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ON APPROXIMATION IN WEIGHTED ORLICZ SPACES

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ABSTRACT. An inverse theorem of the trigonometric approximation theory in Weighted Orlicz spaces is proved and the constructive characterization of the generalized Lipschitz classes defined in these spaces is obtained.

1. Introduction and the main result

A convex and continuous function $M\colon [0,\infty) \to [0,\infty)$, for which $M(0) = 0$, $M(x) > 0$ for $x > 0$ and

$$\lim_{x\to 0} \frac{M(x)}{x} = 0, \qquad \lim_{x\to\infty} \frac{M(x)}{x} = \infty$$

is called a *Young function.* The complementary Young function N of M is defined by

$$N(y) := \max\{xy - M(x) : \ x \ge 0\}$$

for $y \ge 0$.

Let M be a Young function. We denote by L_M the linear space of periodic measurable functions $f\colon [-\pi,\pi] \to \mathbb{R}$, such that

$$\int_{-\pi}^{\pi} M(\lambda |f(x)|)\, \mathrm{d}x < \infty$$

holds for some $\lambda > 0$. Equipped with the norm

$$\|f\|_M := \sup\left\{ \int_{-\pi}^{\pi} |f(x)\, g(x)|\, \mathrm{d}x : \ \int_{-\pi}^{\pi} N(|g(x)|)\, \mathrm{d}x \le 1 \right\},$$

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where N is the complementary function, L_M becomes a Banach space, called the *Orlicz space* generated by M.

The Orlicz spaces are known as the generalization of the Lebesgue spaces; in special case, the Orlicz space generated by the Young function $M_p(x) = \frac{x^p}{p}$, $1 < p < \infty$, is isomorphic to the Lebesgue space L_p. More general information about Orlicz spaces can be found in [13], [20] and [21].

W. Matuszewska and W. Orlicz [17], have associated a pair of indices with a given Orlicz space L_M. A generalization of these, or rather their reciprocals, has been given in the more general context of rearrangement invariant spaces in [3]. Let $M^{-1}\colon [0,\infty) \to [0,\infty)$ be the inverse of the Young function M and let

$$h(t) := \limsup_{x\to\infty} \frac{M^{-1}(x)}{M^{-1}(tx)}, \qquad t > 0.$$

The numbers α_M and β_M defined by

$$\alpha_M := \lim_{t\to\infty} -\frac{\log h(t)}{\log t}, \qquad \beta_M := \lim_{t\to 0^+} -\frac{\log h(t)}{\log t}$$

are called the *lower* and *upper Boyd indices* of the Orlicz space L_M, respectively. It is known that the Boyd indices satisfy

$$0 \le \alpha_M \le \beta_M \le 1$$

and

$$\alpha_N + \beta_M = 1, \qquad \alpha_M + \beta_N = 1.$$

The Orlicz space L_M is reflexive if and only if its Boyd indices are nontrivial, that is $0 < \alpha_M \le \beta_M < 1$.

If $1 \le q < 1/\beta_M \le 1/\alpha_M < p \le \infty$, then $L_p \subset L_M \subset L_q$, where the inclusions are continuous, and hence the relation $L_\infty \subset L_M \subset L_1$ holds. We refer to [1], [2], [3], and [4] for a complete discussion of Boyd indices properties.

A measurable function $\omega\colon [-\pi,\pi] \to [0,\infty]$ is called a weight function if the set $\omega^{-1}(\{0,\infty\})$ has Lebesgue measure zero.

Let ω be a weight function. We denote by $L_{M,\omega}$ the space of the measurable functions $f : [-\pi,\pi] \to \mathbb{R}$ such that $f\omega \in L_M$. The norm on $L_{M,\omega}$ is defined by

$$\|f\|_{M,\omega} := \|f\omega\|_M .$$

The normed space $L_{M,\omega}$ is called a *weighted Orlicz space*.

