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ON QUASI-EINSTEIN WARPED PRODUCTS

BY

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Abstract. We study quasi-Einstein warped product manifolds for arbitrary dimension $n \geq 3$.

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1. Introduction

A Riemannian manifold (M, g), $(n \geq 2)$, is said to be an *Einstein manifold* if its Ricci tensor S satisfies the condition $S = \frac{\tau}{n} g$, where τ denotes the *scalar curvature* of M. A quasi-Einstein manifold was introduced by Chaki and Maity in [1]. A non-flat Riemannian manifold (M, g), $(n \geq 2)$, is defined to be a *quasi-Einstein manifold* if the condition

$$S(X, Y) = \alpha g(X, Y) + \beta A(X)A(Y) \tag{1}$$

is fulfilled on M, where α and β are scalar functions on M with $\beta \neq 0$ and A is a non-zero 1-form such that

$$g(X, U) = A(X), \tag{2}$$

for every vector field X ; $U \in \chi(M)$ being a unit vector field, $\chi(M)$ is the space of vector fields on M. If $\beta = 0$, then the manifold reduces to an Einstein manifold.

By a contraction from the equation (1), it can be easily seen that $\tau = \alpha n + \beta$, where τ is the scalar curvature of M.

Quasi-Einstein manifolds arose during the study of exact solutions of the Einstein field equations as well as during considerations of quasi-umbilical hypersurfaces. For instance, the Robertson-Walker space-times are quasi-Einstein manifolds. For more information about quasi-Einstein manifolds see [4], [5], [6] and [8].

In [2], Chen and Yano introduced the notion of a Riemannian manifold (M, g) of a *quasi-constant sectional curvature* as a Riemannian manifold with the curvature tensor satisfies the condition

$$\begin{aligned} R(X,Y,Z,W) &= a[g(Y,Z)g(X,W)-g(X,Z)g(Y,W)] \\ &\quad +b[g(X,W)\eta(Y)\eta(Z)-g(X,Z)\eta(Y)\eta(W) \\ &\quad +g(Y,Z)\eta(X)\eta(W)-g(Y,W)\eta(X)\eta(Z)], \end{aligned} \tag{3}$$

where a and b are scalar functions with $b \neq 0$, where η is a 1-form denoted by $g(X,E) = \eta(X)$, E is a unit vector field. It can be shown that, if the curvature tensor R is of the form (3), then the manifold is conformally flat. By a contraction from the equation (3), it can be easily seen that every Riemannian manifold of a quasi-constant sectional curvature is a quasi-Einstein manifold.

Let M be an m-dimensional, $m \geq 3$, Riemannian manifold and $p \in M$. Denote by $K(\pi)$ or $K(u,v)$ the sectional curvature of M associated with a plane section $\pi \subset T_pM$, where $\{u,v\}$ is an orthonormal basis of π. For any n-dimensional subspace $L \subseteq T_pM$, $2 \leq n \leq m$, its scalar curvature $\tau(L)$ is denoted in [3] by $\tau(L) = 2\sum_{1\leq i<j\leq n} K(e_i \wedge e_j)$, where $\{e_1, ..., e_n\}$ is any orthonormal basis of L. When $L = T_pM$, then the scalar curvature $\tau(L)$ is just the scalar curvature $\tau(p)$ of M at p.

2. Warped product manifolds

Let (B, g_B) and (F, g_F) be two Riemannian manifolds and f is a positive differentiable function on B. Consider the product manifold $B \times F$ with its projections $\pi : B \times F \to B$ and $\sigma : B \times F \to F$. The *warped product* $B \times_f F$ is the manifold $B \times F$ with the Riemannian structure such that $\|X\|^2 = \|\pi^*(X)\|^2 + f^2(\pi(p))\,\|\sigma^*(X)\|^2$, for any vector field X on M. Thus we have

$$g = g_B + f^2 g_F \tag{4}$$

holds on M. The function f is called the *warping function* of the warped product [10].

