DOI: 10.2478/v10309-012-0004-9
An. Şt. Univ. Ovidius Constanţa Vol. 20(1), 2012, 41–54

SPHERICAL PRODUCT SURFACES IN $\mathbb{E}^4$

Betül Bulca, Kadri Arslan, Bengü (Kılıç) Bayram, Günay Öztürk

Abstract

In the present study we calculate the coefficients of the second fundamental form and curvature ellipse of spherical product surfaces in $\mathbb{E}^4$. Otsuki rotational surfaces and Ganchev-Milousheva rotational surfaces are the special type of spherical product surfaces in $\mathbb{E}^4$. Further, we give necessary and sufficient condition for the origin of N_pM to lie on the curvature ellipse of such surfaces. Finally we get the necessary condition for Ganchev-Milousheva rotational surfaces in $\mathbb{E}^4$ to become flat or Chen type. We also give some examples of the projections of these surfaces in $\mathbb{E}^3$.

1 Introduction

Let M be a smooth surface embedded by $X(u,v)$ in $\mathbb{E}^4$. Given $p \in M$ consider the unit circle in T_pM parametrized by the angle $\theta \in [0, 2\pi]$. Denote by γ_θ, the curve obtained by intersecting M with the hyperplane (3-space) at p composed by the direct sum of the normal plane N_pM and the straight line in tangent direction represented by θ. Such a curve is called *normal section* of M in the direction of θ. The curvature vector η_θ of γ_θ in M lies in N_pM. Varying θ from 0 to 2π, this vector describes an ellipse in N_pM, called the *curvature ellipse* of M at p. A point p in M is said to be hyperbolic, parabolic or elliptic according to whether p lies outside or inside the curvature ellipse of M at p. This ellipse may degenerate on a radial segment of straight line, in which case p is known as an inflection point of the surface. The inflection point is of real

Key Words: Second fundamental form, Curvature ellipse, spherical product, Otsuiki surface.
2010 Mathematics Subject Classification: 53C40, 53C42.
Received: January, 2011.
Revised: March, 2011.
Accepted: January, 2012.

type when p belongs to the curvature ellipse, and of imaginary type when it does not. An inflection point is flat when p is an end point of the curvature ellipse [14].

In [3] B.Y. Chen defined the allied vector field $a(v)$ of a normal vector field v. In particular, the allied mean curvature vector field is orthogonal to H. Further, B.Y. Chen defined the $\mathcal{A}$-surface to be the surfaces for which $a(H)$ vanishes identically. Such surfaces are also called Chen surfaces [7]. The class of Chen surfaces contains all minimal and pseudo-umbilical surfaces, and also all surfaces for which $dimN_1 \leq 1$, where N_1 is the first normal space of M, in particular it includes all hypersurfaces. These Chen surfaces are said to be Trivial $\mathcal{A}$-surfaces [8]. For more details, see also [4], [9], [12] and [16].

Rotational embeddings are special products which are introduced first by N.H. Kuiper in 1970 [11]. Recently the second and third authors studied with these type of embeddings [1]. Spherical products $X = \alpha \otimes \beta$ of two $2D$ curves are the special type of rotational embeddings [10]. Surface of revolution is a simple example of spherical product which is also a rotational embedding. All quadratics and superquadrics can be considered as spherical products of two $2D$ curves. Actually, superquadrics are solid models that can fairly simple parametrization of representing a large variety of standard geometric solids, as well as smooth shapes in between. This makes them much more convenient for representing rounded, blob-like shape parts, typical for object formed by natural process [10].

In the present study we define spherical product $X = \alpha \otimes \beta$ of a $3D$ (space) curve $\alpha(u) = (f_1(u), f_2(u), f_3(u))$ with a $2D$ curve $\beta(v) = (g_1(v),\ g_2(v))$ in $\mathbb{E}^4$. For the case $f_1(u) = 0$ or $f_2(u) = 0$, the patch $X = \alpha \otimes \beta : \mathbb{E}^2 \longrightarrow \mathbb{E}^3$ becomes a spherical product of two $2D$ curves [2]. In [15], T. Otsuki considered the special case $\alpha(u) = (f_1(u), f_2(u), \sin u)$ and $\beta(v) = (\cos v, \sin v)$ such that $X = \alpha \otimes \beta : \mathbb{S}^2 \longrightarrow \mathbb{E}^4$ is a surface patch in $\mathbb{E}^4$. Recently, G. Ganchev and V. Milousheva considered the special case $\alpha(u) = (f_1(u), f_2(u), f_3(u))$ and $\beta(v) = (\cos v, \sin v)$ which is a rotational embedding in $\mathbb{E}^4$ [6]. We calculate the coefficient of the second fundamental form and curvature ellipse of Ganchev-Milousheva surface. Further, we give necessary and sufficient condition for the origin of N_pM to lie on the curvature ellipse of such surfaces. We give necessary condition for the Ganchev-Milousheva surface to become flat or nontrivial Chen surface. Finally, we give some examples of the projections of these surfaces in $\mathbb{E}^3$.

