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On some classes of super quasi-Einstein manifolds

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Abstract

Quasi-Einstein and generalized quasi-Einstein manifolds are the generalizations of Einstein manifolds. In this study, we consider a super quasi-Einstein manifold, which is another generalization of an Einstein manifold. We find the curvature characterizations of a Ricci-pseudosymmetric and a quasi-conformally flat super quasi-Einstein manifolds. We also consider the condition $\widetilde{C} \cdot S = 0$ on a super quasi-Einstein manifold, where $\widetilde{C}$ and S denote the quasi-conformal curvature tensor and Ricci tensor of the manifold, respectively.

1. Introduction

The notion of a quasi-Einstein manifold was introduced by Chaki and Maity in [1]. A non-flat semi-Riemannian manifold (M^n, g), $(n \geqslant 3)$, is called *quasi-Einstein* if its Ricci tensor S is not identically zero and satisfies the condition

$$S(X,Y) = ag(X,Y) + bA(X)A(Y), \tag{1}$$

where a, b are scalars of which $b \neq 0$, A is non-zero 1-form such that

$$g(X,U) = A(X) \quad \forall X$$

and U is a unit vector field. In such a case a, b are called *associated scalars*. A is called the *associated* 1-*form* and U is called the *generator* of the manifold. For more details about quasi-Einstein manifolds see also [4,9].

Quasi-Einstein manifolds arose during the study of exact solutions of the Einstein field equations. There are many studies about Einstein field equations. For example, in [8], El Naschie turned the tables on the theory of elementary particles and showed that we could derive the expectation number of elementary particles of the standard model using Einstein's unified field equation or more precisely his somewhat forgotten strength criteria directly and without resorting to quantum field theory [6]. In [7], possible connections between Gödel's classical solution of Einstein's field equations and E-infinity were discussed.

As a generalization of a quasi-Einstein manifold, in [2], Chaki introduced the notion of a generalized quasi-Einstein manifold. A non-flat semi-Riemannian manifold (M^n, g), $(n \geqslant 3)$, is called *generalized quasi-Einstein* if its Ricci tensor S is not identically zero and satisfies the condition

$$S(X,Y) = ag(X,Y) + bA(X)A(Y) + c[A(X)B(Y) + A(Y)B(X)], \tag{2}$$

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where a, b, c are scalars of which $b \neq 0$, $c \neq 0$, A, B are two non-zero 1-forms such that

$$g(X,U) = A(X), \quad g(X,V) = B(X) \quad \forall X$$

and U, V are two unit vector fields perpendicular to each other. In such a case a, b, c are called *associated scalars*. A, B are called the *associated* 1-*forms* and U, V are called the *generators* of the manifold. It is found that a perfect fluid space-time of general relativity is a four-dimensional semi-Riemannian quasi-Einstein manifold whose associated scalars are $(r/2) + kp$ and $k(\rho + p)$, where k is the gravitational constant, ρ and p are the energy density and the isotropic pressure of the fluid and r is the scalar curvature, the generator of the manifold being the unit timelike velocity vector field of the fluid. The importance of a generalized quasi-Einstein manifold lies in the fact that such a four-dimensional semi-Riemannian manifold is relevant to the study of a general relativistic fluid space-time admitting heat flux (see [2,10,12]). The global properties of such a space-time are under investigation. Study of space-times admitting fluid viscosity and electromagnetic fields require further generalization of the Ricci tensor and is under process. In this study, we shall study super quasi-Einstein manifolds which has another generalization of Ricci tensor.

In Cosmology, the reason for studying various types of space-time models is mainly for the purpose of representing the different phases in the evolution of the universe. The evolution of the universe to its present state can be divided into three phases. The initial phase just after the big bang when the effects of both viscosity and heat flux were quite pronounced. The intermediate phase when the effect of viscosity was no longer significant but the heat flux was still not negligible. The final phase, which extends to the present state of the universe when both the effects of viscosity and the heat flux have become negligible and the matter content of the universe may be assumed to be a perfect fluid. The importance of the study of the generalized quasi-Einstein and quasi-Einstein manifolds lies in the fact that these space-time manifolds represent the second and the third phase, respectively in the evolution of the universe [10].