Since $B \times_f F$ is a warped product, then we have $\nabla_X Z = \nabla_Z X = (X \ln f)Z$ for unit vector fields X, Z on B and F, respectively. Hence, we find $K(X \wedge Z) = g(\nabla_Z \nabla_X X - \nabla_X \nabla_Z X, Z) = (1/f)\{(\nabla_X X)f - X^2 f\}$. If we chose a local orthonormal frame $e_1, ..., e_n$ such that $e_1, ..., e_{n_1}$ are tangent to B and $e_{n_1+1}, ..., e_n$ are tangent to F, then we have

$$\frac{\Delta f}{f} = \sum_{i=1}^{n} K(e_j \wedge e_s), \tag{5}$$

for each $s = n_1 + 1, ..., n$ [10].

We need the following two lemmas from [10], for later use :

Lemma 2.1. *Let $M = B \times_f F$ be a warped product, with Riemannian curvature tensor ${}^M R$. Given fields X, Y, Z on B and U, V, W on F, then:*

(1) ${}^M R(X, Y)Z =^B R(X, Y)Z$,

(2) ${}^M R(V, X)Y = -(H^f(X, Y)/f)V$, *where H^f is the Hessian of f,*

(3) ${}^M R(X, Y)V =^M R(V, W)X = 0$,

(4) ${}^M R(X, V)W = -(g(V, W)/f)\nabla_X(\operatorname{grad} f)$,

(5) ${}^M R(V, W)U =^F R(V, W)U + (\|\operatorname{grad} f\|^2 / f^2)\{g(V, U)W - g(W, U)V\}$.

Lemma 2.2. *Let $M = B \times_f F$ be a warped product, with Ricci tensor ${}^M S$. Given fields X, Y on B and V, W on F, then:*

(1) ${}^M S(X, Y) =^B S(X, Y) - \frac{d}{f} H^f(X, Y)$, *where $d = \dim F$,*

(2) ${}^M S(X, V) = 0$,

(3) ${}^M S(V, W) =^F S(V, W) - g(V, W) \left[\frac{\Delta f}{f} + (d - 1)\frac{\|\operatorname{grad} f\|^2}{f^2}\right]$, *where Δf is the Laplacian of f on B.*

Moreover, the scalar curvature ${}^M\tau$ of the manifold M satisfies the condition

$${}^M\tau =^B \tau + \frac{1}{f^2}{}^F\tau - \frac{2d}{f}\Delta f - \frac{d(d-1)}{f^2}\|\operatorname{grad} f\|^2, \tag{6}$$

where ${}^B\tau$ and ${}^F\tau$ are scalar curvatures of B and F, respectively.

In [7], Gebarowski studied Einstein warped product manifolds and proved the following three theorems:

Theorem 2.3. *Let (M,g) be a warped product $I \times_f F$, $\dim I = 1$, $\dim F = n-1$ $(n \geq 3)$. Then (M,g) is an Einstein manifold if and only if F is Einstein with constant scalar curvature ${}^F\tau$ in the case $n=3$ and f is given by one of the following formulae, for any real number b,*

$$f^2(t) = \left\{ \begin{array}{c} \frac{4}{a}K\sinh^2\frac{\sqrt{a}(t+b)}{2} \quad (a>0), \\ K(t+b)^2 \quad (a=0), \\ -\frac{4}{a}K\sin^2\frac{\sqrt{-a}(t+b)}{2} \quad (a<0), \end{array} \right\}$$

for $K>0$, $f^2(t) = b\exp(at)$ $(a \neq 0)$, for $K=0$, $f^2(t) = -\frac{4}{a}K\cosh^2\frac{\sqrt{a}(t+b)}{2}$, $(a>0)$, for $K<0$, where a is the constant appearing after first integration of the equation $q''e^q + 2K = 0$ and $K = \frac{{}^F\tau}{(n-1)(n-2)}$.

Theorem 2.4. *Let (M,g) be a warped product $B \times_f F$ of a complete connected r-dimensional $(1<r<n)$ Riemannian manifold B and $(n-r)$-dimensional Riemannian manifold F. If (M,g) is a space of constant sectional curvature $K>0$, then B is a sphere of radius $\frac{1}{\sqrt{K}}$.*

Theorem 2.5. *Let (M,g) be a warped product $B \times_f I$ of a complete connected $(n-1)$-dimensional Riemannian manifold B and one-dimensional Riemannian manifold I. If (M,g) is an Einstein manifold with scalar curvature ${}^M\tau > 0$ and the Hessian of f is proportional to the metric tensor g_B, then*

(1) *(B, g_B) is an $(n-1)$-dimensional sphere of radius $\rho = (\frac{{}^B\tau}{(n-1)(n-2)})^{-\frac{1}{2}}$.*

(2) *(M,g) is a space of constant sectional curvature $K = \frac{{}^M\tau}{n(n-1)}$.*

Motivated by the above study by Gebarowski, in the present paper our aim is to generalize Theorem 2.3, Theorem 2.4 and Theorem 2.5 for quasi-Einstein manifolds.