Let $1 < p < \infty$ and $1/p + 1/q = 1$. A weight function ω belongs to the *Muckenhoupt class* A_p if the condition

$$\sup \left(\frac{1}{|J|} \int_J \omega^p(x)\, \mathrm{d}x \right)^{1/p} \left(\frac{1}{|J|} \int_J \omega^{-q}(x)\, \mathrm{d}x \right)^{1/q} < \infty$$

holds, where the supremum is taken over all subintervals J of $[-\pi,\pi]$ and $|J|$ denotes the length of J.

The detailed information about Muckenhoupt weights can be found in [5, pp. 22–68], [8] and [18].

Let L_M be an Orlicz space with nontrivial Boyd indices and $\omega \in A_{1/\alpha_M} \cap A_{1/\beta_M}$. For a given function $f \in L_{M,\omega}$ we define the shift operator σ_h

$$(\sigma_h f)(x) := \frac{1}{2h} \int_{-h}^{h} f(x+t)\,\mathrm{d}t, \qquad 0 < h < \pi, \;\; x \in [-\pi, \pi],$$

and the *modulus of smoothness* $\Omega_k(f, \cdot)_{M,\omega}$ $(k = 1, 2, \dots)$

$$\Omega_k(f, \delta)_{M,\omega} := \sup_{\substack{0<h_i\le\delta \\ 1\le i\le k}} \left\| \prod_{i=1}^{k} (I - \sigma_{h_i}) f \right\|_{M,\omega}, \qquad \delta > 0, \tag{1}$$

where I is the identity operator. This modulus of smoothness is well defined, because the linear operator σ_h is bounded in the space $L_{M,\omega}$ (see [11]). We define the shift operator σ_h and the modulus of smoothness $\Omega_k(f, \cdot)_{M,\omega}$ in such a way, because the space $L_{M,\omega}$ is noninvariant, in general, under the usual shift $f(x) \to f(x+h)$.

In the case of $k = 0$ we assume $\Omega_0(f, \delta)_{M,\omega} := \|f\|_{M,\omega}$ and if $k = 1$ we write $\Omega(f, \delta)_{M,\omega} := \Omega_1(f, \delta)_{M,\omega}$. The modulus of smoothness $\Omega_k(f, \cdot)_{M,\omega}$ is nondecreasing, nonnegative, continuous function and

$$\Omega_k(f+g, \delta)_{M,\omega} \le \Omega_k(f, \delta)_{M,\omega} + \Omega_k(g, \delta)_{M,\omega} \tag{2}$$

for $f, g \in L_{M,\omega}$.

Furthermore (see [11]), if $f \in L_{M,\omega}$ has an absolutely continuous derivative of order $2k-1$ and $f^{(2k)} \in L_{M,\omega}$, then

$$\Omega_k(f, \delta)_{M,\omega} \le c\delta^{2k} \big\| f^{(2k)} \big\|_{M,\omega}. \tag{3}$$

Let $E_n(f)_{M,\omega}$ $(n = 0, 1, 2, \dots)$ be the distance of the function $f \in L_{M,\omega}$ from Π_n (the class of trigonometric polynomials of degree at most n), i.e.,

$$E_n(f)_{M,\omega} := \inf\big\{ \|f - T_n\|_{M,\omega} : \; T_n \in \Pi_n \big\}.$$

Since under the condition $\omega \in A_{1/\alpha_M} \cap A_{1/\beta_M}$ the space $L_{M,\omega}$ becomes a Banach space, it follows that, for example from [7, Theorem 1.1, p. 59], there exists a trigonometric polynomial $T_n^* \in \Pi_n$ such that

$$E_n(f)_{M,\omega} = \|f - T_n^*\|_{M,\omega}, \qquad n = 0, 1, 2, \dots.$$

Let's denote by $W^r_{M,\omega}$ $(r = 1, 2, \dots)$ the set of functions $f \in L_{M,\omega}$ such that $f^{(r-1)}$ is absolutely continuous and $f^{(r)} \in L_{M,\omega}$.

