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Contact CR-warped product submanifolds in generalized Sasakian space forms

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Abstract

We consider a contact CR-warped product submanifold $M = M_{\top} \times_f M_{\perp}$ of a trans-Sasakian generalized Sasakian space form $\widetilde{M}(f_1, f_2, f_3)$. We show that M is a contact CR-product under certain conditions.

Key words and phrases: Warped product manifold, contact CR-warped product submanifold, trans-Sasakian manifold, generalized Sasakian space form

1. Introduction

The notion of a CR-warped product manifold was introduced by B. Y. Chen (see [6] and [7]). He established a sharp relationship between the warping function f of a warped product CR-submanifold of a Kaehler manifold and the squared norm of the second fundamental form. Later, I. Hasegawa and I. Mihai found a similar inequality for contact CR-warped product submanifolds of Sasakian manifolds in [8]. Moreover, I. Mihai [11] improved the same inequality for contact CR-warped products in Sasakian space forms and he gave some applications. A classification of contact CR-warped products in spheres, which satisfy the equality case, identically, was also given.

Furthermore, in [2], K. Arslan, R. Ezentaş, I. Mihai and C. Murathan considered contact CR-warped product submanifolds in Kenmotsu space forms and they obtained sharp estimates for the squared norm of the second fundamental form in terms of the warping function for contact CR-warped products isometrically immersed in Kenmotsu space forms.

Recently, in [3], M. Atçeken studied on the contact CR-warped product submanifolds of a cosymplectic space form and obtained a necessary and sufficient condition for a contact CR-product.

Motivated by the studies of the above authors, in the present study, we consider contact CR-warped product submanifolds of a trans-Sasakian generalized Sasakian space forms and obtain a necessary and sufficient condition for a contact CR-warped product submanifold of a trans-Sasakian generalized Sasakian space form to be a contact CR-product.

The paper is organized as follows: In Section 2, we give a brief information about almost contact metric manifolds. Moreover, in this section the definitions of a generalized Sasakian space form and a contact CR-warped product submanifold are given. In Section 3, warped product manifolds are introduced. In the

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last section, we establish a sharp relationship between the warping function f and the squared norm of the second fundamental form σ of a contact CR-warped product submanifold of a trans-Sasakian manifold and give characterizations for a contact CR-warped product submanifold of a trans-Sasakian generalized Sasakian space form to be a contact CR-product submanifold.

2. Preliminaries

An odd-dimensional Riemannian manifold $\widetilde{M}$ is called an *almost contact metric manifold* if there exist on $\widetilde{M}$ a $(1,1)$-tensor field φ, a vector field ξ (called a *structure vector field*), a 1-form η and the Riemannian metric g on $\widetilde{M}$ such that

$$\varphi^2 = -I + \eta \otimes \xi, \quad \varphi\xi = 0, \quad \eta(\xi) = 1, \quad \eta \circ \varphi = 0, \tag{1}$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2}$$

$$\eta(X) = g(X, \xi), \quad g(\varphi X, Y) = -g(X, \varphi Y), \tag{3}$$

for all vector fields on $\widetilde{M}$ [4].

Such a manifold is said to be a *contact metric manifold* if $d\eta = \Phi$, where $\Phi(X, Y) = g(X, \varphi Y)$ is called the *fundamental 2-form* of $\widetilde{M}$.

On the other hand, the almost contact metric structure of $\widetilde{M}$ is said to be *normal* if $[\varphi, \varphi](X, Y) = -2d\eta(X, Y)\xi$ for any X, Y on $\widetilde{M}$, where $[\varphi, \varphi]$ denotes the Nijenhuis torsion of φ, given by

$$[\varphi, \varphi](X, Y) = \varphi^2[X, Y] + [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y].$$

A normal contact metric manifold is called a *Sasakian manifold* [4]. It is easy to see that an almost contact metric manifold is Sasakian if and only if

$$(\widetilde{\nabla}_X \varphi)Y = g(X, Y)\xi - \eta(Y)X,$$

for any X, Y on $\widetilde{M}$.

In [13], A. Oubiña introduced the notion of a trans-Sasakian manifold. An almost contact metric manifold $\widetilde{M}$ is said to be a *trans-Sasakian manifold* if there exist two functions α and β on $\widetilde{M}$ such that

$$(\widetilde{\nabla}_X \varphi)Y = \alpha[g(X, Y)\xi - \eta(Y)X] + \beta[g(\varphi X, Y)\xi - \eta(Y)\varphi X], \tag{4}$$

for all vector fields on $\widetilde{M}$. If $\beta = 0$ (resp. $\alpha = 0$), then $\widetilde{M}$ is said to be an *α-Sasakian manifold* (resp. *β-Kenmotsu manifold*). Sasakian manifolds (resp. Kenmotsu manifolds) appear as examples of α-Sasakian manifolds (resp. β-Kenmotsu manifolds), with $\alpha = 1$ (resp. $\beta = 1$).