3. Quasi-Einstein warped products

In this section, we consider quasi-Einstein warped product manifolds and prove some results concerning these type manifolds.

Now, let begin with the following theorem:

Theorem 3.1. *Let (M,g) be a warped product $I \times_f F$, $\dim I = 1, \dim F = n-1 (n \geq 3)$, where $U \in \chi(M)$. If (M,g) is a quasi-Einstein manifold with associated scalars α and β, then F is a quasi-Einstein manifold.*

Proof. Denote by $(dt)^2$ the metric on I. Taking $f = \exp\left\{\frac{q}{2}\right\}$ and making use of the Lemma 2.2, we can write

$$ {}^{M}S\left(\frac{\partial}{\partial t},\frac{\partial}{\partial t}\right) = -\frac{n-1}{4}[2q'' + (q')^2] \tag{7} $$

and

$$ {}^{M}S(V,W) = {}^{F}S(V,W) - \frac{1}{4}e^{q}[2q'' + (n-1)(q')^2]g_{F}(V,W), \tag{8} $$

for all vector fields V, W on F.

Since M is quasi-Einstein, from (1) we have

$$ {}^{M}S\left(\frac{\partial}{\partial t},\frac{\partial}{\partial t}\right) = \alpha g\left(\frac{\partial}{\partial t},\frac{\partial}{\partial t}\right) + \beta A\left(\frac{\partial}{\partial t}\right)A\left(\frac{\partial}{\partial t}\right) \tag{9} $$

and

$$ {}^{M}S(V,W) = \alpha g(V,W) + \beta A(V)A(W). \tag{10} $$

Decomposing the vector field U uniquely into its components U_{I} and U_{F} on I and F, respectively, we can write $U = U_{I} + U_{F}$. Since $\dim I = 1$, we can take $U_{I} = \mu\frac{\partial}{\partial t}$ which gives us $U = \mu\frac{\partial}{\partial t} + U_{F}$, where μ is a function on M. Then we can write

$$ A\left(\frac{\partial}{\partial t}\right) = g\left(\frac{\partial}{\partial t}, U\right) = \mu. \tag{11} $$

On the other hand, by the use of (4) and (11), the equations (9) and (10) reduce to

$$ {}^{M}S\left(\frac{\partial}{\partial t},\frac{\partial}{\partial t}\right) = \alpha + \mu^2\beta \tag{12} $$

and

$$ {}^{M}S(V,W) = \alpha e^{q} g_{F}(V,W) + \beta A(V)A(W). \tag{13} $$

Comparing the right hand sides of the equations (7) and (12) we get

$$ \alpha + \mu^2\beta = -\frac{(n-1)}{4}[2q'' + (q')^2]. \tag{14} $$

Similarly, comparing the right hand sides of (8) and (13) we obtain

$$^{F}S(V,W)=\frac{1}{4}e^{q}\left[2q''+(n-1)(q')^{2}+4\alpha\right]g_{F}(V,W)+\beta A(V)A(W),$$

which implies that F is a quasi-Einstein manifold. This completes the proof of the theorem. □

Theorem 3.2. *Let (M,g) be a warped product $B\times_f F$ of a complete connected r-dimensional $(1<r<n)$ Riemannian manifold B and $(n-r)$-dimensional Riemannian manifold F.*

(1) *If (M,g) is a space of quasi-constant sectional curvature, the Hessian of f is proportional to the metric tensor g_B and the associated vector field E is a general vector field on M or $E\in\chi(B)$, then B is a 2-dimensional Einstein manifold.*

(2) *If (M,g) is a space of quasi-constant sectional curvature and the associated vector field $E\in\chi(F)$, then B is an Einstein manifold.*

Proof. Assume that M is a space of quasi-constant sectional curvature. Then from the equation (3) we can write

$$\begin{aligned}^{M}R(X,Y,Z,W) &= a[g(Y,Z)g(X,W)-g(X,Z)g(Y,W)]\\ &\quad+b[g(X,W)\eta(Y)\eta(Z)-g(X,Z)\eta(Y)\eta(W)\\ &\quad+g(Y,Z)\eta(X)\eta(W)-g(Y,W)\eta(X)\eta(Z)],\end{aligned}\tag{15}$$

for all vector fields X,Y,Z,W on B.