In [3], Chaki introduced the notion of a super quasi-Einstein manifold, which is another generalization of a quasi-Einstein manifold. A non-flat semi-Riemannian manifold (M^n, g), $(n \geqslant 3)$, is called *super quasi-Einstein* if its Ricci tensor S is not identically zero and satisfies the condition

$$S(X,Y) = ag(X,Y) + bA(X)A(Y) + c[A(X)B(Y) + A(Y)B(X)] + dD(X,Y), \tag{3}$$

where a, b, c, d are scalars of which $b \neq 0$, $c \neq 0$, $d \neq 0$, A, B are two non-zero 1-forms such that

$$g(X,U) = A(X), \quad g(X,V) = B(X) \quad \forall X \tag{4}$$

and U, V are mutually orthogonal unit vector fields, D is a symmetric $(0,2)$-tensor with zero trace which satisfies the condition

$$D(X,U) = 0 \quad \forall X. \tag{5}$$

In such a case a, b, c, d are called *associated scalars*. A, B are called *the associated main and auxiliary* 1-*forms* and U, V are called *the main and auxiliary generators* of the manifold. D is called *the associated tensor* of the manifold. From (3), by a contraction, one can easily obtain that the scalar curvature r of M^n is

$$r = na + b. \tag{6}$$

In [3] (see proof of Theorem 3), Chaki gave the following nice example of a super quasi-Einstein manifold.

Example 1.1. Let (M^4, g) be a viscous fluid space-time admitting heat flux and satisfying Einstein's equation without cosmological constant. Let U be the unit timelike velocity vector field of the fluid, V be the heat flux vector field and D be the anisotropic pressure tensor of the fluid. Then

$$g(U,U) = -1, \quad g(V,V) = 1, \quad g(U,V) = 0, \tag{7}$$

$$D(X,Y) = D(Y,X), \quad \operatorname{trace} D = 0, \tag{8}$$

$$D(X,U) = 0, \quad \forall X. \tag{9}$$

Let

$$g(X,U) = A(X), \; g(X,V) = B(X), \forall X. \tag{10}$$

Further, let T be the $(0,2)$-type energy momentum tensor describing the matter distribution of such a fluid. Then

$$T(X,Y) = (\sigma + p)A(X)A(Y) + pg(X,Y) + A(X)B(Y) + A(Y)B(X) + D(X,Y), \tag{11}$$

where σ, p denote the density and isotropic pressure and D denotes the anisotropic pressure tensor of the fluid. It is known that [11] Einstein's equation without cosmological constant can be written as follows:

$$S(X,Y) - \frac{1}{2}rg(X,Y) = kT(X,Y), \tag{12}$$

where k is the gravitational constant and T is the energy momentum tensor of type $(0,2)$. In the present case (12) can be written as follows:

$$S(X,Y) - \frac{1}{2}rg(X,Y) = k[(\sigma + p)A(X)A(Y) + pg(X,Y) + A(X)B(Y) + A(Y)B(X) + D(X,Y)].$$

So

$$S(X,Y) = \left(kp + \frac{1}{2}r\right)g(X,Y) + k(\sigma + p)A(X)A(Y) + k[A(X)B(Y) + A(Y)B(X)] + kD(X,Y).$$

Hence the space-time under consideration is a super quasi-Einstein manifold with $kp + \frac{1}{2}r, k(\sigma + p), k, k$ as associated scalars, A and B as associated 1-forms, U, V as generators and D as the associated symmetric $(0,2)$-tensor.

In this study, we consider Ricci-pseudosymmetric and quasi-conformally flat super quasi-Einstein manifolds. We also consider the condition $\widetilde{C} \cdot S = 0$ on a super quasi-Einstein manifold, where $\widetilde{C}$ denotes the quasi-conformal curvature tensor of the manifold. The paper is organized as follows: In Section 2, we find the curvature characterizations of Ricci-pseudosymmetric super quasi-Einstein manifolds. In Section 3, we obtain the necessary condition for a super quasi-Einstein manifold to be quasi-conformally flat. In the final section, we find the characterization of the curvature tensor of a super-quasi Einstein manifold satisfying the condition $\widetilde{C} \cdot S = 0$.

