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APPROXIMATION IN WEIGHTED L^p SPACES

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ABSTRACT. The Lipschitz classes $Lip\,(\alpha, p, w)\,, 0 < \alpha \leq 1$ are defined for the weighted Lebesgue spaces L^p_w with Muckenhoupt weights, and the degree of approximation by matrix transforms of $f \in Lip\,(\alpha, p, w)$ is estimated by $n^{-\alpha}$.

1. Introduction and the main results

A measurable 2π-periodic function $w : \mathbb{R} \to [0, \infty]$ is said to be a weight function if the set $w^{-1}(\{0, \infty\})$ has Lebesgue measure zero. We denote by $L^p_w = L^p_w([0, 2\pi])$, where $1 \leq p < \infty$ and w a weight function, the weighted Lebesgue space of all measurable $2\pi-$periodic functions f, that is, the space of all such functions for which

$$\|f\|_{p,w} = \left(\int_0^{2\pi} |f(x)|^p \, w(x)\, dx \right)^{1/p} < \infty.$$

Let $1 < p < \infty$. A weight function w belongs to the Muckenhoupt class $\mathcal{A}_p = \mathcal{A}_p([0, 2\pi])$ if

$$\sup_I \left(\frac{1}{|I|} \int_I w(x)\, dx \right) \left(\frac{1}{|I|} \int_I [w(x)]^{-1/p-1} \, dx \right)^{p-1} < \infty,$$

where the supremum is taken over all intervals I with length $|I| \leq 2\pi$.

The weight functions belong to the $\mathcal{A}_p$, class introduced by Muckenhoupt ([8]), play a very important role in different fields of Mathematical Analysis.

Denote by M the Hardy-Littlewood maximal operator, defined for $f \in L^1$ by

$$M(f)(x) = \sup_I \frac{1}{|I|} \int_I |f(t)|\, dt, \quad x \in [0, 2\pi],$$

where the supremum is taken over all subintervals I of $[0, 2\pi]$ with $x \in I$.

Let $1 < p < \infty$ and w be a weight function. In [8] it was proved that the maximal operator M is bounded on L^p_w, that is,

$$\|M(f)\|_{p,w} \leq c \, \|f\|_{p,w} \tag{1.3}$$

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for all $f \in L^p_w$, where c is a constant depends only on p, if and only if $w \in \mathcal{A}_p$.

Let $1 < p < \infty$, $w \in \mathcal{A}_p$ and $f \in L^p_w$. The modulus of continuity of the function f is defined by

$$\Omega(f,\delta)_{p,w} = \sup_{|h|\le\delta} \|\Delta_h(f)\|_{p,w}, \quad \delta > 0, \tag{1.4}$$

where

$$\Delta_h(f)(x) := \frac{1}{h}\int_0^h |f(x+t) - f(x)|\, dt. \tag{1.5}$$

The existence of $\Omega(f,\delta)_{p,w}$ follows from (1.3).

The modulus $\Omega(f,\cdot)_{p,w}$ is nonnegative, continuous function such that

$$\lim_{\delta\to 0} \Omega(f,\delta)_{p,w} = 0, \quad \Omega(f_1+f_2,\cdot)_{p,w} \le \Omega(f_1,\cdot)_{p,w} + \Omega(f_2,\cdot)_{p,w}.$$

In the Lebesgue spaces L^p $(1 < p < \infty)$, the classical modulus of continuity $\omega(f,\cdot)_p$ is defined by

$$\omega(f,\delta)_p = \sup_{0<h\le\delta} \|f(\cdot+h) - f\|_p, \quad \delta > 0. \tag{1.6}$$

It is known that in the Lebesgue spaces L^p the moduli of continuity (1.4) and (1.6) are equivalent (see [5]).

We define in the spaces L^p_w the modulus of continuity by using the shift (1.5), because the space L^p_w is not translation invariant. The idea of defining the modulus of continuity by (1.4) was developed in [5].