Using the L_p version of (1), E. A. Gadjieva in [9] proved the direct and inverse theorems of the approximation theory in the weighted L_p spaces, when the weight function satisfies the Muckenhoupt condition. The same problems in the weighted Lebesgue spaces with Muckenhoupt weights were also investigated by N. X. Ky (see [14], [15]) in terms of other modulus of smoothness. In the weighted L_p spaces, for more general class of weights, namely for doubling weights, similar problems were investigated by Mastroianni and Totik in [16]. Also De Bonis, Mastroianni and Russo gave some results for some special weight functions in [6].

The approximation problems in non-weighted Orlicz spaces were investigated by Kokilashvili [12], Ramazanov [19], Garidi [10] and Runovskii [22].

In this work we prove the inverse theorem in the spaces $W^r_{M,\omega}$, namely we give a sufficient condition to assure $f \in W^r_{M,\omega}$ and we estimate the kth modulus of smoothness of the derivative $f^{(r)}$ in terms of the sequence of $E_n(f)_{M,\omega}$ for arbitrary k. Also we define the generalized Lipschitz classes and give a constructive description of these classes.

We shall denote by $c, c_1, c_2, \ldots$ for real constants which are not important for the questions involve in the paper and can be different at each occurence.

The following theorems were proved in [11].

Theorem A. *Let L_M be an Orlicz space with nontrivial Boyd indices α_M and β_M, and let $\omega \in A_{1/\alpha_M} \cap A_{1/\beta_M}$. If $f \in W^r_{M,\omega}$, then*

$$E_n(f)_{M,\omega} \le \frac{c}{n^r} E_n\left(f^{(r)}\right)_{M,\omega}.$$

Theorem B. *Let L_M be an Orlicz space with nontrivial Boyd indices α_M and β_M, and let $\omega \in A_{1/\alpha_M} \cap A_{1/\beta_M}$. Then for $f \in L_{M,\omega}$ the estimate*

$$E_n(f)_{M,\omega} \le c\Omega_k\left(f, \frac{1}{n}\right)_{M,\omega}, \qquad n = 1, 2, \ldots,$$

holds.

The main result of this paper is the following.

Theorem 1. *Let L_M be an Orlicz space with nontrivial Boyd indices α_M and β_M, and let $\omega \in A_{1/\alpha_M} \cap A_{1/\beta_M}$. If for $f \in L_{M,\omega}$*

$$\sum_{\nu=1}^{\infty} \nu^{r-1} E_\nu(f)_{M,\omega} < \infty$$

holds for some natural number r, then $f \in W^r_{M,\omega}$. Furthermore, for any natural number k, and $n = 1, 2, \ldots,$ we have

$$\Omega_k\left(f^{(r)}, \frac{1}{n}\right)_{M,\omega} \le c\left\{\frac{1}{n^{2k}} \sum_{\nu=0}^{n} (\nu+1)^{2k+r-1} E_\nu(f)_{M,\omega} + \sum_{\nu=n+1}^{\infty} \nu^{r-1} E_\nu(f)_{M,\omega}\right\}$$

with a constant $c > 0$, depending only on k and r.

When r is an even natural number, the first part of Theorem 1 was proved in [11]. The second part of this theorem, in the case $r = 0$ was proved also in [11].

Corollary 1. *If*

$$E_n(f)_{M,\omega} = \mathcal{O}\left(\frac{1}{n^{r+\alpha}}\right), \qquad \alpha > 0, \quad n = 1, 2, \ldots,$$

then $f \in W^r_{M,\omega}$ *and*

$$\Omega_k\left(f^{(r)}, \delta\right)_{M,\omega} = \begin{cases} \mathcal{O}\left(\delta^\alpha\right), & k > \alpha/2, \\ \mathcal{O}\left(\delta^\alpha \log\left(1/\delta\right)\right), & k = \alpha/2, \\ \mathcal{O}\left(\delta^{2k}\right), & k < \alpha/2. \end{cases}$$

For $\alpha > 0$, let $k = \left[\frac{\alpha}{2}\right] + 1$. We define the generalized Lipschitz classes $\mathrm{Lip}^*_{M,\omega}\,\alpha$ and $W^{r,\alpha}_{M,\omega}$ as

$$\mathrm{Lip}^*_{M,\omega}\,\alpha := \left\{f \in L_{M,\omega} : \ \Omega_k(f, \delta)_{M,\omega} \le c\delta^\alpha, \ \ \delta > 0\right\}$$

and

$$W^{r,\alpha}_{M,\omega} := \left\{f \in W^r_{M,\omega} : \ f^{(r)} \in \mathrm{Lip}^*_{M,\omega}\,\alpha\right\}.$$

By virtue of Corollary 1 we obtain the following result.