2. Ricci-pseudosymmetric super quasi-Einstein manifolds

An n-dimensional semi-Riemannian manifold (M^n, g) is called *Ricci-pseudosymmetric* [5] if the tensors $R \cdot S$ and $Q(g,S)$ are linearly dependent, where

$$(R(X,Y) \cdot S)(Z,W) = -S(R(X,Y)Z,W) - S(Z,R(X,Y)W), \tag{13}$$

$$Q(g,S)(Z,W;X,Y) = -S((X \wedge Y)Z,W) - S(Z,(X \wedge Y)W) \tag{14}$$

and

$$(X \wedge Y)Z = g(Y,Z)X - g(X,Z)Y \tag{15}$$

for vector fields X, Y, Z, W on M^n, R denotes the curvature tensor of M^n.

The condition of Ricci-pseudosymmetry is equivalent to

$$(R(X,Y) \cdot S)(Z,W) = L_S Q(g,S)(Z,W;X,Y) \tag{16}$$

holding on the set

$$U_S = \left\{x \in M : S \neq \frac{r}{n}g \text{ at } x\right\}$$

where L_S is some function on U_S. If $R \cdot S = 0$ then M^n is called *Ricci-semisymmetric*. Every Ricci-semisymmetric manifold is Ricci-pseudosymmetric but the converse is not true [5].

Now we consider a Ricci-pseudosymmetric super quasi-Einstein manifold.

So we have the following theorem.

Theorem 2.1. *Let (M^n, g), $(n \geqslant 3)$, be a super quasi-Einstein manifold. If M^n is Ricci-pseudosymmetric then the following conditions hold on M^n:*

$$R(X,Y,U,V) = L_S\{A(Y)B(X) - A(X)B(Y)\}, \tag{17}$$

$$D(R(X,Y)U,V) = L_S\{A(Y)D(X,V) - A(X)D(Y,V)\} \tag{18}$$

and

$$D(R(X,Y)V,V) = L_S\{B(Y)D(X,V) - B(X)D(Y,V)\} \tag{19}$$

for all vector fields X, Y on M^n, where U, V are the generators of the manifold M^n.

Proof. Assume that M^n is Ricci-pseudosymmetric. Then by the use of (13)–(16) we can write

$$S(R(X,Y)Z,W)+S(Z,R(X,Y)W)=L_S\{g(Y,Z)S(X,W)-g(X,Z)S(Y,W)+g(Y,W)S(X,Z)-g(X,W)S(Y,Z)\}. \tag{20}$$

Since M^n is also super quasi-Einstein, using the well-known properties of the curvature tensor R we get

$$\begin{aligned}
&b[A(R(X,Y)Z)A(W)+A(Z)A(R(X,Y)W)]+c[A(R(X,Y)Z)B(W)+A(W)B(R(X,Y)Z)+A(Z)B(R(X,Y)W)\\
&\quad+A(R(X,Y)W)B(Z)]+d[D(R(X,Y)Z,W)+D(Z,R(X,Y)W)]\\
&\quad=L_S\{b[g(Y,Z)A(X)A(W)-g(X,Z)A(Y)A(W)+g(Y,W)A(X)A(Z)-g(X,W)A(Y)A(Z)]\\
&\qquad+c[g(Y,Z)A(X)B(W)+g(Y,Z)A(W)B(X)-g(X,Z)A(Y)B(W)-g(X,Z)A(W)B(Y)\\
&\qquad+g(Y,W)A(X)B(Z)+g(Y,W)A(Z)B(X)-g(X,W)A(Y)B(Z)-g(X,W)A(Z)B(Y)]\\
&\qquad+d[g(Y,Z)D(X,W)-g(X,Z)D(Y,W)+g(Y,W)D(X,Z)-g(X,W)D(Y,Z)]\}.
\end{aligned}\tag{21}$$