Let $1 < p < \infty$, $w \in \mathcal{A}_p$, $f \in L^p_w$ and $0 < \alpha \le 1$. We define the Lipschitz class $Lip(\alpha,p,w)$ as

$$Lip(\alpha,p,w) = \left\{ f \in L^p_w : \Omega(f,\delta)_{p,w} = O(\delta^\alpha), \delta > 0 \right\}.$$

Let $f \in L^1$ has the Fourier series

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx). \tag{1.7}$$

Denote by $S_n(f)(x)$, $n = 0, 1, \ldots$ the nth partial sums of the series (1.7) at the point x, that is,

$$S_n(f)(x) = \sum_{k=0}^{n} u_k(f)(x),$$

where

$$u_0(f)(x) = \frac{a_0}{2}, \quad u_k(f)(x) = a_k \cos kx + b_k \sin kx, \quad k = 1, 2, \ldots .$$

Let (p_n) be a sequence of positive numbers. The Nörlund means of the series (1.7) with respect to the sequence (p_n) are defined by

$$N_n(f)(x) = \frac{1}{P_n}\sum_{k=0}^{n} p_{n-k} S_k(f)(x), \tag{1.8}$$

where $P_n = \sum_{k=0}^{n} p_k$, and $p_{-1} = P_{-1} := 0$.

If $p_n = 1$ for $n = 0, 1, \ldots$, then $N_n(f)(x)$ coincides with the Cesàro means

$$\sigma_n(f)(x) = \frac{1}{n+1}\sum_{k=0}^{n} S_k(f)(x).$$

The sequence (p_n) is called almost monotone decreasing (increasing) if there exists a constant K, depending only on (p_n), such that $p_n \le K p_m$ $(p_m \le K p_n)$ for $n \ge m$.

In the non-weighted Lebesgue spaces L^p, the following results were obtained recently.

Theorem A ([1])**.** *Let $f \in Lip(\alpha, p)$ and (p_n) be a sequence of positive numbers such that $(n+1)p_n = O(P_n)$. If either*

(i) $p > 1$, $0 < \alpha \le 1$ and (p_n) is monotonic

or

(ii) $p = 1$, $0 < \alpha < 1$ and (p_n) is non-decreasing,

then

$$\|f - N_n(f)\|_p = O\left(n^{-\alpha}\right).$$

Theorem B ([6])**.** *Let $f \in Lip(\alpha, p)$ and (p_n) be a sequence of positive numbers. If one of the conditions*

(i) $p > 1$, $0 < \alpha < 1$ and (p_n) is almost monotone decreasing,

(ii) $p > 1$, $0 < \alpha < 1$, (p_n) is almost monotone increasing and $(n+1)p_n = O(P_n)$,

(iii) $p > 1$, $\alpha = 1$ and $\sum_{k=1}^{n-1} k\,|p_k - p_{k+1}| = O(P_n)$,

(iv) $p > 1$, $\alpha = 1$ and $\sum_{k=0}^{n-1} |p_k - p_{k+1}| = O(P_n/n)$,

(v) $p = 1$, $0 < \alpha < 1$ and $\sum_{k=-1}^{n-1} |p_k - p_{k+1}| = O(P_n/n)$

maintains, then

$$\|f - N_n(f)\|_p = O\left(n^{-\alpha}\right).$$

It is clear that Theorem B is more general than Theorem A.

In the weighted Lebesgue spaces L^p_w, where $1 < p < \infty$ and $w \in \mathcal{A}_p$ an analogue of Theorem A was proved in [3].

In the paper [7], the authors extended Theorem A to more general classes of triangular matrix methods.

Let $A = (a_{n,k})$ be an infinite lower triangular regular matrix with nonnegative entries and let $s_n^{(A)}$ $(n = 0, 1, \ldots)$ denote the row sums of this matrix, that is $s_n^{(A)} = \sum_{k=0}^{n} a_{n,k}$.

The matrix $A = (a_{n,k})$ is said to has monotone rows if, for each n, $(a_{n,k})$ is either non-increasing or non-decreasing with respect to k, $0 \le k \le n$.

