(Z, \nabla_Y W) - (\overline{\nabla}_{\nabla_X Y} h)(Z, W).\end{aligned} \tag{3}$$

Then we have

$$\begin{aligned}&(\overline{\nabla}_X \overline{\nabla}_Y h)(Z, W) - (\overline{\nabla}_Y \overline{\nabla}_X h)(Z, W) = (\overline{R}(X, Y) \cdot h)(Z, W)\\ &= R^{\perp}(X, Y)h(Z, W) - h(R(X, Y)Z, W) - h(Z, R(X, Y)W),\end{aligned} \tag{4}$$

where $\overline{R}$ is the curvature tensor belonging to the connection $\overline{\nabla}$. The basic equations of Gauss and Ricci are

$$\begin{aligned}R(X, Y, Z, W) = g(R(X, Y)Z, W) &= \tilde{g}(\tilde{R}(X, Y)Z, W)\\ +\widetilde{g}(h(Y, Z), h(X, W)) &- \widetilde{g}(h(X, Z), h(Y, W)),\end{aligned} \tag{5}$$

$$\widetilde{g}(R^{\perp}(X, Y)\xi, \eta) = g([A_{\xi}, A_{\eta}]X, Y); \quad \xi, \eta \in N(M), \tag{6}$$

respectively, and $N(M)$ denotes the normal bundle of M. Here $\tilde{R}$ and $R^{\perp}$ denote the curvature operator of CP^n and the normal connection defined by

$$\tilde{g}(\tilde{R}(X, Y)Z, W) = g(Y, Z)g(X, W) - g(X, Z)g(Y, W)$$

and

$$R^{\perp}(X, Y)Z = \nabla^{\perp}_X \nabla^{\perp}_Y Z - \nabla^{\perp}_Y \nabla^{\perp}_X Z - \nabla^{\perp}_{[X, Y]} Z,$$

respectively. The *Weyl conformal curvature tensor* of an n-dimensional Riemannian manifold (M, g) is defined by

$$\begin{aligned}C(X, Y, Z, W) = R(X, Y, Z, W) &- \tfrac{1}{n-2}\{S(Y, Z)g(X, W)\\ -S(X, Z)g(Y, W) + S(X, W)g(Y, Z) &- S(Y, W)g(X, Z)\}\\ +\tfrac{\tau}{(n-1)(n-2)}\{g(Y, Z)g(X, W) &- g(X, Z)g(Y, W)\}.\end{aligned} \tag{7}$$

For $n \geq 4$, if $C = 0$, then M is called *conformally flat* [**15**].

We choose local field of orthonormal frames $\{e_1, e_2, \ldots, e_n, Je_1 = e_{1^*}, \ldots, Je_n = e_{n^*}\}$ in CP^n such that, restricted to M, the vectors $e_1, e_2, \ldots, e_n$ are tangent to M. Then for $1 \leq i, j \leq n$, the components of the second fundamental form h are given by

$$h(e_i, e_j) = \sum h^{k^*}_{ij} e_{k^*} \tag{8}$$

and satisfy

$$h^{k^*}_{ji} = h^{k^*}_{ij} = h^{i^*}_{kj}. \tag{9}$$

Similarly, the components of the first and the second covariant derivative of h are given by

$$h^{\alpha}_{ijk} = g((\overline{\nabla}_{e_k} h)(e_i, e_j), e_\alpha) = \overline{\nabla}_{e_k} h^{\alpha}_{ij} \tag{10}$$

and

$$\begin{aligned} h^{\alpha}_{ijkl} &= g((\overline{\nabla}_{e_l}\overline{\nabla}_{e_k} h)(e_i, e_j), e_\alpha) \\ &= \overline{\nabla}_{e_l} h^{\alpha}_{ijk} \\ &= \overline{\nabla}_{e_l}\overline{\nabla}_{e_k} h^{\alpha}_{ij}, \end{aligned} \tag{11}$$

respectively.