2 Basic Concepts

Let M be a smooth surface immersed in $\mathbb{E}^4$ with the Riemannian metric induced by the standard Riemannian metric of $\mathbb{E}^4$. For each $p \in M$, consider the decomposition $T_p\mathbb{E}^4 = T_pM \oplus N_pM$ where N_pM is the orthogonal complement of T_pM in $\mathbb{E}^4$. Let $\widetilde{\nabla}$ be the Riemannian connection of $\mathbb{E}^4$. Given local vector fields e_1, e_2 on M . The induced connection on M is defined by $\nabla_{e_1} e_2 = \left(\widetilde{\nabla}_{e_1} e_2\right)^T$.

Let $\chi(M)$ and $N(M)$ be the space of the smooth vector fields tangent to M and the space of the smooth vector fields normal to M, respectively. Consider the second fundamental map:

$$h : \chi(M) \times \chi(M) \to N(M), \quad h(e_1, e_2) = \widetilde{\nabla}_{e_1} e_2 - \nabla_{e_1} e_2. \tag{1}$$

This map is well defined, symmetric and bilinear. Recall the shape operator

$$A_v : T_pM \to T_pM, \ A_v e_1 = -\left(\widetilde{\nabla}_{e_1} e_2\right)^T \tag{2}$$

where v is the normal vector field at $p \in M$ and T means the tangent component. This operator is bilinear, self-adjoint and for any $e_1, e_2 \in T_pM$ satisfies $\langle A_v e_1, e_2\rangle = \langle h(e_1, e_2), v\rangle$. We choose a local field of orthonormal frame e_1, e_2, e_3, e_4 on M such that, restricted to e_1, e_2 are tangent and e_3, e_4 are normal to M. It is well-known that the coefficients of the second fundamental form h satisfy

$$h^r_{ij} = \langle h(e_i, e_j), e_r\rangle \,, \quad i, j = 1, 2, \quad r = 3, 4. \tag{3}$$

Recall that a submanifold of a Riemannian manifold is said to be *minimal* if its *mean curvature vector* $H =\frac{1}{2}(h(e_1, e_1) + h(e_2, e_2))$ vanishes identically (see, for instance, [3]). In the case under consideration, $X(u, v)$ is *minimal* if and only if $h(e_1, e_1) + h(e_2, e_2) = 0$, where h denotes the *second fundamental form* of M, or equivalently $< h(e_1, e_1) + h(e_2, e_2), e_r >= 0, r = 3, 4$.

For a smooth surface M in $\mathbb{E}^4$, let γ_θ be the normal section of M in the direction of θ. Given an orthonormal basis $\{e_1, e_2\}$ of the tangent space T_pM at $p \in M$ denote $\gamma'_\theta = X = \cos\theta e_1 + \sin\theta e_2$ the unit vector of the normal section. The subset of the normal space defined as

$$\{h(X, X) : X \in TpM, \|X\| = 1\}$$

is called the curvature ellipse of M and denoted by $E(p)$, where h is the second fundamental form of the surface patch $X(u, v)$. To see that this is an ellipse, we just have to look at the following formula for:

$$X = \cos\theta e_1 + \sin\theta e_2$$

the unit vector that

$$h(X, X) = H + cos2\theta B + \sin 2\theta C, \tag{4}$$

where $H = \frac{1}{2}(h(e_1, e_1) + h(e_2, e_2))$ is the mean curvature vector of M at p and

$$B = \frac{1}{2}(h(e_1, e_1) - h(e_2, e_2)), C = h(e_1, e_2), \tag{5}$$

are the normal vectors. This shows that when X goes once around the unit tangent circle, the vector $h(X, X)$ goes twice around an ellipse centered at H, the curvature ellipse $E(p)$ of $X(u, v)$ at p. Clearly $E(p)$ can degenerate into a line segment or a point. It follows from (4) that $E(p)$ is a circle if and only if for some (and hence for any) orthonormal basis of T_pM it holds that $< B, C >= 0$ and $\|B\| = \|C\|$ [5]. General aspects of the curvature ellipse of surfaces in $\mathbb{E}^4$ studied by Wong [17]. For more details see also [13], [14], and [16].

We have the following functions associated to the coefficients of the second fundamental form :

$$\Delta(p) = \frac{1}{4}\det\begin{bmatrix} h_{11}^3 & 2h_{12}^3 & h_{22}^3 & 0 \\ h_{11}^4 & 2h_{12}^4 & h_{22}^4 & 0 \\ 0 & h_{11}^3 & 2h_{12}^3 & h_{22}^3 \\ 0 & h_{11}^4 & 2h_{12}^4 & h_{22}^4 \end{bmatrix}(p) \tag{6}$$

$$K(p) = \frac{1}{4}(h_{11}^3 h_{22}^3 - (h_{12}^3)^2 + h_{11}^4 h_{22}^4 - (h_{12}^4)^2)(p). \tag{7}$$

(Gaussian curvature of M) and the matrix

$$\alpha(p) = \begin{bmatrix} h_{11}^3 & h_{12}^3 & h_{22}^3 \\ h_{11}^4 & h_{12}^4 & h_{22}^4 \end{bmatrix}(p). \tag{8}$$

By identifying p with the origin of N_pM it can be shown that:

a) $\Delta(p) < 0 \Rightarrow p$ lies outside of the curvature ellipse (such a point is said to be a hyperbolic point of M),

b) $\Delta(p) > 0 \Rightarrow p$ lies inside the curvature ellipse (elliptic point),

c) $\Delta(p) = 0 \Rightarrow p$ lies on the curvature ellipse (parabolic point).