For a given infinite lower triangular regular matrix $A = (a_{n,k})$ with nonnegative entries we consider the matrix transform

$$T_n^{(A)}(f)(x) = \sum_{k=0}^{n} a_{n,k} S_k(f)(x). \tag{1.9}$$

Theorem C ([7])**.** *Let $f \in Lip(\alpha, p)$, A has monotone rows and satisfy $\left|s_n^{(A)} - 1\right| = O(n^{-\alpha})$. If one of the conditions*

(i) $p > 1$, $0 < \alpha < 1$ and $(n+1)\max\{a_{n,0}, a_{n,r}\} = O(1)$ where $r = [n/2]$,

(ii) $p > 1$, $\alpha = 1$ and $(n+1)\max\{a_{n,0}, a_{n,r}\} = O(1)$ where $r = [n/2]$,

(iii) $p = 1$, $0 < \alpha < 1$ and $(n+1)\max\{a_{n,0}, a_{n,n}\} = O(1)$,

holds, then

$$\left\| f - T_n^{(A)}(f) \right\|_p = O\left(n^{-\alpha}\right).$$

For a given positive sequence (p_n), if we consider the lower triangular matrix with entries $a_{n,k} = p_{n-k}/P_n$, then the Nörlund transform (1.8) can be regarded as a matrix transform of the form (1.9). Further, in this case the conditions of Theorem A implies conditions of Theorem C and hence Theorem C is more general than Theorem A (see [7]).

In the present paper we give generalizations of Theorems B and C in weighted Lebesgue spaces.

We call the matrix $A = (a_{n,k})$ has almost monotone increasing (decreasing) rows if there exists a constant K, depending only on A, such that $a_{n,k} \leq K a_{n,m}$ ($a_{n,m} \leq K a_{n,k}$) for each n and $0 \leq k \leq m \leq n$.

Our main results are the following.

Theorem 1. *Let $1 < p < \infty$, $w \in \mathcal{A}_p$, $0 < \alpha < 1$, $f \in Lip(\alpha, p, w)$ and $A = (a_{n,k})$ be a lower triangular regular matrix with $\left|s_n^{(A)} - 1\right| = O(n^{-\alpha})$. If one of the conditions*

(i) A has almost monotone decreasing rows and $(n+1)a_{n,0} = O(1)$,

(ii) A has almost monotone increasing rows and $(n+1)a_{n,r} = O(1)$ where $r := [n/2]$,

holds, then

$$\left\| f - T_n^{(A)}(f) \right\|_{p,w} = O\left(n^{-\alpha}\right).$$

Theorem 2. *Let $1 < p < \infty$, $w \in \mathcal{A}_p$, $f \in Lip(1, p, w)$ and $A = (a_{n,k})$ be a lower triangular regular matrix with $\left|s_n^{(A)} - 1\right| = O\left(n^{-1}\right)$. If one of the conditions*

(i) $\sum_{k=1}^{n-1} |a_{n,k-1} - a_{n,k}| = O\left(n^{-1}\right)$,

(ii) $\sum_{k=1}^{n-1} (n-k)|a_{n,k-1} - a_{n,k}| = O(1)$,

holds, then

$$\left\| f - T_n^{(A)}(f) \right\|_{p,w} = O\left(n^{-1}\right).$$

Let (p_n) be a sequence of positive numbers, $0<\alpha<1$ and $1<p<\infty$. Consider the lower triangular matrix $A=(a_{n,k})$ with $a_{n,k}=p_{n-k}/P_n$. It is clear that in this case $s_n^{(A)}=1$.

If (p_n) is almost monotone decreasing, then the Nörlund matrix A has almost monotone increasing rows and

$$(n+1)\,a_{n,r}\leq (n+1)\,K a_{n,n}=K\,(n+1)\,\frac{p_0}{P_n}\leq 1,$$

where $r=[n/2]$. Thus, A satisfies the condition (ii) of Theorem 1.

If (p_n) is almost monotone increasing and $(n+1)\,p_n=O\,(P_n)$, then A has almost monotone decreasing rows and

$$(n+1)\,a_{n,0}=(n+1)\,\frac{p_n}{P_n}=\frac{1}{P_n}O\,(P_n)=O\,(1)\,.$$

Thus, A satisfies the condition (i) of Theorem 1.

Hence part (ii) of Theorem 1 is general than part (i) of Theorem B and and part (i) of Thorem 1 is general than part (ii) of Theorem B even in the case $w\,(x)\equiv 1$.