Corollary 2. *If*

$$E_n(f)_{M,\omega} = \mathcal{O}\left(\frac{1}{n^{r+\alpha}}\right), \qquad \alpha > 0, \quad n = 1, 2, \ldots,$$

then $f \in W^{r,\alpha}_{M,\omega}$.

Theorem A, Theorem B and the definition of $W^{r,\alpha}_{M,\omega}$ yield the following result.

Corollary 3. *If* $f \in W^{r,\alpha}_{M,\omega}$, *then*

$$E_n(f)_{M,\omega} = \mathcal{O}\left(\frac{1}{n^{r+\alpha}}\right).$$

Combining this with Corollary 2 we get the following constructive description of the classes $W^{r,\alpha}_{M,\omega}$.

Theorem 2. *Let* L_M *be an Orlicz space with nontrivial Boyd indices* α_M, β_M, *and let* $\omega \in A_{1/\alpha_M} \cap A_{1/\beta_M}$. *Then, for* $\alpha > 0$ *and* $r = 1, 2, \ldots,$ *the following assertions are equivalent.*

(i) $f \in W^{r,\alpha}_{M,\omega}$,

(ii) $E_n(f)_{M,\omega} = \mathcal{O}\left(\frac{1}{n^{r+\alpha}}\right)$, $n = 1, 2, \ldots$.

In the case $r = 0$, this result was obtained in [11].

In the nonweighted Orlicz spaces, under some restrictive conditions on the function M, Theorem 1 was obtained in [12]. Similar theorem for the nonweighted Lebesgue spaces was given by A. F. Timan which can be found in [23, pp. 334-336].

2. Proof of Theorem 1

Let $T_n = T_n(f, x)$ $(n = 0, 1, 2, \dots)$ be the trigonometric polynomial of best approximation to f in the space $L_{M,\omega}$. For $l = 0, 1, \dots, r$, we consider the series

$$T_1^{(l)} + \sum_{\nu=0}^{\infty} \left\{ T_{2^{\nu+1}}^{(l)} - T_{2^\nu}^{(l)} \right\}. \tag{4}$$

Using the Bernstein inequality for weighted Orlicz spaces ([11]), we obtain

$$\left\| T_{2^{\nu+1}}^{(l)} - T_{2^\nu}^{(l)} \right\|_{M,\omega} \le c_1 2^{(\nu+1)l} \left\| T_{2^{\nu+1}} - T_{2^\nu} \right\|_{M,\omega} \le c_2 2^{(\nu+1)l} E_{2^\nu}(f)_{M,\omega}.$$

From this and the inequality

$$2^{(\nu+1)l} E_{2^\nu}(f)_{M,\omega} \le 2^{2l} \sum_{\mu=2^{\nu-1}+1}^{2^\nu} \mu^{l-1} E_\mu(f)_{M,\omega}, \qquad \nu \ge 1,$$

we get

$$\begin{aligned} & \left\| T_1^{(l)} \right\|_{M,\omega} + \sum_{\nu=0}^{\infty} \left\| T_{2^{\nu+1}}^{(l)} - T_{2^\nu}^{(l)} \right\|_{M,\omega} \\ \le \quad & \left\| T_1^{(l)} \right\|_{M,\omega} + c_3 \sum_{\nu=0}^{\infty} 2^{(\nu+1)l} E_{2^\nu}(f)_{M,\omega} \\ \le \quad & \left\| T_1^{(l)} \right\|_{M,\omega} + c_3 2^r E_1(f)_{M,\omega} + c_3 2^{2l} \sum_{\mu=2}^{\infty} \mu^{r-1} E_\mu(f)_{M,\omega} < \infty. \end{aligned}$$