Moreover, the components R_{ijkh} of the curvature tensor R, the components $S_{ik} = \sum R_{ijkj}$ of the Ricci tensor S and the scalar curvature $\tau = \sum S_{ii}$ are given by

$$R_{ijkh} = (\delta_{ik}\delta_{jh} - \delta_{ih}\delta_{jk}) + \sum \left(h^{l^*}_{ik}h^{l^*}_{jh} - h^{l^*}_{ih}h^{l^*}_{jk}\right), \tag{12}$$

$$S_{ik} = (n-1)\delta_{ik} + \sum (trA_{l^*}) g(A_{l^*}e_i, e_k) - \sum g(A_{l^*}e_i, A_{l^*}e_k) \tag{13}$$

and

$$\tau = n(n-1) + \sum \left(trA_{l^*}\right)^2 - \|h\|^2, \tag{14}$$

respectively, where

$$\|h\|^2 = \sum tr\left(A^2_{l^*}\right) = \sum \left(h^{l^*}_{ik}\right)^2. \tag{15}$$

Proof of Theorem 1. It was proven in **[3]** that the second fundamental form of the immersion satisfies

$$\begin{aligned} \frac{1}{2}\Delta\|h\|^2 = \|\overline{\nabla} h\|^2 &+ \sum tr(A_{i^*}A_{j^*} - A_{j^*}A_{i^*})^2 \\ &- \sum tr(A_{i^*}A_{j^*})^2 + (n+1)\|h\|^2. \end{aligned} \tag{16}$$

Since

$$\sum tr(A_{i^*}A_{j^*} - A_{j^*}A_{i^*})^2 = -\sum \left(\sum \left(h^{i^*}_{km}h^{j^*}_{lm} - h^{j^*}_{km}h^{i^*}_{lm}\right)\right)^2,$$

by the use of Gauss equation we have

$$\sum tr(A_{i^*}A_{j^*} - A_{j^*}A_{i^*})^2 = \|R\|^2 + 4\tau - 2n(n-1) \tag{17}$$

and

$$\sum tr(A_{i^*}A_{j^*})^2 = \|S\|^2 - 2(n-1)\tau + n(n-1)^2, \tag{18}$$

(see [**12**]). In view of (17) and (18), equation (16) can be written as

$$\frac{1}{2}\Delta\|h\|^2 = \|\overline{\nabla}h\|^2 - \|R\|^2 - \|S\|^2 + (n+1)\tau. \tag{19}$$

Furthermore, it is known that (see [**2**])

$$\|R\|^2 \geq \frac{4}{n-2}\|S\|^2 - \frac{2\tau^2}{(n-1)(n-2)}, \tag{20}$$

equality holding if and only if M is conformally flat.

Since M is semi-parallel, then by definition the condition

$$\overline{R}(e_l, e_k)\cdot h = 0 \tag{21}$$

is fulfilled for $1 \leq k, l \leq n$.

By (4), we have

$$(\overline{R}(e_l, e_k)\cdot h)(e_i, e_j) = (\overline{\nabla}_{e_l}\overline{\nabla}_{e_k}h)(e_i, e_j) - (\overline{\nabla}_{e_k}\overline{\nabla}_{e_l}h)(e_i, e_j). \tag{22}$$

By the use of (10) and (11) the semi-parallelity condition (21) turns into

$$h^{\alpha}_{ijkl} = h^{\alpha}_{ijlk}, \tag{23}$$

where $g(e_i, e_j) = \delta_{ij}$ and $1 \leq i, j, k, l \leq n,\ \ n+1 \leq \alpha \leq 2n$.

Recall that the Laplacian Δh^{α}_{ij} of h^{α}_{ij} is defined by

$$\Delta h^{\alpha}_{ij} = \sum_{i,j,k=1}^{n} h^{\alpha}_{ijkk}. \tag{24}$$

Then we obtain

$$\frac{1}{2}\Delta(\|h\|^2) = \sum_{i,j,k=1}^{n}\sum_{\alpha=n+1}^{2n} h^{\alpha}_{ij}h^{\alpha}_{ijkk} + \|\overline{\nabla}h\|^2, \tag{25}$$