Also, it is clear that parts (i) and (ii) of Theorem 1 are general than corresponding parts of Theorem C.

Now let $p>1$, $\alpha=1$ and $\sum\limits_{k=1}^{n-1}k\,|p_k-p_{k+1}|=O\,(P_n)$. Then,

$$\begin{aligned}\sum_{k=1}^{n-1}(n-k)\,|a_{n,k-1}-a_{n,k}| &= \sum_{k=1}^{n-1}(n-k)\left|\frac{p_{n-k+1}}{P_n}-\frac{p_{n-k}}{P_n}\right| \\ &= \frac{1}{P_n}\sum_{k=1}^{n-1}k\,|p_k-p_{k+1}|=\frac{1}{P_n}O\,(P_n) \\ &= O\,(1)\,.\end{aligned}$$

Thus, the Nörlund matrix $A=(p_{n-k}/P_n)$ satisfies the condition (ii) of Theorem 2. Hence, part (iii) of Theorem B is a special case of part (ii) of Theorem 2. Similarly, one can easily show that part (i) of Theorem 2 is general than part (iv) of Theorem B even if $w\,(x)\equiv 1$.

2. LEMMAS

Lemma 1 ([3]). *Let* $1<p<\infty$, $w\in\mathcal{A}_p$ and $0<\alpha\leq 1$. *Then for every* $f\in Lip\,(\alpha,p,w)$ *the estimate*

$$\|f-S_n\,(f)\|_{p,w}=O\left(n^{-\alpha}\right),\quad n=1,2,\ldots \tag{2.1}$$

holds.

Lemma 2 ([3]). *Let* $1<p<\infty$, $w\in\mathcal{A}_p$, $0<\alpha\leq 1$ *and* $f\in Lip\,(1,p,w)$. *Then for* $n=1,2,\ldots$ *the estimate*

$$\|S_n\,(f)-\sigma_n\,(f)\|_{p,w}=O\left(n^{-1}\right) \tag{2.2}$$

holds.

In the non-weighted Lebesgue spaces L^p, $1 < p < \infty$, the analogue of Lemma 2 was proved in [9].

Lemma 3. *Let $A = (a_{n,k})$ be an infinite lower triangular matrix and $0 < \alpha < 1$. If one of the conditions*

(i) A has almost monotone decreasing rows and $(n+1)\, a_{n,0} = O(1)$,

(ii) A has almost monotone increasing rows, $(n+1)\, a_{n,r} = O(1)$ where $r := [n/2]$, and $\left| s_n^{(A)} - 1 \right| = O\left(n^{-\alpha}\right)$,

holds, then

$$\sum_{k=1}^{n} k^{-\alpha} a_{n,k} = O\left(n^{-\alpha}\right). \tag{2.3}$$

Proof.

(i) Since $\sum\limits_{k=1}^{n} k^{-\alpha} = O\left(n^{1-\alpha}\right)$ and $a_{n,k} \leq K a_{n,0}$ for $k = 1, \ldots, n$, we get

$$\begin{aligned} \sum_{k=1}^{n} k^{-\alpha} a_{n,k} &\leq K a_{n,0} \sum_{k=1}^{n} k^{-\alpha} \\ &= O\left(\frac{1}{n+1}\right) O\left(n^{1-\alpha}\right) \\ &= O\left(n^{-\alpha}\right). \end{aligned}$$

(ii) Since $a_{n,k} \leq K a_{n,r}$ for $k = 1, \ldots, r$ and $\left| s_n^{(A)} - 1 \right| = O\left(n^{-\alpha}\right)$,

$$\begin{aligned} \sum_{k=1}^{n} k^{-\alpha} a_{n,k} &= \sum_{k=1}^{r} k^{-\alpha} a_{n,k} + \sum_{k=r+1}^{n} k^{-\alpha} a_{n,k} \\ &\leq K a_{n,r} \sum_{k=1}^{r} k^{-\alpha} + (r+1)^{-\alpha} \sum_{k=r+1}^{n} a_{n,k} \\ &\leq K a_{n,r} \sum_{k=1}^{n} k^{-\alpha} + (r+1)^{-\alpha} \sum_{k=0}^{n} a_{n,k} \\ &= O\left(\frac{1}{n+1}\right) O\left(n^{1-\alpha}\right) + O\left(n^{-\alpha}\right) s_n^{(A)} \\ &= O\left(n^{-\alpha}\right). \blacksquare \end{aligned}$$