This implies that the sequence $S_{n,l} = S_{n,l}(f, x)$ $(l = 0, 1, 2, \dots)$ of the nth partial sums of the series (4) converges in the norm of $L_{M,\omega}$. Denote its limit function by f_l. $S_{n,l}$ converges in the L_1 norm and hence it has a subsequence $S_{n_i,l}$, which converges almost everywhere to the function f_l for $l = 0, 1, 2, \dots, r$. It is clear that $f_0 = f$ almost everywhere on $[-\pi, \pi]$.

Let x_0 be a point of convergence of $S_{n_i,l}$ for all $l = 0, 1, \ldots, r$. Since

$$\begin{aligned} & f_{l-1}(x) - f_{l-1}(x_0) - \int_{x_0}^{x} f_l(t)\, \mathrm{d}t \\ = \quad & f_{l-1}(x) - S_{n_i,l-1}(x) - f_{l-1}(x_0) + S_{n_i,l-1}(x_0) - \int_{x_0}^{x} \{f_l(t) - S_{n_i,l}(t)\}\, \mathrm{d}t, \end{aligned}$$

we obtain

$$\begin{aligned} & \left\| f_{l-1} - f_{l-1}(x_0) - \int_{x_0}^{(\cdot)} f_l(t)\, \mathrm{d}t \right\|_{M,\omega} \\ \leq \ & \|f_{l-1} - S_{n_i,l-1}\|_{M,\omega} + \left| f_{l-1}(x_0) - S_{n_i,l-1}(x_0) \right| \|\omega\|_M \\ & \qquad + \left\| \int_{x_0}^{(\cdot)} \{f_l(t) - S_{n_i,l}(t)\}\, \mathrm{d}t \right\|_{M,\omega}. \end{aligned}$$

Since the right side tends to zero as $i \to \infty$, we get

$$\left\| f_{l-1} - f_{l-1}(x_0) - \int_{x_0}^{(\cdot)} f_l(t)\, \mathrm{d}t \right\|_{M,\omega} = 0$$

and hence

$$f_{l-1}(x) - f_{l-1}(x_0) = \int_{x_0}^{x} f_l(t)\, \mathrm{d}t, \qquad l = 1, 2, \ldots, r,$$

for almost all x. This implies that f_{l-1} is differentiable and

$$f'_{l-1}(x) = f_l(x) \tag{5}$$

almost everywhere.

Since

$$f(x) = f_0(x) = f_0(x_0) + \int_{x_0}^{x} f_1(t)\, \mathrm{d}t$$

almost everywhere, we have

$$f'(x) = f_1(x)$$

for almost all x. Considering (5), we obtain recursively

$$f^{(r-1)}(x) = f_{r-1}(x) = f_{r-1}(x_0) + \int_{x_0}^{x} f_r(t)\, \mathrm{d}t$$

almost everywhere. Hence $f^{(r-1)}$ is absolutely continuous with derivative $f^{(r)} = f_r$ almost everywhere. This implies that $f \in W^r_{M,\omega}$.

Let m and n be arbitrary natural numbers. Consider the mth partial sum $S_{m,r}$ of the series

$$T_1^{(r)} + \sum_{\nu=0}^{\infty} \left\{ T_{2^{\nu+1}}^{(r)} - T_{2^\nu}^{(r)} \right\}.$$

By (2) we have

$$\Omega_k \left(f^{(r)}, \frac{1}{n} \right)_{M,\omega} \leq \Omega_k \left(f^{(r)} - S_{m,r}, \frac{1}{n} \right)_{M,\omega} + \Omega_k \left(S_{m,r}, \frac{1}{n} \right)_{M,\omega}.$$