From the above equation, for a trans-Sasakian manifold we also have

$$\widetilde{\nabla}_X \xi = -\alpha\varphi X + \beta[X - \eta(X)\xi]. \tag{5}$$

A plane section in the tangent space $T_x\widetilde{M}$ at $x \in \widetilde{M}$ is called a *φ-section* if it is spanned by a vector X orthogonal to ξ and φX. The sectional curvature $K(X \wedge \varphi X)$ with respect to a φ-section denoted by a vector X is called a *φ-sectional curvature.* A Sasakian manifold with constant φ-sectional curvature c is a *Sasakian space form* [4] and its Riemannian curvature tensor is given by

$$\begin{aligned} \widetilde{R}(X,Y)Z &= \frac{1}{4}(c+3)\{g(Y,Z)X - g(X,Z)Y\} \\ &+\frac{1}{4}(c-1)\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\ &+g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi \\ &+g(X,\varphi Z)\varphi Y - g(Y,\varphi Z)\varphi X + 2g(X,\varphi Y)\varphi Z\}. \end{aligned} \tag{6}$$

Given an almost contact metric manifold $\widetilde{M}$, it is said to be a *generalized Sasakian space form* [1] if there exist three functions f_1, f_2 and f_3 on $\widetilde{M}$ such that

$$\begin{aligned} \widetilde{R}(X,Y)Z &= f_1\{g(Y,Z)X - g(X,Z)Y\} \\ &+f_2\{g(X,\varphi Z)\varphi Y - g(Y,\varphi Z)\varphi X + 2g(X,\varphi Y)\varphi Z\} \\ &+f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\ &+g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\}, \end{aligned} \tag{7}$$

for any vector fields X, Y, Z on $\widetilde{M}$, where $\widetilde{R}$ denotes the curvature tensor of $\widetilde{M}$. If $f_1 = \frac{c+3}{4}$, $f_2 = f_3 = \frac{c-1}{4}$, then $\widetilde{M}$ is a Sasakian space form [4], if $f_1 = \frac{c-3}{4}$, $f_2 = f_3 = \frac{c+1}{4}$, then $\widetilde{M}$ is a Kenmotsu space form [9], if $f_1 = f_2 = f_3 = \frac{c}{4}$, then $\widetilde{M}$ is a cosymplectic space form [10].

Let $f : M \longrightarrow \widetilde{M}$ be an isometric immersion of an n-dimensional Riemannian manifold M into an $(n+d)$-dimensional Riemannian manifold $\widetilde{M}$. We denote by ∇ and $\widetilde{\nabla}$ the Levi-Civita connections of M and $\widetilde{M}$, respectively. Then we have the Gauss and Weingarten formulas

$$\widetilde{\nabla}_X Y = \nabla_X Y + \sigma(X,Y) \tag{8}$$

and

$$\widetilde{\nabla}_X N = -A_N X + \nabla^{\perp}_X N, \tag{9}$$

where $\nabla^{\perp}$ denotes the normal connection on $T^{\perp}M$ of M and A_N is the shape operator of M, for $X, Y \in \chi(M)$ and a normal vector field N on M. We call σ the *second fundamental form* of the submanifold M. If $\sigma = 0$ then the submanifold is said to be *totally geodesic.* The second fundamental form σ and A_N are related by

$$g(A_N X, Y) = g(\sigma(X,Y), N),$$

for any vector fields X, Y tangent to M.

The equation of Gauss and Codazzi are defined by

$$(\widetilde{R}(X,Y)Z)^{\top} = R(X,Y)Z + A_{\sigma(X,Z)}Y - A_{\sigma(Y,Z)}X \tag{10}$$

and

$$(\widetilde{R}(X,Y)Z)^{\perp} = (\overline{\nabla}_X \sigma)(Y,Z) - (\overline{\nabla}_Y \sigma)(X,Z), \tag{11}$$

for all vector fields X, Y, Z on $\widetilde{M}$, where $(\widetilde{R}(X,Y)Z)^{\top}$ and $(\widetilde{R}(X,Y)Z)^{\perp}$ denote the tangent and normal components of $\widetilde{R}(X,Y)Z$, respectively.

Moreover, the first derivative $\overline{\nabla}\sigma$ of the second fundamental form σ is given by

$$(\overline{\nabla}_X \sigma)(Y,Z) = \nabla_X^{\perp}\sigma(Y,Z) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z), \tag{12}$$

where $\overline{\nabla}$ is called the *van der Waerden-Bortolotti connection* of M [5].