3. Proofs of the main results

Proof of Theorem 1. By definition of $T_n^{(A)}(f)$, we have

$$\begin{aligned}
T_n^{(A)}(f)(x) - f(x) &= \sum_{k=0}^{n} a_{n,k} S_k(f)(x) - f(x) \\
&= \sum_{k=0}^{n} a_{n,k} S_k(f)(x) - f(x) + s_n^{(A)} f(x) - s_n^{(A)} f(x) \\
&= \sum_{k=0}^{n} a_{n,k} \left(S_k(f)(x) - f(x)\right) + \left(s_n^{(A)} - 1\right) f(x).
\end{aligned}$$

Hence, by (2.1) and (2.3) we obtain

$$\begin{aligned}
\left\| f - T_n^{(A)}(f) \right\|_{p,w} &\leq \sum_{k=1}^{n} a_{n,k} \left\| S_k(f) - f \right\|_{p,w} + a_{n,0} \left\| S_0(f) - f \right\|_{p,w} \\
&\quad + \left| s_n^{(A)} - 1 \right| \left\| f \right\|_{p,w} \\
&= \sum_{k=1}^{n} a_{n,k} k^{-\alpha} + O\left(\frac{1}{n+1}\right) + O\left(n^{-\alpha}\right) \\
&= O\left(n^{-\alpha}\right),
\end{aligned}$$

since $\left| s_n^{(A)} - 1 \right| = O\left(n^{-\alpha}\right)$. ■

Proof of Theorem 2. By (2.1),

$$\begin{aligned}
\left\| f - T_n^{(A)}(f) \right\|_{p,w} &\leq \left\| S_n(f) - T_n^{(A)}(f) \right\|_{p,w} + \left\| f - S_n(f) \right\|_{p,w} \\
&= \left\| S_n(f) - T_n^{(A)}(f) \right\|_{p,w} + O\left(n^{-1}\right).
\end{aligned}$$

Thus, we have to show that

$$\left\| S_n(f) - T_n^{(A)}(f) \right\|_{p,w} = O\left(n^{-1}\right). \tag{3.1}$$

Set $A_{n,k} := \sum_{m=k}^{n} a_{n,m}$. Hence,

$$\begin{aligned}
T_n^{(A)}(f)(x) &= \sum_{k=0}^{n} a_{n,k} S_k(f)(x) = \sum_{k=0}^{n} a_{n,k} \left(\sum_{m=0}^{k} u_m(f)(x) \right) \\
&= \sum_{k=0}^{n} \left(\sum_{m=k}^{n} a_{n,m} \right) u_k(f)(x) = \sum_{k=0}^{n} A_{n,k} u_k(f)(x).
\end{aligned}$$

On the other hand,

$$\begin{aligned} S_n(f)(x) &= \sum_{k=0}^{n} u_k(f)(x) = A_{n,0}\sum_{k=0}^{n} u_k(f)(x) + (1 - A_{n,0})\sum_{k=0}^{n} u_k(f)(x) \\ &= \sum_{k=0}^{n} A_{n,0} u_k(f)(x) + \left(1 - s_n^{(A)}\right) S_n(f)(x). \end{aligned}$$

Thus,

$$T_n^{(A)}(f)(x) - S_n(f)(x) = \sum_{k=1}^{n} (A_{n,k} - A_{n,0}) u_k(f)(x) + \left(s_n^{(A)} - 1\right) S_n(f)(x).$$

By boundedness of the partial sums in the space L_w^p (see [4]) we get

$$\begin{aligned} \left\| S_n(f) - T_n^{(A)}(f) \right\|_{p,w} &\leq \left\| \sum_{k=1}^{n} (A_{n,k} - A_{n,0}) u_k(f) \right\|_{p,w} + \left| s_n^{(A)} - 1 \right| \|f\|_{p,w} \qquad (3.2) \\ &= \left\| \sum_{k=1}^{n} (A_{n,k} - A_{n,0}) u_k(f) \right\|_{p,w} + O\left(n^{-1}\right). \end{aligned}$$