Decomposing the vector field E uniquely into its components E_B and E_F on B and F, respectively, we have

$$E=E_B+E_F.\tag{16}$$

By making use of (4) and (16), we can write

$$\eta(Y)=g(Y,E)=g(Y,E_B)=g_B(Y,E_B).\tag{17}$$

In view of Lemma 2.1 and by the use of (4) and (17), we obtain

$$\begin{aligned}&^{B}R(X,Y,Z,W)=a[g_B(Y,Z)g_B(X,W)-g_B(X,Z)g_B(Y,W)]\\ &+b[g_B(X,W)g_B(Y,E_B)g_B(Z,E_B)-g_B(X,Z)g_B(Y,E_B)g_B(W,E_B)\\ &+g_B(Y,Z)g_B(X,E_B)g_B(W,E_B)-g_B(Y,W)g_B(X,E_B)g_B(Z,E_B)].\end{aligned}\tag{18}$$

By a contraction from the last equation over X and W and making use of the equation (17) again, we get

$$ {}^{B}S(Y,Z) = [a(r-1) + bg_{B}(E_{B}, E_{B})]g_{B}(Y,Z) + b(r-2)\eta(Y)\eta(Z), \tag{19} $$

which shows us B is a quasi-Einstein manifold. Contracting from (19) over Y and Z, it can be easily seen that

$$ {}^{B}\tau = (r-1)[ar + 2bg_{B}(E_{B}, E_{B})]. \tag{20} $$

Since M is a space of quasi-constant sectional curvature, in view of (5) and (18) we get

$$ \frac{\Delta f}{f} = \frac{ar + bg_{B}(E_{B}, E_{B})}{2}. \tag{21} $$

On the other hand, since the Hessian of f is proportional to the metric tensor g_B, it can be written as follows

$$ H^{f}(X,Y) = \frac{\Delta f}{r} g_{B}(X,Y). \tag{22} $$

Then, by the use of (20) and (21) in (22) we obtain $H^f(X,Y) + Kfg_B(X,Y) = 0$, where $K = \frac{(r-1)bg_B(E_B,E_B) - {}^{B}\tau}{2r(r-1)}$ holds on B. So by OBATA's theorem [9], B is isometric to the sphere of radius $\frac{1}{\sqrt{K}}$ in the $(r+1)$-dimensional Euclidean space. This gives us B is an Einstein manifold. Since $b \neq 0$ this implies that $r = 2$. Hence B is a 2-dimensional Einstein manifold.

Assume that the associated vector field $E \in \chi(B)$. Then in view of Lemma 2.1 and by making use of (4) and (15) we can write

$$ \begin{aligned} {}^{B}R(X,Y,Z,W) &= a[g_{B}(Y,Z)g_{B}(X,W) - g_{B}(X,Z)g_{B}(Y,W)] \\ &+ b[g_{B}(X,W)g_{B}(Y,E)g_{B}(Z,E) - g_{B}(X,Z)g_{B}(Y,E)g_{B}(W,E) \\ &+ g_{B}(Y,Z)g_{B}(X,E)g_{B}(W,E) - g_{B}(Y,W)g_{B}(X,E)g_{B}(Z,E)]. \end{aligned} \tag{23} $$

By a contraction from the last equation over X and W, we obtain

$$ {}^{B}S(Y,Z) = [a(r-1) + b]g_{B}(Y,Z) + b(r-2)g_{B}(Y,E)g_{B}(Z,E), \tag{24} $$

which gives us B is a quasi-Einstein manifold.