An m-dimensional Riemannian submanifold M of a trans-Sasakian manifold $\widetilde{M}$, where ξ is tangent to M, is called a *contact CR-submanifold* if it admits an invariant distribution D whose orthogonal complementary distribution $D^{\perp}$ is anti-invariant, that is

$$TM = D \oplus D^{\perp} \oplus sp\{\xi\}$$

with $\varphi D_x \subseteq D_x$ and $\varphi D_x^{\perp} \subseteq T_x^{\perp}M$ for each $x \in M$, where $sp\{\xi\}$ denotes 1-dimensional distribution which is spanned by ξ.

Let us denote the orthogonal complementary of $\varphi D^{\perp}$ in $T^{\perp}M$ by υ. Then we have

$$T^{\perp}M = \varphi D^{\perp} \oplus \upsilon.$$

It is obvious that $\varphi\upsilon = \upsilon$.

For any vector field X tangent to M, we can write

$$\varphi X = TX + NX,$$

where TX (resp. NX) denotes tangential (resp. normal) component of φX.

Similarly, for any vector field N normal to M, we put

$$\varphi N = BN + CN,$$

where BN (resp. CN) denotes the tangential (resp. normal) component of φN.

3. Warped product manifolds

Let (B, g_B) and (F, g_F) be two Riemannian manifolds and f is a positive differentiable function on B. Consider the product manifold $B \times F$ with its projections $\pi : B \times F \to B$ and $\sigma : B \times F \to F$. The *warped product* $B \times_f F$ is the manifold $B \times F$ with the Riemannian structure such that

$$\|X\|^2 = \|\pi^*(X)\|^2 + f^2(\pi(p))\,\|\sigma^*(X)\|^2,$$

for any vector field X on M. Thus we have

$$g = g_B + f^2 g_F, \tag{13}$$

holds on M. The function f is called the *warping function* of the warped product [12].

We need the following lemma from [12], for later use :

Lemma 3.1 *Let us consider* $M = B \times_f F$ *and denote by* ∇, ${}^B\nabla$ *and* ${}^F\nabla$ *the Riemannian connections on* M, B *and* F, *respectively. If* X, Y *are vector fields on* B *and* V, W *on* F, *then:*

(*i*) $\nabla_X Y$ *is the lift of* ${}^B\nabla_X Y$,

(*ii*) $\nabla_X V = \nabla_V X = (Xf/f)V$,

(*iii*) *The component of* $\nabla_V W$ *normal to the fibers is* $-(g(V,W)/f)gradf$,

(*iv*) *The component of* $\nabla_V W$ *tangent to the fibers is the lift of* ${}^F\nabla_V W$.

Let we chose a local orthonormal frame $e_1, ..., e_n$ such that $e_1, ..., e_{n_1}$ are tangent to B and $e_{n_1+1}, ..., e_n$ are tangent to F. The gradient and Hessian form of f are defined by

$$X(f) = g(\text{grad}f, X) \tag{14}$$

and

$$H^f(X,Y) = X(Y(f)) - (\nabla_X Y)f = g(\nabla_X \text{grad}f, Y), \tag{15}$$

for any vector fields X, Y on M, respectively.

Moreover, the Laplacian of f is given by

$$\Delta f = \sum_{i=1}^{n} \{(\nabla_{e_i} e_i)f - e_i(e_i(f))\} = -\sum_{i=1}^{n} g(\nabla_{e_i} \text{grad}f, e_i), \tag{16}$$

(see [12]).

From the Green Theory for compact orientable Riemannian manifolds without boundary, it is well-known that

$$\int_M \Delta f dV = 0, \tag{17}$$

where dV denotes the volume element of M.

4. Contact CR-warped product submanifolds

In this section, we establish a sharp relationship between the warping function f and the squared norm of the second fundamental form σ of a contact CR-warped product submanifold of a trans-Sasakian manifold and give characterizations for a contact CR-warped product submanifold of a trans-Sasakian generalized Sasakian space form to be a contact CR-product submanifold.

Now, let's begin with the following lemma.

Lemma 4.1 *Let* $M = M_\top \times_f M_\perp$ *be a contact CR-warped product submanifold of a trans-Sasakian manifold* $\widetilde{M}$. *Then we have*

$$g(\sigma(\varphi X, Y), \varphi Y) = X(\ln f) g(Y,Y), \tag{18}$$

$$g(\sigma(X,Y), \varphi Y) = -\varphi X(\ln f) g(Y,Y) \tag{19}$$

and

$$g(\sigma(\varphi X, Z), \varphi Y) = 0, \tag{20}$$

for any vector fields X, Z *on* $M_\top$ *and* Y *on* $M_\perp$.