More detailed study of this case allows us to distinguish among the following possibilities:

d) $\Delta(p) = 0$, $K(p) > 0 \Rightarrow p$ is an inflection point of imaginary type,

e) $\Delta(p) = 0$, $K(p) < 0$ and $\begin{cases} \text{rank}\alpha(p) = 2 \Rightarrow \text{ellipse is non-degenerate} \\ \text{rank}\alpha(p) = 1 \Rightarrow p \text{ is an inflection point} \\ \text{of real type,} \end{cases}$

f) $\Delta(p) = 0$, $K(p) = 0 \Rightarrow p$ is an inflection point of flat type [14].

3 Spherical Product Surfaces in $\mathbb{E}^4$

Let $f : M \longrightarrow \mathbb{E}^{m+d}$ be an embedding of an m-dimensional manifold M into $(m+d)$-dimensional Euclidean space $\mathbb{E}^{m+d}$ and $g : S^n \longrightarrow \mathbb{E}^{n+1}$ be standard embedding on n-sphere . We define an embedding $x : M\times\ S^n \longrightarrow \mathbb{E}^{m+n+d}$ by

$$X(u,v) = (f_1(u), f_2(u), .., f_{m+d-1}(u), f_{m+d}(u)g(v)) \tag{9}$$

($f_1(u) \neq 0$ for all $u \in M$), $v \in S^n$. We call it rotational embedding. Here X is obtained from f by rotating $\mathbb{E}^n$ about $\mathbb{E}^{m+d-1}$ in $\mathbb{E}^{m+n+d}$ [11].

Definition 3.1. *Let α, β: $\mathbb{R} \longrightarrow \mathbb{E}^2$ be Euclidean plane curves. Put $\alpha(u) = (f_1(u), f_2(u))$ and $\beta(v) = (g_1(v), g_2(v))$. Then their spherical product patch is given by*

$$X = \alpha \otimes \beta : \mathbb{E}^2 \longrightarrow \mathbb{E}^3;\ X(u,v) = (f_1(u), f_2(u)g_1(v), f_2(u)g_2(v)); \tag{10}$$

$u \in I = (u_0, u_1), v \in J = (v_0, v_1)$, which is a surface in $\mathbb{E}^3$.

Superquadrics are a family of shapes that includes not only superellipsoids, but also superhyperboloids of one piece and superhyperboloids of two pieces, as well as supertoroids [10]. In computer vision literature, it is common to refer to superellipsoids by the more generic terms of superquadrics. The following position vector X defines a superquadric surface (see, [2]):

$$\begin{aligned} X(u,v) &= \alpha(u) \otimes \beta(v) = \begin{bmatrix} a_1 \sin^{\epsilon_1} u \\ \cos^{\epsilon_1} u \end{bmatrix} \otimes \begin{bmatrix} a_2 \cos^{\epsilon_2} v \\ a_3 \sin^{\epsilon_2} v \end{bmatrix} \\ &= \begin{bmatrix} a_1 \sin^{\epsilon_1} u \\ a_2 \cos^{\epsilon_1} u \cos^{\epsilon_2} v \\ a_3 \cos^{\epsilon_1} u \sin^{\epsilon_2} v \end{bmatrix}, -\frac{\pi}{2} < u < \frac{\pi}{2},\ \ -\pi \leq v < \pi. \end{aligned} \tag{11}$$

where a_1, a_2 and a_3 are scaling factors along the three coordinate axes. ϵ_1 and ϵ_2 are derived from the exponents of the two original superellipses. ϵ_2 determines the shape of the superellipsoid cross section parallel to the (x, y) plane, while ϵ_1 determines the shape of the superellipsoid cross section in a plane perpendicular to the (x, y) plane and containing z axis. Similarly, we define the spherical product patch of $\mathbb{E}^4$ as follows;

Definition 3.2. *Let $\alpha : \mathbb{R} \longrightarrow \mathbb{E}^3$ be an Euclidean space curve and $\beta : \mathbb{R} \longrightarrow \mathbb{E}^2$ Euclidean plane curve. Put $\alpha(u) = (f_1(u), f_2(u), f_3(u))$ and $\beta(v) = (g_1(v), g_2(v))$. Then their spherical product patch is given by*

$$X = \alpha\otimes\beta : \mathbb{E}^2 \longrightarrow \mathbb{E}^4;\ X(u,v) = (f_1(u), f_2(u), f_3(u)g_1(v), f_3(u)g_2(v)); \tag{12}$$

$u \in I = (u_0, u_1), v \in J = (v_0, v_1)$, which is a surface in $\mathbb{E}^4$. For the case $f_1(u) = 0$ or $f_2(u) = 0$, the patch $X = \alpha \otimes \beta : \mathbb{E}^2 \longrightarrow \mathbb{E}^3$ becomes a spherical product of two $2D$ curves.