By a contraction from (24) over Y and Z, we get

$$ {}^{B}\tau = (r-1)[ar + 2b]. \tag{25} $$

Since M is a space of quasi-constant sectional curvature, in view of (5) and (23) we have

$$\frac{\Delta f}{f} = \frac{ar+b}{2}. \tag{26}$$

On the other hand, since the Hessian of f is proportional to the metric tensor g_B, it can be written as follows

$$H^f(X,Y) = \frac{\Delta f}{r} g_B(X,Y). \tag{27}$$

Then, by the use of (25) and (26) in (27) we obtain $H^f(X,Y)+Kfg_B(X,Y) = 0$, where $K = \frac{(r-1)b-{}^B\tau}{2r(r-1)}$ holds on B. So by OBATA's theorem [9], B is isometric to the sphere of radius $\frac{1}{\sqrt{K}}$ in the $(r+1)$-dimensional Euclidean space. This shows us B is an Einstein manifold. Since $b \neq 0$ this implies that $r = 2$. Hence B is a 2-dimensional Einstein manifold.

Assume that the associated vector field $E \in \chi(F)$, then the equation (15) reduces to

$${}^M R(X,Y,Z,W) = a[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)].$$

In view of Lemma 2.1 and by the use of (4), the above equation can be written as follows

$${}^B R(X,Y,Z,W) = a[g_B(Y,Z)g_B(X,W) - g_B(X,Z)g_B(Y,W)].$$

By a contraction from the above equation over X and W, we get ${}^B S(Y,Z) = a(r-1)g_B(Y,Z)$, which implies that B is an Einstein manifold with the scalar curvature ${}^B\tau = ar(r-1)$. Hence, the proof of the theorem is completed. □

Theorem 3.3. *Let (M,g) be a warped product $B \times_f I$ of a complete connected $(n-1)$-dimensional Riemannian manifold B and one-dimensional Riemannian manifold I. If (M,g) is a quasi-Einstein manifold with constant associated scalars α and β, $U \in \chi(M)$ and the Hessian of f is proportional to the metric tensor g_B, then (B,g_B) is an $(n-1)$-dimensional sphere of radius $\rho = \frac{n-1}{\sqrt{{}^B\tau+\alpha}}$.*

Proof. Assume that M is a warped product manifold. Then by the use of the Lemma 2.2 we can write

$$^{B}S(X,Y) = ^{M} S(X,Y) + \frac{1}{f}H^{f}(X,Y) \tag{28}$$

for any vector fields X, Y on B. On the other hand, since M is quasi-Einstein we have

$$^{M}S(X,Y) = \alpha g(X,Y) + \beta A(X)A(Y). \tag{29}$$

Decomposing the vector field U uniquely into its components U_B and U_I on B and I, respectively, we get

$$U = U_B + U_I. \tag{30}$$

In view of (2), (4), (29) and (30) the equation (28) can be written as

$$^{B}S(X,Y) = \alpha g_B(X,Y) + \beta g_B(X,U_B)g_B(Y,U_B) + \frac{1}{f}H^{f}(X,Y).$$

By a contraction from the above equation over X and Y, we find

$$^{B}\tau = \alpha(n-1) + \beta g_B(U_B,U_B) + \frac{\Delta f}{f}. \tag{31}$$

On the other hand, we know from the equation (29) that

$$^{M}\tau = \alpha n + \beta g_B(U_B,U_B). \tag{32}$$

By the use of (32) in (31) we get $^{B}\tau = ^{M}\tau - \alpha + \frac{\Delta f}{f}$. In view of Lemma 2.2 we also know that

$$-\frac{^{M}\tau}{n} = \frac{\Delta f}{f}. \tag{33}$$

The last two equations give us $^{B}\tau = \frac{(n-1)}{n}{}^{M}\tau - \alpha$. On the other hand, since the Hessian of f is proportional to the metric tensor g_B, we can write $H^{f}(X,Y) = \frac{\Delta f}{n-1}g_B(X,Y)$. As a consequence of the equation (33) we have $\frac{\Delta f}{n-1} = -\frac{1}{n(n-1)}{}^{M}\tau f$, which implies that

$$H^{f}(X,Y) + \frac{^{B}\tau + \alpha}{(n-1)^2} f g_B(X,Y) = 0.$$

So B is isometric to the $(n-1)$-dimensional sphere of radius $\sqrt{\frac{n-1}{^{B}\tau+\alpha}}$ (see Obata [9]). Thus our theorem is proved. □

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