Proof. Assume that M is a contact CR-warped product submanifold of a trans-Sasakian manifold $\widetilde{M}$. From the Gauss formula we can write

$$\widetilde{\nabla}_Y \varphi X = \nabla_Y \varphi X + \sigma(\varphi X, Y), \tag{21}$$

for vector fields X on $M_\top$ and Y on $M_\perp$. Taking the inner product of the above equation with φY we get

$$g(\sigma(\varphi X, Y), \varphi Y) = g(\widetilde{\nabla}_Y \varphi X, \varphi Y). \tag{22}$$

Since $\widetilde{M}$ is a trans-Sasakian manifold, from (4) we have

$$(\widetilde{\nabla}_Y \varphi)X = \alpha[g(X,Y)\xi - \eta(X)Y] + \beta[g(\varphi Y, X)\xi - \eta(X)\varphi Y]. \tag{23}$$

By the use of M is a contact CR-warped product submanifold, the equation (23) reduces to

$$(\widetilde{\nabla}_Y \varphi)X = 0,$$

which implies that

$$\widetilde{\nabla}_Y \varphi X = \varphi \widetilde{\nabla}_Y X. \tag{24}$$

In view of (24) in (22), we obtain

$$g(\sigma(\varphi X, Y), \varphi Y) = g(\varphi \widetilde{\nabla}_Y X, \varphi Y).$$

Using (2), the last equation turns into

$$g(\sigma(\varphi X, Y), \varphi Y) = g(\widetilde{\nabla}_Y X, Y).$$

By making use of the Gauss equation again, we get

$$g(\sigma(\varphi X, Y), \varphi Y) = g(\nabla_Y X, Y).$$

Since $\nabla_X Y - \nabla_Y X = [X,Y] = 0$ for vector fields X on $M_\top$ and Y on $M_\perp$, from [12], the above equation can be written as

$$g(\sigma(\varphi X, Y), \varphi Y) = g(\nabla_X Y, Y). \tag{25}$$

So by virtue of the Lemma 3.1, (25) gives us (18).

Similarly by the use of the Gauss formula we can write

$$g(\sigma(X, Y), \varphi Y) = g(\widetilde{\nabla}_Y X, \varphi Y).$$

From (3), the last equation shows us

$$g(\sigma(X, Y), \varphi Y) = -g(\varphi \widetilde{\nabla}_Y X, Y).$$

In view of (24), we get

$$g(\sigma(X, Y), \varphi Y) = -g(\widetilde{\nabla}_Y \varphi X, Y).$$

Then, by the use of the Gauss formula and Lemma 3.1 we obtain (19).

Similar to the proof of (18) and (19) we can easily show that

$$g(\sigma(\varphi X, Z), \varphi Y) = g(\nabla_Z X, Y),$$

for any vector fields X, Z on $M_\top$ and Y on $M_\perp$. Since $M_\top$ is totally geodesic in M, the above equation gives us (20). Hence, we finish the proof of the lemma. □

Lemma 4.2 *Let $M = M_\top \times_f M_\perp$ be a contact CR-warped product submanifold of a trans-Sasakian manifold $\widetilde{M}$. Then we have*

$$g(\sigma(\varphi X, Y), \varphi\sigma(X, Y)) = \|\sigma(X, Y)\|^2 - [\varphi X(\ln f)]^2 \|Y\|^2, \tag{26}$$

for any vector fields X on $M_\top$ and Y on $M_\perp$.

Proof. Taking the inner product of (21) with $\varphi\sigma(X, Y)$ we get

$$g(\sigma(\varphi X, Y), \varphi\sigma(X, Y)) = g(\widetilde{\nabla}_Y \varphi X - \nabla_Y \varphi X, \varphi\sigma(X, Y)),$$

for any vector fields X on $M_\top$ and Y on $M_\perp$.

Since the ambient space $\widetilde{M}$ is trans-Sasakian, by the use of (24) and Lemma 3.1 we find

$$g(\sigma(\varphi X, Y), \varphi\sigma(X, Y)) = g(\varphi\widetilde{\nabla}_Y X, \varphi\sigma(X, Y)) - g(\varphi X(\ln f)Y, \varphi\sigma(X, Y)). \tag{27}$$

In view of (2) and (3), the equation (27) reduces to

$$g(\sigma(\varphi X, Y), \varphi\sigma(X, Y)) = g(\widetilde{\nabla}_Y X, \sigma(X, Y)) + \varphi X(\ln f) g(\varphi Y, \sigma(X, Y)).$$