Taking $Z=U$ and $W=V$ in (21), in view of (4) and (5), we obtain

$$0=b[R(X,Y,V,U)-L_S\{A(X)B(Y)-A(Y)B(X)\}]+d[D(R(X,Y)U,V)-L_S\{A(Y)D(X,V)-A(X)D(Y,V)\}]. \tag{22}$$

Taking $Z=W=U$ in (21) we get

$$c[R(X,Y,U,V)-L_S\{A(Y)B(X)-A(X)B(Y)\}]=0.$$

Since $c\neq 0$ we have

$$R(X,Y,U,V)-L_S\{A(Y)B(X)-A(X)B(Y)\}=0, \tag{23}$$

which gives us (17). Similarly, taking $Z=W=V$ in (21) we get

$$0=c[R(X,Y,V,U)-L_S\{A(X)B(Y)-A(Y)B(X)\}]+d[D(R(X,Y)V,V)-L_S\{B(Y)D(X,V)-B(X)D(Y,V)\}]. \tag{24}$$

Since $d\neq 0$, from (22) and (23) we have (18). So using (23) and (24) we obtain (19). Our theorem is thus proved. □

3. Quasi-conformally flat super quasi-Einstein manifolds

The quasi-conformal curvature tensor was defined by Yano and Sawaki (see [13]) as

$$\widetilde{C}(X,Y)Z=\lambda R(X,Y)Z+\mu\{S(Y,Z)X-S(X,Z)Y+g(Y,Z)QX-g(X,Z)QY\}-\frac{r}{n}\left[\frac{\lambda}{n-1}+2\mu\right][g(Y,Z)X-g(X,Z)Y], \tag{25}$$

where λ and μ are nonzero constants, Q is the Ricci operator defined by $S(X,Y)=g(QX,Y)$. An n-dimensional semi-Riemannian manifold (M^n,g), $n>3$, is called *quasi-conformally flat* if $\widetilde{C}=0$. If $\lambda=1$ and $\mu=-\frac{1}{n-2}$, then the quasi-conformal curvature tensor is reduced to the conformal curvature tensor. In [3], Chaki studied conformally flat super quasi-Einstein manifolds. In the present section, we study quasi-conformally flat super quasi-Einstein manifolds.

Let us define

$$g(lX,Y)=D(X,Y). \tag{26}$$

Now we can state the following theorem:

Theorem 3.1. *Let (M^n,g) $(n>3)$ be a super quasi-Einstein manifold. If M^n is quasi-conformally flat then the curvature tensor R of M^n satisfies the following property*:

$$R(X,Y)Z=\left[\frac{a(n-2)-b}{(n-1)(n-2)}\right][g(Y,Z)X-g(X,Z)Y]+\frac{c}{2-n}[g(X,Z)B(Y)-g(Y,Z)B(X)]U+\frac{d}{2-n}[D(X,Z)Y-D(Y,Z)X+g(X,Z)lY-g(Y,Z)lX] \tag{27}$$

for all vector fields $X,Y,Z\in U^{\perp}$ on M^n.

Proof. Since M^n is quasi-conformally flat $\widetilde{C} = 0$. Then from (25) we have

$$\lambda R(X,Y,Z,W) = \mu[S(X,Z)g(Y,W) - S(Y,Z)g(X,W) + g(X,Z)S(Y,W) - g(Y,Z)S(X,W)] + \frac{r}{n}\left[\frac{\lambda}{n-1} + 2\mu\right][g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] \tag{28}$$

for all vector fields $X,\ Y,\ Z,\ W$ on M^n. Since M^n is super quasi-Einstein $\lambda + (n-2)\mu = 0$. Because when $\lambda + (n-2)\mu \neq 0$ for a quasi-conformally flat manifold one can easily see that M^n becomes an Einstein manifold. So in view of (2) and (28), we can write

$$\begin{aligned}\lambda R(X,Y,Z,W) = {} & \left[-2a\mu + \frac{r\mu}{n}\left(\frac{2-n}{n-1} + 2\right)\right][g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] + b\mu[g(Y,W)A(X)A(Z) \\ & + g(X,Z)A(Y)A(W) - g(X,W)A(Y)A(Z) - g(Y,Z)A(X)A(W)] + c\mu[g(Y,W)(A(X)B(Z) \\ & + A(Z)B(X)) + g(X,Z)(A(Y)B(W) + A(W)B(Y)) - g(X,W)(A(Y)B(Z) + A(Z)B(Y)) \\ & - g(Y,Z)(A(X)B(W) + A(W)B(X))] + d\mu[g(Y,W)D(X,Z) + g(X,Z)D(Y,W) \\ & - g(X,W)D(Y,Z) - g(Y,Z)D(X,W)].\end{aligned} \tag{29}$$