where

$$\|h\|^2 = \sum_{i,j,k=1}^{n}\sum_{\alpha=n+1}^{2n} (h^{\alpha}_{ij})^2 \tag{26}$$

and

$$\|\overline{\nabla}h\|^2 = \sum_{i,j,k=1}^{n}\sum_{\alpha=n+1}^{2n} (h^{\alpha}_{ijkk})^2 \tag{27}$$

are the squares of the lengths of the second and third fundamental forms of M, respectively. In addition, using (8) and (11), we obtain

$$\begin{aligned} h^{\alpha}_{ij}h^{\alpha}_{ijkk} &= g(h(e_i, e_j), e_\alpha)g((\overline{\nabla}_{e_k}\overline{\nabla}_{e_k}h)(e_i, e_j), e_\alpha) \\ &= g((\overline{\nabla}_{e_k}\overline{\nabla}_{e_k}h)(e_i, e_j)g(h(e_i, e_j), e_\alpha), e_\alpha) \\ &= g((\overline{\nabla}_{e_k}\overline{\nabla}_{e_k}h)(e_i, e_j), h(e_i, e_j)). \end{aligned} \tag{28}$$

Therefore due to (28), equation (25) becomes

$$\frac{1}{2}\Delta(\|h\|^2) = \sum_{i,j,k=1}^{n} g((\overline{\nabla}_{e_k}\overline{\nabla}_{e_k}h)(e_i, e_j), h(e_i, e_j)) + \|\overline{\nabla}h\|^2. \tag{29}$$

Furthermore by definition

$$\begin{aligned} \|h\|^2 &= \sum_{i,j=1}^{n} g(h(e_i, e_j), h(e_i, e_j)), \\ H^\alpha &= \sum_{k=1}^{n} h^{\alpha}_{kk}, \\ \|H\|^2 &= \frac{1}{n^2}\sum_{\alpha=n+1}^{2n}(H^\alpha)^2, \end{aligned}$$

and using equations (23)–(25), we get

$$\frac{1}{2}\Delta(\|h\|^2) = \sum_{i,j,k=1}^{n}\sum_{\alpha=n+1}^{2n} h^{\alpha}_{ij}(\overline{\nabla}_{e_i}\overline{\nabla}_{e_j}H^\alpha) + \|\overline{\nabla}h\|^2. \tag{30}$$

Using minimality condition, equation (30) reduces to

$$\frac{1}{2}\Delta(\|h\|^2) = \|\overline{\nabla}h\|^2. \tag{31}$$

So comparing equations (19) and (31) we obtain

$$\|R\|^2 + \|S\|^2 - (n+1)\tau = 0, \tag{32}$$

which gives us, from (32) and (20),

$$\left(\frac{n+2}{n-2}\right)\|S\|^2 - \frac{2\tau^2}{(n-1)(n-2)} - (n+1)\tau \leq 0. \tag{33}$$

Using (18), equation (33) turns into

$$\begin{aligned} &\left(\frac{n+2}{n-2}\right)(2(n-1)\tau - n(n-1)^2 + \sum tr(A_{i^*}A_{j^*})^2) \\ &\quad -\frac{2\tau^2}{(n-1)(n-2)} - (n+1)\tau \leq 0, \end{aligned} \tag{34}$$

which gives us

$$-\frac{2}{(n-1)(n-2)}\tau^2+\frac{n^2+3n-2}{n-2}\tau-\frac{n(n-1)^2(n+2)}{n-2}\leq 0.$$

If τ is between $n(n-1)$ and $\frac{n^3-3n+2}{2}$, then $\tau = n(n-1)$ or $\tau = \frac{n^3-3n+2}{2}$. If $\tau = n(n-1)$, then using (14) we have

$$n(n-1)=n(n-1)-\|h\|^2,$$

which implies that M is totally geodesic. If $\tau = \frac{n^3-3n+2}{2}$, then using (14), we have

$$\frac{n^3-3n+2}{2}=n(n-1)-\|h\|^2.$$

But this contradicts the fact that $\|h\|^2 \geq 0$. Hence in view of Theorem C, M is an open part of real projective space RP^n. This completes proof of the theorem. □