Example 3.3. *In 1966, T. Otsuki considered the special case* $\alpha(u) = (f_1(u), f_2(u), \sin u)$ *and* $\beta(v) = (\cos v, \sin v)$ *such that*

$$X = \alpha \otimes \beta : \mathbb{S}^2 \longrightarrow \mathbb{E}^4;\ X(u,v) = (f_1(u), f_2(u), \sin u \cos v, \sin u \sin v); \quad (13)$$

$(\ u \in I,\ 0 \leq v < 2\pi)$ *is a surface patch in* $\mathbb{E}^4$*, where* $(f_1{}')^2 + (f_2{}')^2 = \sin^2 u$. *In the same paper T. Otsuki consider the following cases;*

$$\begin{aligned} a)\ f_1(u) &= \frac{4}{3}\cos^3(\frac{u}{2}),\ f_2(u) = \frac{4}{3}\sin^3(\frac{u}{2}),\ f_3(u) = \sin u, \quad (14) \\ b)\ f_1(u) &= \frac{1}{2}\sin^2 u \cos(2u),\ f_2(u) = \frac{1}{2}\sin^2 u \sin(2u),\ f_3(u) = \sin u. \quad (15) \end{aligned}$$

For the case a) the patch X is called Otsuki (non-round) sphere in $\mathbb{E}^4$ *which does not lie in a 3-dimensional subspace of* $\mathbb{E}^4$*. It has been shown that these surfaces have constant Gaussian curvature [15].*

Example 3.4. *Recently, G. Ganchev and V. Milousheva considered the general product of the space curve* $\alpha(u) = (f_1(u), f_2(u), f_3(u))$ *with the circle* $\beta(v) = (\cos v, \sin v)$ *such that*

$$X(u,v) = \alpha(u) \otimes \beta(v) = (f_1(u), f_2(u), f_3(u)\cos v, f_3(u)\sin v); \quad (16)$$

$u \in I,\ 0 \leq v < 2\pi$*, where* $\alpha(u)$ *is parametrized with respect to the arc-length, i.e.* $(f_1{}')^2 + (f_2{}')^2 + (f_3{}')^2 = 1$ *and* $f_3(u) > 0$*, [6].*

We give an extension of the superquadrics in $\mathbb{E}^4$.

Example 3.5. *The following position vector X defines a superquadric surface in* $\mathbb{E}^4$.

$$\begin{aligned} X(u,v) &= \alpha(u) \otimes \beta(v) = \begin{bmatrix} a_1 \cos^{2\epsilon_1} u \\ a_2 \cos^{\epsilon_1} u \sin^{\epsilon_1} u \\ \sin^{\epsilon_1} u \end{bmatrix} \otimes \begin{bmatrix} a_3 \cos^{\epsilon_2} v \\ a_4 \sin^{\epsilon_2} v \end{bmatrix} \\ &= \begin{bmatrix} a_1 \cos^{2\epsilon_1} u \\ a_2 \cos^{\epsilon_1} u \sin^{\epsilon_1} u \\ a_3 \sin^{\epsilon_1} u \cos^{\epsilon_2} v \\ a_4 \sin^{\epsilon_1} u \sin^{\epsilon_2} v \end{bmatrix}, -\frac{\pi}{2} < u < \frac{\pi}{2},\ -\pi \leq v < \pi \quad (17) \end{aligned}$$

By eliminating parameter u and v using equality $cos^2\alpha + sin^2\alpha = 1$*, the following implicit equation can be obtained*

$$\left(\left|\frac{x_3}{a_3}\right|^{\frac{2}{\epsilon_2}} + \left|\frac{x_4}{a_4}\right|^{\frac{2}{\epsilon_2}} \right)^{\frac{\epsilon_2}{\epsilon_1}} + \left|\frac{x_1}{a_1}\right|^{\frac{2}{\epsilon_1}} + \left|\frac{x_2}{a_2}\right|^{\frac{2}{\epsilon_1}} = 1 \quad (18)$$

where a_4 *is a positive real number.*

Consequently we have the following result.

Theorem 3.6. *Let M Ganchev-Milousheva rotation surface given by the parametrization (16).*

i) *If $\kappa_1 \neq 0$ then p lies outside of the curvature ellipse (such a point is said to be a hyperbolic point of M),*

ii) *If $\kappa_1 = 0$ then p lies on the curvature ellipse (parabolic point), which is an inflection point of real type ,*

iii) *If $\kappa_1 = 0$ and $f_3''(u) = 0$ then p is an inflection point of flat type,*

where p is the origin of N_pM and $\kappa_1 = f_1'f_2''(u) - f_1''f_2'(u)$ is the curvature of the projection of the curve α on the Oe_1e_2- plane .