Then, from the Gauss formula and the equation (19) we obtain

$$g(\sigma(\varphi X, Y), \varphi\sigma(X, Y)) = g(\sigma(X, Y), \sigma(X, Y)) - [\varphi X(\ln f)]^2 g(Y, Y),$$

which gives us (26). Thus, the proof of the lemma is completed. □

Lemma 4.3 *Let $M = M_\top \times_f M_\perp$ be a contact CR-warped product submanifold of a trans-Sasakian generalized Sasakian space form $\widetilde{M}(f_1, f_2, f_3)$. Then we have*

$$\begin{aligned} 2\|\sigma(X, Y)\|^2 &= \{H^{\ln f}(X, X) + H^{\ln f}(\varphi X, \varphi X) \\ &\quad + 2[\varphi X(\ln f)]^2 + 2 f_2 \|X\|^2\} \|Y\|^2, \end{aligned} \tag{28}$$

for any vector fields X on $M_\top$ and Y on $M_\perp$.

Proof. In view of the equation (11), we can write

$$g(\widetilde{R}(X, \varphi X)Y, \varphi Y) = g((\overline{\nabla}_X \sigma)(\varphi X, Y) - (\overline{\nabla}_{\varphi X} \sigma)(X, Y), \varphi Y), \tag{29}$$

for any vector fields X on $M_\top$ and Y on $M_\perp$. Then, by the use of (12) the equation (29) reduces to

$$\begin{aligned} g(\widetilde{R}(X,\varphi X)Y,\varphi Y) &= g(\nabla_X^\perp\sigma(\varphi X,Y)-\sigma(\nabla_X\varphi X,Y)-\sigma(\nabla_XY,\varphi X),\varphi Y) \\ &\quad -g(\nabla_{\varphi X}^\perp\sigma(X,Y)+\sigma(\nabla_{\varphi X}X,Y)+\sigma(\nabla_{\varphi X}Y,X),\varphi Y). \end{aligned}$$

By making use of the Weingarten formula in the above equation, we get

$$\begin{aligned} g(\widetilde{R}(X,\varphi X)Y,\varphi Y) &= g(\widetilde{\nabla}_X\sigma(\varphi X,Y),\varphi Y)-g(\sigma(\nabla_X\varphi X,Y),\varphi Y) \\ &\quad -g(\sigma(\nabla_XY,\varphi X),\varphi Y)-g(\widetilde{\nabla}_{\varphi X}\sigma(X,Y),\varphi Y) \\ &\quad +g(\sigma(\nabla_{\varphi X}X,Y),\varphi Y)+g(\sigma(\nabla_{\varphi X}Y,X),\varphi Y). \end{aligned}$$

By virtue of the properties of the Levi-Civita connection $\widetilde{\nabla}$, the above equation can be written as follows

$$\begin{aligned} g(\widetilde{R}(X,\varphi X)Y,\varphi Y) &= X[g(\sigma(\varphi X,Y),\varphi Y)]-g(\sigma(\varphi X,Y),\widetilde{\nabla}_X\varphi Y) \\ &\quad -g(\sigma(\nabla_X\varphi X,Y),\varphi Y)-g(\sigma(\nabla_XY,\varphi X),\varphi Y) \\ &\quad -\varphi X[g(\sigma(X,Y),\varphi Y)]+g(\sigma(X,Y),\widetilde{\nabla}_{\varphi X}\varphi Y) \\ &\quad +g(\sigma(\nabla_{\varphi X}X,Y),\varphi Y)+g(\sigma(\nabla_{\varphi X}Y,X),\varphi Y). \end{aligned}$$

Then, in view of Lemma 3.1, Lemma 4.1 and (24), the last equation turns into

$$\begin{aligned} g(\widetilde{R}(X,\varphi X)Y,\varphi Y) &= X[X(\ln f)g(Y,Y)]-g(\sigma(\varphi X,Y),\varphi\widetilde{\nabla}_XY) \\ &\quad +\varphi\nabla_X\varphi X(\ln f)g(Y,Y)-X(\ln f)g(\sigma(\varphi X,Y),\varphi Y) \\ &\quad +\varphi X[\varphi X(\ln f)g(Y,Y)]+g(\sigma(X,Y),\varphi\widetilde{\nabla}_{\varphi X}Y) \\ &\quad -\varphi\nabla_{\varphi X}X(\ln f)g(Y,Y)+\varphi X(\ln f)g(\sigma(X,Y),\varphi Y). \end{aligned} \tag{30}$$