Thus, the problem reduced to proving that

$$\left\| \sum_{k=1}^{n} (A_{n,k} - A_{n,0}) u_k(f) \right\|_{p,w} = O\left(n^{-1}\right). \qquad (3.3)$$

If we set

$$b_{n,k} := \frac{A_{n,k} - A_{n,0}}{k}, \quad k = 1, ..., n,$$

Abel transform yields

$$\begin{aligned} \sum_{k=1}^{n} (A_{n,k} - A_{n,0}) u_k(f) &= \sum_{k=1}^{n} b_{n,k} k u_k(f) \\ &= b_{n,n} \sum_{m=1}^{n} m u_m(f) + \sum_{k=1}^{n-1} (b_{n,k} - b_{n,k+1}) \left(\sum_{m=1}^{k} m u_m(f) \right). \end{aligned}$$

Hence,

$$\begin{aligned} \left\| \sum_{k=1}^{n} (A_{n,k} - A_{n,0}) u_k(f) \right\|_{p,w} &\leq |b_{n,n}| \left\| \sum_{m=1}^{n} m u_m(f) \right\|_{p,w} \\ &\quad + \sum_{k=1}^{n-1} |b_{n,k} - b_{n,k+1}| \left(\left\| \sum_{m=1}^{k} m u_m(f) \right\|_{p,w} \right). \end{aligned}$$

Considering (2.2), we have

$$\begin{aligned}\left\|\sum_{m=1}^{n} m u_m(f)\right\|_{p,w} &= (n+1)\left\|S_n(f)-\sigma_n(f)\right\|_{p,w}\\ &= (n+1)\,O\left(n^{-1}\right)=O(1).\end{aligned}$$

This and the previous inequality yield

$$\left\|\sum_{k=1}^{n}\left(A_{n,k}-A_{n,0}\right)u_k(f)\right\|_{p,w}=O(1)\left|b_{n,n}\right|+O(1)\sum_{k=1}^{n-1}\left|b_{n,k}-b_{n,k+1}\right|. \tag{3.4}$$

Since $\left|s_n^{(A)}-1\right|=O\left(n^{-1}\right)$,

$$\begin{aligned}\left|b_{n,n}\right| &= \frac{\left|A_{n,n}-A_{n,0}\right|}{n}=\frac{\left|a_{n,n}-s_n^{(A)}\right|}{n}\\ &= \frac{1}{n}\left(s_n^{(A)}-a_{n,n}\right)\le\frac{1}{n}s_n^{(A)}\\ &= \frac{1}{n}O(1)=O\left(n^{-1}\right).\end{aligned} \tag{3.5}$$

Therefore, it is remained to prove that

$$\sum_{k=1}^{n-1}\left|b_{n,k}-b_{n,k+1}\right|=O\left(n^{-1}\right). \tag{3.6}$$

A simple calculation yields

$$b_{n,k}-b_{n,k+1}=\frac{1}{k(k+1)}\left\{(k+1)\,a_{n,k}-\sum_{m=0}^{k}a_{n,m}\right\}.$$

(i) Let $\sum\limits_{k=1}^{n-1}\left|a_{n,k-1}-a_{n,k}\right|=O\left(n^{-1}\right)$.

Let's verify by induction that

$$\left|\sum_{m=0}^{k}a_{n,m}-(k+1)\,a_{n,k}\right|\le\sum_{m=1}^{k}m\left|a_{n,m-1}-a_{n,m}\right| \tag{3.7}$$

for $k=1,\ldots,n$.

If $k=1$, then

$$\left|\sum_{m=0}^{1}a_{n,m}-2a_{n,1}\right|=\left|a_{n,0}-a_{n,1}\right|,$$

thus (3.7) holds. Now let us assume that (3.7) is true for $k=\nu$. For $k=\nu+1$,