Proof. The tangent space of $Im(X) = M$ is spanned by the vector fields

$$\begin{aligned}
\frac{\partial X}{\partial u} &= (f_1'(u), f_2'(u), f_3'(u)\cos v, f_3'(u)\sin v), \\
\frac{\partial X}{\partial v} &= (0, 0, -f_3(u)\sin v, f_3(u)\cos v).
\end{aligned}$$

We choose a moving frame e_1, e_2, e_3, e_4 such that e_1, e_2 are tangent to M and e_3, e_4 are normal to M as given the following:

$$\begin{aligned}
e_1 &= \frac{\frac{\partial X}{\partial u}}{\left\|\frac{\partial X}{\partial u}\right\|}, \quad e_2 = \frac{\frac{\partial X}{\partial v}}{\left\|\frac{\partial X}{\partial v}\right\|} \\
e_3 &= \frac{1}{\kappa}(f_1''(u), f_2''(u), f_3''(u)\cos v, f_3''(u)\sin v) \\
e_4 &= \frac{1}{\kappa}(f_2'f_3''(u) - f_2''f_3'(u), f_1''f_3'(u) - f_1'f_3''(u), \\
&\quad (f_1'f_2''(u) - f_1''f_2'(u))\cos v, (f_1'f_2''(u) - f_1''f_2'(u))\sin v
\end{aligned}$$

where $\kappa = \sqrt{(f_1'')^2 + (f_2'')^2 + (f_3'')^2}$ is the curvature of the space curve $\alpha(u)$.

Hence, the coefficients of the first fundamental form of the surface are

$$\begin{aligned}
E &= \ < X_u(u,v), X_u(u,v) > \ = 1, \\
F &= \ < X_u(u,v), X_v(u,v) > \ = 0, \\
G &= \ < X_v(u,v), X_v(u,v) > \ = f_3{}^2(u),
\end{aligned}$$

where $\langle , \rangle$ is the standard scalar product in $\mathbb{E}^4$. Since $EG - F^2 = f_3{}^2(u)$ does not vanishes then the surface patch $X(u,v)$ is regular.

The second partial derivatives of $X(u,v)$ are expressed as follows

$$\begin{aligned}
X_{uu}(u,v) &= (f_1''(u), f_2''(u), f_3''(u)\cos v, f_3''(u)\sin v), \\
X_{uv}(u,v) &= (0, 0, -f_3'(u)\sin v, f_3'(u)\cos v), \\
X_{vv}(u,v) &= (0, 0, -f_3(u)\cos v, -f_3(u)\sin v).
\end{aligned}$$

Using (1) and (3) we can get that the coefficients of the second fundamental form h are as follows:

$$\begin{aligned} h_{11}^3 &= \frac{< X_{uu}(u,v), e_3 >}{E} = \kappa, \quad h_{12}^3 = \frac{< X_{uv}(u,v), e_3 >}{\sqrt{EG}} = 0, \\ h_{22}^3 &= \frac{< X_{vv}(u,v), e_3 >}{G} = \frac{-f_3{}''}{\kappa f_3}, \\ h_{11}^4 &= \frac{< X_{uu}(u,v), e_4 >}{E} = 0, \quad h_{12}^4 = \frac{< X_{uv}(u,v), e_4 >}{\sqrt{EG}} = 0, \\ h_{22}^4 &= \frac{< X_{vv}(u,v), e_4 >}{G} = \frac{-\kappa_1}{\kappa f_3}, \end{aligned} \tag{19}$$

where κ is the curvature of the curve α and $\kappa_1 = f_1{}' f_2{}''(u) - f_1{}'' f_2{}'(u)$ is the curvature of the projection of the curve α on the Oe_1e_2- plane.

Thus, by the use of equations (6)-(8), we have

$$\Delta(p) = -\frac{1}{4}\frac{\kappa_1^2}{f_3^2}, \quad K(p) = \frac{-f_3{}''}{f_3}; \quad f_3(u) \neq 0, \tag{20}$$

and

$$\alpha(p) = \begin{bmatrix} \kappa & 0 & \frac{-f_3{}''}{\kappa f_3} \\ 0 & 0 & \frac{-\kappa_1}{\kappa f_3} \end{bmatrix}(p). \tag{21}$$

So, $\kappa_1 = 0$ implies $\Delta(p) = 0$, (and rank $(\alpha(p)) = 1$), and $f_3{}'' = 0$ implies $K = 0$. Hence, by identifying p with the origin of N_pM and using (20) with (21) we get the result. □