Taking into account of the covariant derivative and the Gauss formula in (30) we obtain

$$\begin{aligned} g(\widetilde{R}(X,\varphi X)Y,\varphi Y) &= X(X(\ln f))g(Y,Y)+2X(\ln f)g(\nabla_XY,Y) \\ &\quad -g(\sigma(\varphi X,Y),\varphi\nabla_XY)-g(\sigma(\varphi X,Y),\varphi\sigma(X,Y)) \\ &\quad +\varphi\nabla_X\varphi X(\ln f)g(Y,Y)-X(\ln f)g(\sigma(\varphi X,Y),\varphi Y) \\ &\quad +\varphi X(\varphi X(\ln f))g(Y,Y)+2\varphi X(\ln f)g(\nabla_{\varphi X}Y,Y) \\ &\quad +g(\sigma(X,Y),\varphi\nabla_{\varphi X}Y)+g(\sigma(X,Y),\varphi\sigma(X,Y)) \\ &\quad -\varphi\nabla_{\varphi X}X(\ln f)g(Y,Y)+\varphi X(\ln f)g(\sigma(X,Y),\varphi Y). \end{aligned}$$

By the use of Lemma 3.1, Lemma 4.1 and Lemma 4.2 in the above equation we get

$$\begin{aligned} g(\widetilde{R}(X,\varphi X)Y,\varphi Y) &= \{X(X(\ln f))+\varphi\nabla_X\varphi X(\ln f) \\ &\quad -\varphi\nabla_{\varphi X}X(\ln f)+\varphi X(\varphi X(\ln f)) \\ &\quad +2[\varphi X(\ln f)]^2\}g(Y,Y)-2\left\|\sigma(X,Y)\right\|^2. \end{aligned} \tag{31}$$

Since $M_\top$ is totally geodesic in M and it is an invariant submanifold of a trans-Sasakian manifold $\widetilde{M}$, from (4) we have

$$\varphi\nabla_X\varphi X = -\nabla_X X \tag{32}$$

and

$$\varphi\nabla_{\varphi X}X = \nabla_{\varphi X}\varphi X + \beta g(X,X)\xi. \tag{33}$$

By making use of (32) and (33) in (31), we obtain

$$\begin{aligned} g(\widetilde{R}(X,\varphi X)Y,\varphi Y) &= \{X(X(\ln f)) - \nabla_X X(\ln f) - \nabla_{\varphi X}\varphi X(\ln f) \\ &\quad -\beta g(X,X)\xi(\ln f) + \varphi X(\varphi X(\ln f)) \\ &\quad +2[\varphi X(\ln f)]^2\}g(Y,Y) - 2\left\|\sigma(X,Y)\right\|^2. \end{aligned}$$

Since $\xi(\ln f) = 0$, the above equation reduces to

$$\begin{aligned} g(\widetilde{R}(X,\varphi X)Y,\varphi Y) &= \{X(X(\ln f)) - \nabla_X X(\ln f) \\ &\quad +\varphi X(\varphi X(\ln f)) - \nabla_{\varphi X}\varphi X(\ln f) \\ &\quad +2[\varphi X(\ln f)]^2\}g(Y,Y) - 2\left\|\sigma(X,Y)\right\|^2, \end{aligned}$$

which gives us

$$\begin{aligned} g(\widetilde{R}(X,\varphi X)Y,\varphi Y) &= \{H^{\ln f}(X,X) + H^{\ln f}(\varphi X,\varphi X) \\ &\quad +2[\varphi X(\ln f)]^2\}g(Y,Y) - 2\left\|\sigma(X,Y)\right\|^2. \end{aligned} \tag{34}$$

On the other hand, since $\widetilde{M}$ is a generalized Sasakian space form, in view of (7) we get

$$g(\widetilde{R}(X,\varphi X)Y,\varphi Y) = -2f_2 g(X,X)g(Y,Y). \tag{35}$$

Hence, comparing the right hand sides of the equations (34) and (35) we can write

$$\begin{aligned} 2\left\|\sigma(X,Y)\right\|^2 &= \{H^{\ln f}(X,X) + H^{\ln f}(\varphi X,\varphi X) \\ &\quad +2[\varphi X(\ln f)]^2 + 2f_2 g(X,X)\}g(Y,Y). \end{aligned}$$

Thus, the proof of the lemma is completed. □

Theorem 4.4 *Let $M = M_\top \times_f M_\perp$ be a compact contact CR-warped product submanifold of a trans-Sasakian generalized Sasakian space form $\widetilde{M}(f_1,f_2,f_3)$. Then M is a contact CR-product if*

$$\sum_{i=1}^{p}\sum_{j=1}^{q}\left\|\sigma_\upsilon(e_i,e^j)\right\|^2 \geq f_2\cdot p\cdot q,$$

where σ_υ denotes the component of σ in υ, $(2p+1)$-dim$(TM_\top)$ and q-dim$(TM_\perp)$.