$$\begin{aligned}
\left|\sum_{m=0}^{\nu+1} a_{n,m}-(\nu+2)\,a_{n,\nu+1}\right| &= \left|\sum_{m=0}^{\nu} a_{n,m}-(\nu+1)\,a_{n,\nu+1}\right| \\
&\leq \left|\sum_{m=0}^{\nu} a_{n,m}-(\nu+1)\,a_{n,\nu}\right| \\
&\quad +\left|(\nu+1)\,a_{n,\nu}-(\nu+1)\,a_{n,\nu+1}\right| \\
&\leq \sum_{m=1}^{\nu} m\left|a_{n,m-1}-a_{n,m}\right|+(\nu+1)\left|a_{n,\nu}-a_{n,\nu+1}\right| \\
&= \sum_{m=1}^{\nu+1} m\left|a_{n,m-1}-a_{n,m}\right|,
\end{aligned}$$

and hence (3.7) holds for $k=1,\ldots,n$. Therefore,

$$\begin{aligned}
\sum_{k=1}^{n-1}\left|b_{n,k}-b_{n,k+1}\right| &= \sum_{k=1}^{n-1}\left|\frac{1}{k\,(k+1)}\left\{(k+1)\,a_{n,k}-\sum_{m=0}^{k} a_{n,m}\right\}\right| \\
&= \sum_{k=1}^{n-1}\frac{1}{k\,(k+1)}\left|\sum_{m=0}^{k} a_{n,m}-(k+1)\,a_{n,k}\right| \\
&\leq \sum_{k=1}^{n-1}\frac{1}{k\,(k+1)}\sum_{m=1}^{k} m\left|a_{n,m-1}-a_{n,m}\right| \\
&= \sum_{m=1}^{n-1} m\left|a_{n,m-1}-a_{n,m}\right|\sum_{k=m}^{n-1}\frac{1}{k\,(k+1)} \\
&\leq \sum_{m=1}^{n-1} m\left|a_{n,m-1}-a_{n,m}\right|\sum_{k=m}^{\infty}\frac{1}{k\,(k+1)} \\
&= \sum_{m=1}^{n-1}\left|a_{n,m-1}-a_{n,m}\right| \\
&= O\left(n^{-1}\right).
\end{aligned}$$

(ii) Let $\sum\limits_{k=1}^{n-1}(n-k)\left|a_{n,k-1}-a_{n,k}\right|=O(1)$.

By (3.7),

$$\begin{aligned}\sum_{k=1}^{n-1}|b_{n,k}-b_{n,k+1}| &\leq \sum_{k=1}^{n-1}\frac{1}{k(k+1)}\sum_{m=1}^{k}m|a_{n,m-1}-a_{n,m}| \\ &\leq \sum_{k=1}^{r}\frac{1}{k(k+1)}\sum_{m=1}^{k}m|a_{n,m-1}-a_{n,m}| \\ &\quad+\sum_{k=r}^{n-1}\frac{1}{k(k+1)}\sum_{m=1}^{k}m|a_{n,m-1}-a_{n,m}|,\end{aligned}$$

where $r:=[n/2]$. By Abel transform,

$$\begin{aligned}\sum_{k=1}^{r}\frac{1}{k(k+1)}\sum_{m=1}^{k}m|a_{n,m-1}-a_{n,m}| &\leq \sum_{k=1}^{r}|a_{n,k-1}-a_{n,k}| \\ &= \sum_{k=1}^{r}\frac{1}{n-k}(n-k)|a_{n,k-1}-a_{n,k}| \\ &\leq \frac{1}{n-r}\sum_{k=1}^{r}(n-k)|a_{n,k-1}-a_{n,k}| \\ &= \frac{1}{n-r}O(1)=O\left(n^{-1}\right).\end{aligned}$$

On the other hand

$$\begin{aligned}&\sum_{k=r}^{n-1}\frac{1}{k(k+1)}\sum_{m=1}^{k}m|a_{n,m-1}-a_{n,m}| \\ \leq\ &\sum_{k=r}^{n-1}\frac{1}{k(k+1)}\left\{\sum_{m=1}^{r}m|a_{n,m-1}-a_{n,m}|+\sum_{m=r}^{k}m|a_{n,m-1}-a_{n,m}|\right\} \\ =\ &\sum_{k=r}^{n-1}\frac{1}{k(k+1)}\sum_{m=1}^{r}m|a_{n,m-1}-a_{n,m}|+\sum_{k=r}^{n-1}\frac{1}{k(k+1)}\sum_{m=r}^{k}m|a_{n,m-1}-a_{n,m}| \\ =\ &:I_{n1}+I_{n2}.\end{aligned}$$