Definition 3.7. *Let M be an n-dimensional smooth submanifold of m-dimensional Riemannian manifold N and ζ be a normal vector field of M. Let ξ_x be $m-n$ mutually orthogonal unit normal vector fields of M such that $\zeta = \|\zeta\|\,\xi_1$. In [3] B.Y. Chen defined the allied vector field $a(\zeta)$ of a normal vector field ζ by the formula*

$$a(v) = \frac{\|\zeta\|}{n}\sum_{x=2}^{m-n}\{tr(A_1 A_x)\}\,\xi_x$$

where $A_x = A_{\xi_x}$ is the shape operator. In particular, the allied mean curvature vector field of the mean curvature vector H is a well-defined normal vector field orthogonal to H. If the allied mean vector $a(H)$ vanishes identically, then the submanifold M is called $\mathcal{A}$-submanifold of N. Furthermore, $\mathcal{A}$-submanifolds are also called Chen submanifolds [7]

For the case M is a smooth surface of $\mathbb{E}^4$ the allied vector $a(H)$ becomes

$$a(H) = \frac{\|H\|}{2}\left\{tr(A_{e_3} A_{e_4})\right\} e_4 \tag{22}$$

where $\{e_3, e_4\}$ is an orthonormal basis of N_pM.

Theorem 3.8. *[9] Let M be a non-trivial $\mathcal{A}$-surface in $\mathbb{E}^4$ with e_3 in the direction of H and e_1, e_2 are principal directions of A_{e_3}.*

i) *If the coefficients h^3_{11} and h^3_{22} are the same sign (resp. different sign) then the origin of N_pM lies outside (resp. inside) of the curvature ellipse of M.*

ii) *If one of the coefficients h^3_{11} or h^3_{22} is identically zero then the origin of N_pM lies on the curvature ellipse of M.*

We prove the following result.

Theorem 3.9. *Let M be Ganchev-Milousheva surface given by the parametrization (16). If M is a nontrivial Chen surface then the following equation fulfilled*

$$\kappa_1(\kappa^4 {f_3}^2 - \kappa_1^2 - ({f_3}'')^2) = 0. \tag{23}$$

Proof. Suppose M is a Ganchev-Milousheva rotational surface given by the parametrization (16). The mean curvature vector of M becomes

$$H = \frac{1}{2}(h(e_1, e_1) + h(e_2, e_2)) = \frac{1}{2}\left\{(\kappa - \frac{{f_3}''}{\kappa f_3})e_3 - \frac{\kappa_1}{\kappa f_3}e_4\right\}. \tag{24}$$

Since H is not parallel e_3, we can define another orthogonal frame field $\{n_1, n_2\}$ of M such that

$$n_1 = (\kappa - \frac{{f_3}''}{\kappa f_3})e_3 - \frac{\kappa_1}{\kappa f_3}e_4, \quad n_2 = \frac{\kappa_1}{\kappa f_3}e_3 + (\kappa - \frac{{f_3}''}{\kappa f_3})e_4.$$

For simplicity let us denote,

$$\lambda = \kappa - \frac{{f_3}''}{\kappa f_3}, \ \mu = \frac{\kappa_1}{\kappa f_3}, W^2 = \lambda^2 + \mu^2. \tag{25}$$

So, we can get the orthonormal frame field $\{\widetilde{e}_3, \widetilde{e}_4\}$ of M

$$\widetilde{e}_3 = \frac{n_1}{\|n_1\|} = \frac{\lambda e_3 - \mu e_4}{W}, \qquad \widetilde{e}_4 = \frac{n_2}{\|n_2\|} = \frac{\mu e_3 + \lambda e_4}{W}.$$

Using (1) and (3) we can get that the coefficients of the second fundamental form h are as following:

$$\begin{aligned}
\widetilde{h}_{11}^{3} &= \frac{< X_{uu}(u,v), \widetilde{e}_3 >}{E} = \frac{\lambda\kappa}{W}, \quad \widetilde{h}_{12}^{3} = \frac{< X_{uv}(u,v), \widetilde{e}_3 >}{\sqrt{EG}} = 0, \\
\widetilde{h}_{11}^{4} &= \frac{< X_{uu}(u,v), \widetilde{e}_4 >}{E} = \frac{\mu\kappa}{W}, \quad \widetilde{h}_{12}^{4} = \frac{< X_{uv}(u,v), \widetilde{e}_4 >}{\sqrt{EG}} = 0, \\
\widetilde{h}_{22}^{3} &= \frac{< X_{vv}(u,v), \widetilde{e}_3 >}{G} = \frac{-\lambda f_3{}''}{W\kappa f_3} + \frac{\mu^2}{W} = \beta, \\
\widetilde{h}_{22}^{4} &= \frac{< X_{vv}(u,v), \widetilde{e}_4 >}{G} = \frac{-\mu f_3{}''}{W\kappa f_3} - \frac{\lambda\mu}{W} = \gamma.
\end{aligned} \tag{26}$$

By the use of (26) the shape operator matrices with respect to $\{\widetilde{e}_3, \widetilde{e}_4\}$ become

$$A_{\widetilde{e}_3} = \begin{bmatrix} \frac{\lambda\kappa}{W} & 0 \\ 0 & \beta \end{bmatrix}, A_{\widetilde{e}_4} = \begin{bmatrix} \frac{\mu\kappa}{W} & 0 \\ 0 & \gamma \end{bmatrix}.$$