Proof. Let $\{e_0 = f, e_1, e_2, ..., e_p, \varphi e_1, \varphi e_2, ..., \varphi e_p, e^1, e^2, ..., e^q\}$ be an orthonormal basis of $\chi(M)$ such that $e_0, e_1, e_2, ..., e_p, \varphi e_1, \varphi e_2, ..., \varphi e_p$ are tangent to $M_\top$ and $e^1, e^2, ..., e^q$ are tangent to $M_\perp$. Similarly, let $\{\varphi e^1, \varphi e^2, ..., \varphi e^q, N_1, N_2, ..., N_{2r}\}$ be an orthonormal basis of $\chi^\perp(M)$ such that $\varphi e^1, \varphi e^2, ..., \varphi e^q$ are tangent to $\varphi(T(M_\perp))$ and $N_1, N_2, ..., N_{2r}$ are tangent to $\chi(\upsilon)$.

In view of (16), we can write

$$\begin{aligned}\Delta \ln f &= -\sum_{i=1}^{p} g(\nabla_{e_i} \mathrm{grad} \ln f, e_i) - \sum_{i=1}^{p} g(\nabla_{\varphi e_i} \mathrm{grad} \ln f, \varphi e_i) \\ &\quad -\sum_{j=1}^{q} g(\nabla_{e^j} \mathrm{grad} \ln f, e^j) - g(\nabla_\xi \mathrm{grad} \ln f, \xi).\end{aligned}$$

Since $\widetilde{M}$ is trans-Sasakian, the induced connection is Levi-Civita and $\mathrm{grad} f \in \chi(M_\top)$ we have $g(\nabla_\xi \mathrm{grad} \ln f, \xi) = 0$. Hence, by the use of (15), the above equation can be written as

$$\Delta \ln f = -\sum_{i=1}^{p} \{H^{\ln f}(e_i, e_i) + H^{\ln f}(\varphi e_i, \varphi e_i)\} - \sum_{j=1}^{q} g(\nabla_{e^j} \mathrm{grad} \ln f, e^j).$$

Then, similar to the proof of the Theorem 3.4 in [3] we get

$$\begin{aligned}\Delta \ln f &= -\sum_{i=1}^{p} \{H^{\ln f}(e_i, e_i) + H^{\ln f}(\varphi e_i, \varphi e_i)\} \\ &\quad -\sum_{j=1}^{q} \left\{ e^j \left(\frac{g(\mathrm{grad} f, e^j)}{f} \right) - \frac{1}{f} g(\nabla_{e^j} e^j, \mathrm{grad} f) \right\}.\end{aligned}$$

By the use of Lemma 3.1, since $\mathrm{grad} f \in \chi(M_\top)$, we obtain

$$\Delta \ln f = -\sum_{i=1}^{p} \{H^{\ln f}(e_i, e_i) + H^{\ln f}(\varphi e_i, \varphi e_i)\} - q \left\| \mathrm{grad} \ln f \right\|^2. \tag{36}$$

On the other hand, taking $X = e_i$ and $Y = e^j$ in (28), where $1 \leq i \leq p$ and $1 \leq j \leq q$, we can write

$$\begin{aligned}2\sum_{i=1}^{p} \sum_{j=1}^{q} \left\| \sigma(e_i, e^j) \right\|^2 &= q\{\sum_{i=1}^{p} \{H^{\ln f}(e_i, e_i) + H^{\ln f}(\varphi e_i, \varphi e_i) \\ &\quad +2\sum_{i=1}^{p} [\varphi e_i(\ln f)]^2 + 2f_2 \cdot p\}.\end{aligned} \tag{37}$$

Comparing the equations (36) and (37), it can be easily seen that

$$-\Delta \ln f = \frac{2}{q} \sum_{i=1}^{p} \sum_{j=1}^{q} \left\| \sigma(e_i, e^j) \right\|^2 - 2\sum_{i=1}^{p} [\varphi e_i(\ln f)]^2 + q \left\| \mathrm{grad} \ln f \right\|^2 - 2f_2 \cdot p. \tag{38}$$

Furthermore, we can write the second fundamental form σ as follows

$$\sigma(e_i,e^j)=\sum_{k=1}^{q}g(\sigma(e_i,e^j),\varphi e^k)\varphi e^k+\sum_{l=1}^{2r}g(\sigma(e_i,e^j),N_l)N_l,$$

for each $1\leq i\leq p$ and $1\leq j\leq q$. Taking the inner product of the above equation with $\sigma(e_i,e^j)$ we get

$$\sum_{i=1}^{p}\sum_{j=1}^{q}g(\sigma(e_i,e^j),\sigma(e_i,e^j))=\sum_{i=1}^{p}\sum_{j,k=1}^{q}g(\sigma(e_i,e^j),\varphi e^k)^2+\sum_{i=1}^{p}\sum_{j=1}^{q}\sum_{l=1}^{2r}g(\sigma(e_i,e^j),N_l)^2.$$