Let $U^{\perp}$ denote the $(n-1)$-dimensional distribution orthogonal to U in a quasi conformally flat super quasi-Einstein manifold M^n. Then $g(X,U) = 0$ if and only if $X \in U^{\perp}$. Hence from (29), when $X,Y,Z \in U^{\perp}$ we have (27). This completes the proof of the theorem. □

4. Super quasi-Einstein manifolds satisfying the condition $\widetilde{C} \cdot S = 0$

In this section, we consider super quasi-Einstein manifolds (M^n, g), $(n > 3)$, satisfying the condition

$$(\widetilde{C}(X,Y) \cdot S)(Z,W) = -S(\widetilde{C}(X,Y)Z,W) - S(Z,\widetilde{C}(X,Y)W) = 0$$

for all vector fields $X,\ Y,\ Z,\ W$ on M^n.

Then we have the following theorem:

Theorem 4.1. *Let (M^n, g) $(n > 3)$ be a super quasi-Einstein manifold. If the condition $\widetilde{C} \cdot S = 0$ holds on M^n then the curvature tensor R of M^n satisfies the following property*:

$$\lambda R(X,Y,U,V) = \left[\frac{na+b}{n}\left(\frac{\lambda}{n-1} + 2\mu\right) - \mu(2a+b)\right]\{A(Y)B(X) - A(X)B(Y)\} - d\mu\{D(X,V)A(Y) - D(Y,V)A(X)\} \tag{30}$$

for all vector fields $X,\ Y$ on M^n, where $U,\ V$ are the generators of the manifold M^n.

Proof. Since the condition $\widetilde{C} \cdot S = 0$ holds on M^n we have

$$S(\widetilde{C}(X,Y)Z,W) + S(Z,\widetilde{C}(X,Y)W) = 0.$$

Since M^n is super quasi-Einstein, using (3) and the well-known properties of the quasi-conformal curvature tensor we get

$$\begin{aligned}0 = {} & b[A(\widetilde{C}(X,Y)Z)A(W) + A(Z)A(\widetilde{C}(X,Y)W)] + c[A(\widetilde{C}(X,Y)Z)B(W) + A(W)B(\widetilde{C}(X,Y)Z) \\ & + A(Z)B(\widetilde{C}(X,Y)W) + A(\widetilde{C}(X,Y)W)B(Z)] + d[D(\widetilde{C}(X,Y)Z,W) + D(Z,\widetilde{C}(X,Y)W)].\end{aligned} \tag{31}$$

Taking $Z = W = U$ in (31) and using (5) we have

$$B(\widetilde{C}(X,Y)U) = \widetilde{C}(X,Y,U,V) = 0.$$

Since $c \neq 0$ by the use of (3) and (6) we obtain (30). Hence we get the result as required. □

5. Conclusions

Study of space-times admitting fluid viscosity and electromagnetic fields require some generalizations of Einstein manifolds and is under process. The first generalization of an Einstein manifold is a quasi-Einstein manifold. Quasi-Einstein manifolds arose during the study of exact solutions of the Einstein field equations. Another generalization of an Einstein manifold is a generalized quasi-Einstein manifold. The importance of a generalized quasi-Einstein manifold is presented in the introduction. In the present paper, we consider a super quasi-Einstein manifold, which is another generalization of an Einstein manifold. A physical example of a super quasi-Einstein manifold is given in the introduction. We consider Ricci-pseudosymmetric and quasi-conformally flat super quasi-Einstein manifolds. We also consider super quasi-Einstein manifolds satisfying the condition $\widetilde{C} \cdot S = 0$.

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