Further, the trace of the product matrix becomes

$$tr(A_{\widetilde{e}_3} A_{\widetilde{e}_4}) = \beta\gamma + \frac{\lambda\mu\kappa^2}{W^2}. \tag{27}$$

Suppose, M is a nontrivial Chen surface then $tr(A_{\widetilde{e}_3} A_{\widetilde{e}_4}) = 0$. So, using the equations (26) with (22) we get

$$\beta\left(\frac{-\mu f_3{}''}{W\kappa f_3} - \frac{\lambda\mu}{W}\right) + \frac{\lambda\mu\kappa^2}{W^2} = 0 \tag{28}$$

Hence, substituting (25) and (26) into (28) we obtain (23). □

Consequently, by the use of (23) we get the following result.

Corollary 3.10. *Let M be Ganchev-Milousheva (rotational) surface given by the parametrization (16).*

i) *If $\kappa_1 = 0$ and $\kappa^2 = \frac{f_3{}''}{f_3}$ then M is a trivial Chen surface (i.e. M is minimal),*

ii) *If $\kappa_1 = 0$ and $\kappa^2 \neq \frac{f_3{}''}{f_3}$ then M is a non-trivial Chen surface,*

iii) *If $\kappa_1 \neq 0$ and $\kappa^2 = \mp\frac{k_1}{f_3}$ then M is a non-trivial Chen surface of flat type (i.e. $K(p) = 0$).*

4 Visualization

The geometric modeling of the 3D-surfaces are very important in surface modeling systems such as; CAD/CAM systems and NC-processing. In this paper, a method of spherical product surface in $\mathbb{E}^4$ of a 3D curve with a 2D curve is investigated. For demonstrating the performance of the proposed method, the projection of Otsuki surfaces were constructed in $\mathbb{E}^3$. In fact, these projections can be considered as the spherical product surface in $\mathbb{E}^3$ which are the simple parametrization of representing a large variety of standard geometric solids as well as smooth shapes in between. This makes them much more convenient for representing rounded, blob-like shape parts, typical for object formed by natural process.

In the sequel we construct some 3D geometric shape models by using spherical product surfaces given in the Equation (13). First, we construct the geometric model of the Otsuki surfaces defined in Example 3.3 as follows;

$$\begin{aligned} a)\ \ f_1(u) &= \frac{4}{3}\cos^3(\frac{u}{2}),\ \ f_2(u) = \frac{4}{3}\sin^3(\frac{u}{2}),\ \ f_3(u) = \sin u, \\ b)\ \ f_1(u) &= \frac{1}{2}\sin^2 u\cos(2u),\ \ f_2(u) = \frac{1}{2}\sin^2 u\sin(2u),\ \ f_3(u) = \sin u. \end{aligned}$$

We plot the graph of the projection of these surfaces in $\mathbb{E}^3$ by the use of following plotting command respectively (see Figure 1) ;

$$plot3d([f_1(x) + f_2(x), f_3(x)\cos(y), f_3(x)sin(y)], x = a..b, y = c..d]); \qquad (29)$$

Further, we construct a geometric model of the following Ganchev-Milousheva

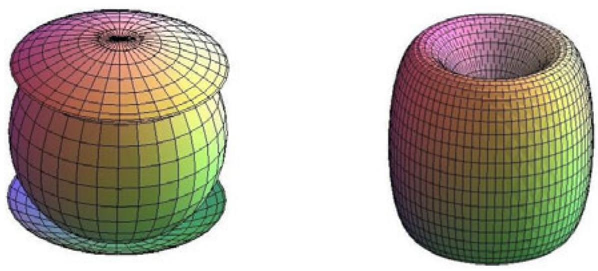

Figure 1: The projections of Otsuki surfaces in $\mathbb{E}^3$

rotation surfaces in $\mathbb{E}^4$;

$$\begin{aligned} c)\ f_1(x) &= \exp(x),\ f_2(x) = \cos x,\ f_3(x) = 3x+1, \\ d)\ f_1(x) &= \sin(x),\ f_2(x) = 3\sin(x)+5,\ f_3(x) = 3x+5, \\ e)\ f_1(x) &= 3\sin(x),\ f_2(x) = x+5,\ f_3(x) = 3x+5. \end{aligned}$$

By Theorem 3.6, the above surfaces satisfy the conditions $\kappa_1 = 0$ and $K \neq 0$ (case a), $\kappa_1 = 0$ and $K = 0$ (case b), or $\kappa_1 \neq 0$ and $K = 0$ (case c). So by Corollary 3.10 all of them are non-trivial Chen surfaces.