Then by making use of Lemma 4.1, the last equation turns into

$$\sum_{i=1}^{p}\sum_{j=1}^{q}\left\|\sigma(e_i,e^j)\right\|^2=q\sum_{i=1}^{p}[\varphi e_i(\ln f)]^2+\sum_{i=1}^{p}\sum_{j=1}^{q}\left\|\sigma_\upsilon(e_i,e^j)\right\|^2. \tag{39}$$

So, comparing the equations (38) and (39) we obtain

$$-\Delta\ln f=\frac{2}{q}\sum_{i=1}^{p}\sum_{j=1}^{q}\left\|\sigma_\upsilon(e_i,e^j)\right\|^2+q\left\|\operatorname{grad}\ln f\right\|^2-2f_2\cdot p.$$

Since M is a compact submanifold, by virtue of (17) we can write

$$\int_M\left\{\sum_{i=1}^{p}\sum_{j=1}^{q}\left\|\sigma_\upsilon(e_i,e^j)\right\|^2+\frac{q^2}{2}\left\|\operatorname{grad}\ln f\right\|^2-f_2\cdot p\cdot q\right\}dV=0. \tag{40}$$

If

$$\sum_{i=1}^{p}\sum_{j=1}^{q}\left\|\sigma_\upsilon(e_i,e^j)\right\|^2\geq f_2\cdot p\cdot q,$$

then (40) gives us $\operatorname{grad}f=0$, which means that f is a constant on M. So, M is a contact CR-product. Hence, we finish the proof of the theorem. □

Proposition 4.5 *Let $M=M_\top\times_f M_\perp$ be a compact contact CR-warped product submanifold of a trans-Sasakian generalized Sasakian space form $\widetilde{M}(f_1,f_2,f_3)$. Then M is a contact CR-product if and only if*

$$\sum_{i=1}^{p}\sum_{j=1}^{q}\left\|\sigma_\upsilon(e_i,e^j)\right\|^2=f_2\cdot p\cdot q. \tag{41}$$

Proof. Assume that M is a compact contact CR-warped product submanifold of trans-Sasakian generalized Sasakian space form $\widetilde{M}$ satisfying

$$\sum_{i=1}^{p}\sum_{j=1}^{q}\left\|\sigma_\upsilon(e_i,e^j)\right\|^2=f_2\cdot p\cdot q.$$

Then, from (40) it is easy to see that f is a constant on M, which implies that M is a contact CR-product.

Conversely, if M is a contact CR-product, then f is a constant on M. So we get

$$g(\sigma(X,Y),\varphi Y) = -\varphi X(\ln f)g(Y,Y) = 0,$$

for any vector fields X on $M_\top$ and Y on $M_\perp$. So, the last equation can be written as

$$g(\varphi\sigma(X,Y),Y) = 0,$$

which gives us $B\sigma(X,Y) = 0$, i. e. $\sigma(X,Y) \in \chi(\upsilon)$. Hence, we obtain (41). □

As a consequence of the above proposition, we can give the following corollaries.

Corollary 4.6 [8] *Let $M = M_\top \times_f M_\perp$ be a compact contact CR-warped product submanifold of a Sasakian space form $\widetilde{M}(c)$. Then M is a contact CR-product if and only if*

$$\sum_{i=1}^{p}\sum_{j=1}^{q}\left\|\sigma_\upsilon(e_i,e^j)\right\|^2 = \frac{(c-1)}{4}p.q.$$

Corollary 4.7 [2] *Let $M = M_\top \times_f M_\perp$ be a compact contact CR-warped product submanifold of a Kenmotsu space form $\widetilde{M}(c)$. Then M is a contact CR-product if and only if*

$$\sum_{i=1}^{p}\sum_{j=1}^{q}\left\|\sigma_\upsilon(e_i,e^j)\right\|^2 = \frac{(c+1)}{4}p.q.$$

Corollary 4.8 [3] *Let $M = M_\top \times_f M_\perp$ be a compact contact CR-warped product submanifold of a cosymplectic space form $\widetilde{M}(c)$. Then M is a contact CR-product if and only if*

$$\sum_{i=1}^{p}\sum_{j=1}^{q}\left\|\sigma_\upsilon(e_i,e^j)\right\|^2 = \frac{c}{4}p.q.$$

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