Proof of Theorem 2. From (33), since $\|S\|^2 \geq 0$, we get

$$\tau\left(\frac{2\tau}{(n-1)(n-2)}+(n+1)\right)\geq 0. \tag{35}$$

If τ is between $\frac{-(n-1)(n-2)(n+1)}{2}$ and 0 we have $\tau = \frac{-(n-1)(n-2)(n+1)}{2}$ or $\tau = 0$. If $\tau = \frac{-(n-1)(n-2)(n+1)}{2}$, then using (33) we get $S = 0$. This contradicts $\tau = \frac{-(n-1)(n-2)(n+1)}{2}$. If $\tau = 0$, then using (32) we get $R = 0$. Hence in view of Theorem C, M is an open part of the Clifford torus $T^n \subset CP^n$. So we get the result as required. □

There are examples of semi-parallel minimal submanifolds of totally real submanifolds of CP^n except RP^n and T^n. We give the following example:

EXAMPLE 2.1. *The submanifolds*
(i) $SU(p)/Z_p$, $n = p^2 - 1$,
(ii) $SU(p)/SO(p)Z_p$, $n = (p-1)(p+2)/2$,
(iii) $SU(2p)/Sp(p)Z_{2p}$, $n = (p-1)(2p+1)$, *and*
(iv) E_6/F_4Z_3, $n = 26$,

are n-dimensional compact totally real minimal submanifolds embedded in CP^n with parallel second fundamental forms [**1**]. *It is well known that every submanifold with parallel second fundamental form is semi-parallel. So the submanifolds* (*i*)–(*iv*) *are semi-parallel.*

REFERENCES

1. A. Amarzaya and Y. Ohnita, Hamiltonian stability of certain minimal Lagrangian submanifolds in complex projective spaces, *Tohoku Math. J.* **55**(4) (2003), 583–610.

2. E. Calabi, Improper affine hyperspheres of convex type and a generalization of a theorem by K. Jörgens, *Mich. Math. J.* **5** (1958), 105–126.

3. B. Y. Chen and K. Ogiue, On totally real submanifolds, *Trans. Am. Math. Soc.* **193** (1974), 257–266.

4. J. Deprez, Semi-parallel surfaces in Euclidean space, *J. Geom.* **25** (1985), 192–200.

5. J. Deprez, Semi-parallel hypersurfaces, *Rend. Sem. Mat. Univers. Politecn. Torino*, **44** (1986), 303–316.

6. F. Dillen, Semi-parallel hypersurfaces of a real space form, *Israel J. Math.* **75** (1991), 193–202.

7. N. Ejiri, Totally real minimal submanifolds in a complex projective space, *Proc. Am. Math. Soc.* **86** (1982), no. 3, 496–497.

8. D. Ferus, Immersions with parallel second fundamental form, *Math. Z.* **140** (1974), 87–93.

9. A. M. Li and G. Zhao, Totally real minimal submanifolds in CP^n, *Arch. Math. (Basel)* **62** (1994), 562–568.

10. Ü. Lumiste, Semi-symmetric submanifolds as the second order envelope of symmetric submanifolds, *Proc. Eston. Acad. Sci. Phys. Math.* **39** (1990), 1–8.

11. H. Naitoh, Totally real parallel submanifolds in $P^n(C)$, *Tokyo J. Math.* **4**(2) (1981), 279–306.

12. D. Perrone, n-dimensional totally real minimal submanifolds of CP^n, *Arch. Math. (Basel)* **68** (1997), 347–352.

13. M. Takeuchi, Parallel submanifolds of space forms, Manifolds and Lie groups (Notre Dame, IN, 1980), pp. 429–447, Progr. Math., 14 (Birkhäuser, Boston, MA, 1981).

14. K. Yano and M. Kon, *Anti-invariant submanifolds*, Lecture Notes in Pure and Applied Mathematics, No. 21. (Marcel Dekker, New York and Basel, 1976).

15. K. Yano and M. Kon, *Structures on manifolds*, Series in Pure Mathematics, 3. (World Scientific, Singapore, 1984).