We plot the graph of the projection of these surfaces in $\mathbb{E}^3$ by the use of plotting command (29) respectively, (see Figure 2);

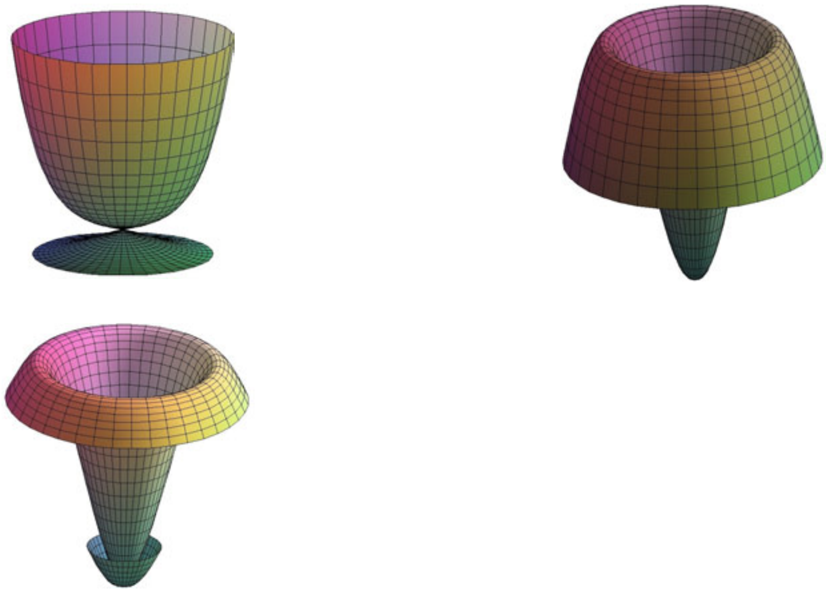

Figure 2: The projections of Ganchev-Milousheva rotation surfaces in $\mathbb{E}^3$

References

[1] K. Arslan and B. Kılıç, *Product Submanifolds and Their Types*, Far East J. Math. Sci. **6**(1) (1998), 125-134.

[2] B. Bulca, K. Arslan, B. (Kılıç) Bayram, G. Öztürk and H. Ugail, *On Spherical Product surfaces in* $\mathbb{E}^3$, IEEE Computer Society, 2009 Int. Conference on CYBERWORLDS.

[3] B. Y. Chen, *Geometry of Submanifols*, Dekker, New York(1973).

[4] U. Dursun, *On Product k-Chen Type Submanifolds*, Glasgow Math. J., **39**(1997), 243-249.

[5] M. Dajczer and R. Tojeiro, *All superconformal surfaces in* $\mathbb{R}^4$ *in terms of minimal surfaces*, Mathematische Zeitschrift, **261**(2009), 869-890.

[6] G. Ganchev and V. Milousheva, *On the Theory of Surfaces in the Four-dimensional Euclidean Space*, Kodai Math. J., **31** (2008), 183-198.

[7] F. Geysens, L. Verheyen, and L. Verstraelen, *Sur les Surfaces A on les Surfaces de Chen*, C.R. Acad. Sc. Paris, I 211(1981).

[8] F. Geysens, L. Verheyen, and L. Verstraelen, *Characterization and Examples of Chen submanifolds*, Journal of geometry, **20**(1983), 47-62.

[9] E. Iyigün, K. Arslan, and G. Öztürk, *A characterization of Chen Surfaces in* $\mathbb{E}^4$, Bull. Malays. Math. Sci. Soc., **31**(2) (2008), 209-215

[10] A. Jaclic, A. Leonardis, and F. Solina, *Segmentation and Recovery of Superquadrics*, Kluwer Academic Publishers, **20** (2000).

[11] N. H. Kuiper, *Minimal Total Absolute Curvature for Immersions*, Invent. Math., **10**(1970), 209-238.

[12] S.J. Li, *Null 2-type Chen Surfaces*, Glasgow Math. J., **37**(1995),233-242.

[13] L. F. Mello, *Orthogonal Asymptotic lines on Surfaces Immersed in* $\mathbb{R}^4$, Rocky Mountain Journal of Math., **39**(2009), 1597-1612.

[14] D. K. H. Mochida, M.D.C.R Fuster, and M.A.S Ruas, *The Geometry of Surfaces in 4-Space From a Contact Viewpoint*, Geometriae Dedicata **54**(1995), 323-332.

[15] T. Otsuki, *Surfaces in the 4-dimensional Euclidean Space Isometric to a Sphere*, Kodai Math. Sem. Rep., **18**(1966), 101-115.

[16] B. Rouxel, *A-submanifolds in Euclidean Space*, Kodai Math. J., **4**(1981), 181-188.

[17] Y.C. Wong, *Contributions to the theory of surfaces in 4-space of constant curvature*, Trans. Amer. Math. Soc, **59** (1946), 467-507.

Betül Bulca, Kadri Arslan,
Uludağ University, Department of Mathematics,
16059 Bursa, TURKEY.
Email: bbulca@uludag.edu.tr, arslan@uludag.edu.tr

Bengü (Kılıç) Bayram,
Balıkesir University, Department of Mathematics,
Balıkesir, TURKEY.
Email: benguk@balikesir.edu.tr

Günay Öztürk,
Kocaeli University, Department of Mathematics,
41380 Kocaeli, TURKEY.
Email: ogunay@kocaeli.edu.tr