By [11, Corollary 3] and the Bernstein inequality,

$$\begin{aligned} \Omega_k \left(f^{(r)} - S_{m,r}, \frac{1}{n} \right)_{M,\omega} &\leq c_4 \left\| f^{(r)} - S_{m,r} \right\|_{M,\omega} \\ &= c_4 \left\| \sum_{\nu=m+1}^{\infty} \left\{ T_{2^{\nu+1}}^{(r)} - T_{2^\nu}^{(r)} \right\} \right\|_{M,\omega} \\ &\leq c_5 \sum_{\nu=m+1}^{\infty} 2^{(\nu+1)r} \left\| T_{2^{\nu+1}} - T_{2^\nu} \right\|_{M,\omega} \\ &\leq c_6 \sum_{\nu=m+1}^{\infty} 2^{(\nu+1)r} E_{2^\nu}(f)_{M,\omega} \\ &\leq c_6 \sum_{\nu=m+1}^{\infty} \left\{ 2^{2r} \sum_{\mu=2^{\nu-1}+1}^{2^\nu} \mu^{r-1} E_\mu(f)_{M,\omega} \right\} \\ &= c_7 \sum_{\nu=2^m+1}^{\infty} \nu^{r-1} E_\nu(f)_{M,\omega}. \end{aligned}$$

Using (2), (3), the Bernstein inequality and the inequality

$$2^{\nu(2k+r)} E_{2^\nu}(f)_{M,\omega} \leq 2^{2k+r} \sum_{\mu=2^{\nu-1}+1}^{2^\nu} \mu^{2k+r-1} E_\mu(f)_{M,\omega}, \qquad \nu \geq 1,$$

we obtain

$$\begin{aligned} &\Omega_k \left(S_{m,r}, \frac{1}{n} \right)_{M,\omega} \\ &\leq \Omega_k \left(T_1^{(r)}, \frac{1}{n} \right)_{M,\omega} + \sum_{\nu=0}^{m} \Omega_k \left(T_{2^{\nu+1}}^{(r)} - T_{2^\nu}^{(r)}, \frac{1}{n} \right)_{M,\omega} \end{aligned}$$

$$\leq c_8 \frac{1}{n^{2k}} \left\| T_1^{(2k+r)} - T_0^{(2k+r)} \right\|_{M,\omega} + c_8 \frac{1}{n^{2k}} \sum_{\nu=0}^{m} \left\| T_{2^{\nu+1}}^{(2k+r)} - T_{2^\nu}^{(2k+r)} \right\|_{M,\omega}$$

$$\leq c_9 \frac{1}{n^{2k}} \left\| T_1 - T_0 \right\|_{M,\omega} + c_{10} \frac{1}{n^{2k}} \sum_{\nu=0}^{m} 2^{\nu(2k+r)} E_{2^\nu}(f)_{M,\omega}$$

$$\leq c_{11} \frac{1}{n^{2k}} \left\{ E_0(f)_{M,\omega} + E_1(f)_{M,\omega} + \sum_{\nu=1}^{m} 2^{2k+r} \sum_{\mu=2^{\nu-1}+1}^{2^\nu} \mu^{2k+r-1} E_\mu(f)_{M,\omega} \right\}$$

$$\leq c_{12} \frac{1}{n^{2k}} \sum_{\nu=0}^{2^m} (\nu+1)^{2k+r-1} E_\nu(f)_{M,\omega}.$$

Combining this inequality with

$$\Omega_k \left(f^{(r)} - S_{m,r}, \frac{1}{n} \right)_{M,\omega} \leq c_7 \sum_{\nu=2^m+1}^{\infty} \nu^{r-1} E_\nu(f)_{M,\omega},$$

yield

$$\Omega_k \left(f^{(r)}, \frac{1}{n} \right)_{M,\omega}$$
$$\leq c_{13} \left\{ \frac{1}{n^{2k}} \sum_{\nu=0}^{2^m} (\nu+1)^{2k+r-1} E_\nu(f)_{M,\omega} + \sum_{\nu=2^m+1}^{\infty} \nu^{r-1} E_\nu(f)_{M,\omega} \right\}.$$

Finally, if we choose m such that $2^m \leq n < 2^{m+1}$, the last inequality finishes the proof of Theorem 1.

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