Since $\sum_{k=1}^{r}|a_{n,k-1}-a_{n,k}|=O\left(n^{-1}\right)$,

$$\begin{aligned}I_{n1} &\leq \sum_{k=r}^{n-1}\frac{1}{k+1}\sum_{m=1}^{r}|a_{n,m-1}-a_{n,m}| \\ &= O\left(n^{-1}\right)\sum_{k=r}^{n-1}\frac{1}{k+1} \\ &= O\left(n^{-1}\right)(n-r)\frac{1}{r+1} \\ &= O\left(n^{-1}\right).\end{aligned}$$

Let's also estimate I_{n2}.

$$\begin{aligned}
I_{n2} &= \sum_{k=r}^{n-1} \frac{1}{k(k+1)} \sum_{m=r}^{k} m\left|a_{n,m-1}-a_{n,m}\right| \\
&\leq \sum_{k=r}^{n-1} \frac{1}{k+1} \sum_{m=r}^{k} \left|a_{n,m-1}-a_{n,m}\right| \\
&\leq \frac{1}{r+1} \sum_{k=r}^{n-1} \left(\sum_{m=r}^{k} \left|a_{n,m-1}-a_{n,m}\right| \right) \\
&\leq \frac{2}{n} \sum_{k=r}^{n-1} \left(\sum_{m=r}^{k} \left|a_{n,m-1}-a_{n,m}\right| \right) \\
&= \frac{2}{n} \sum_{k=n-r}^{n-1} (n-k)\left|a_{n,k-1}-a_{n,k}\right| \\
&\leq \frac{2}{n} \sum_{k=1}^{n-1} (n-k)\left|a_{n,k-1}-a_{n,k}\right| \\
&= \frac{2}{n} O(1) = O\left(n^{-1}\right).
\end{aligned}$$

Thus

$$\sum_{k=r}^{n-1} \frac{1}{k(k+1)} \sum_{m=1}^{k} m\left|a_{n,m-1}-a_{n,m}\right| = O\left(n^{-1}\right),$$

and hence

$$\sum_{k=1}^{n-1} \left|b_{n,k}-b_{n,k+1}\right| = O\left(n^{-1}\right).$$

Therefore, (3.6) is verified both in cases (i) and (ii). Finally, combining (3.1), (3.2), (3.3), (3.4), (3.5) and (3.6) finishes the proof. ■

References

[1] P. Chandra, *Trigonometric approximation of functions in L_p-norm*, J. Math. Anal. Appl. **275** (2002), 13–26. 13

[2] R. A. Devore, G. G. Lorentz, Constructive Approximation, Springer-Verlag (1993).

[3] A. Guven, *Trigonometric approximation of functions in weighted L^p spaces*, Sarajevo J. Math. **5** (17) (2009), 99–108. 13, 15

[4] R. Hunt, B. Muckenhoupt, R. Wheeden, *Weighted Norm Inequalities for the Conjugate Function and Hilbert Transform*, Trans. Amer. Math. Soc. **176** (1973), 227–251. 18

[5] N. X. Ky, *Moduli of mean smoothness and approximation with A_p-weights*, Annales Univ. Sci. Budapest **40** (1997), 37–48. 12

[6] L. Leindler, *Trigonometric approximation in L_p-norm*, J. Math. Anal. Appl. **302** (2005), 129–136. 13

[7] M. L. Mittal, B. E. Rhoades, V. N. Mishra, U. Singh, *Using infinite matrices to approximate functions of class $Lip(\alpha, p)$ using trigonometric polynomials*, J. Math. Anal. Appl. **326** (2007), 667–676. 13, 14

[8] B. Muckenhoupt, *Weighted norm inequalities for the Hardy maximal function,* Trans. Amer. Math. Soc. **165** (1972), 207–226. 11

[9] E. S. Quade, *Trigonometric approximation in the mean*, Duke Math. J. **3** (1937), 529–542. 16

[10] A. Zygmund, Trigonometric Series, Vol I, Cambridge Univ. Press, 2nd edition, (1959).

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