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CONVOLUTION AND APPROXIMATION IN WEIGHTED LORENTZ SPACES

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Abstract. In this paper, it is defined a convolution type transform in the weighted Lorentz spaces with Muckenhoupt weights and investigated the relationship among this transform and the best trigonometric approximation in this spaces.

1. Introduction and main results

This paper deals with certain modified versions of the convolution transform in weighted Lorentz spaces with Muckenhoupt weights. These modifications take mainly into account the presence of the weight function and the consequent lack of translation invariance in this spaces. The convolution type transforms is very important in many branches of theoretic and applied mathematics. Especially, these transforms are very convenient in trigonometric approximation theory for the buildings of the approximating polynomials. Thereby, we need to investigate the relations among these transforms and the sequences of the best approximations numbers in function spaces.

A measurable function $\omega : [-\pi, \pi] \to [0, \infty]$ is called a weight function if the set $\omega^{-1}(\{0, \infty\})$ has Lebesgue measure zero. Let $\mathbb{T} := [-\pi, \pi]$ and ω be a weight function. Given a weight function ω and a measurable set e we put

$$\omega(e) = \int_e \omega(x)dx. \tag{1.1}$$

We define the decreasing rearrangement $f_\omega^*(t)$ of $f : \mathbb{T} \to \mathbb{R}$ with respect to the Borel measure (1.1) by

$$f_\omega^*(t) = \inf\{\tau \geq 0 : \omega(x \in \mathbb{T} : |f(x)| > \tau) \leq t\}.$$

The weighted Lorentz space $L_\omega^{pq}(\mathbb{T})$ is defined [4, p.20], [2, p.219] as

$$L_\omega^{pq}(\mathbb{T}) = \left\{ f \in \mathbf{M}(\mathbb{T}) : \|f\|_{pq,\omega} = \left(\int_{\mathbb{T}} (f^{**}(t))^q \, t^{\frac{q}{p}} \frac{dt}{t} \right)^{1/q} < \infty, \ 1 < p, q < \infty \right\},$$

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where $\mathbf{M}(\mathbb{T})$ is the set of 2π periodic integrable functions on $\mathbb{T}$ and

$$f^{**}(t) = \frac{1}{t}\int\limits_0^t f_\omega^*(u)du.$$

The weighted Lorentz space $L_\omega^{pq}(\mathbb{T})$ is a Banach space with this norm. If $p = q$, $L_\omega^{pq}(\mathbb{T})$ turns into the weighted Lebesgue space $L_\omega^p(\mathbb{T})$ [4, p.20].

A weight function $\omega : \mathbb{T} \to [0,\infty]$ belongs to the Muckenhoupt class A_p [6], $1 < p < \infty$, if

$$\sup \frac{1}{|I|}\int\limits_I \omega(x)dx \left(\frac{1}{|I|}\int\limits_I \omega^{1-p'}(x)dx\right)^{p-1} = C_{A_p} < \infty, \qquad p' := \frac{p}{p-1}$$

with a finite constant C_{A_p} independent of I, where the supremum is taken with respect to all intervals I with length $\leq 2\pi$ and $|I|$ denotes the length of I. The constant C_{A_p} is called the Muckenhoupt constant of ω.

For $f \in L_\omega^{pq}(\mathbb{T})$, $1 < p,q < \infty$, $\omega \in A_p$, the operator σ_h is defined as

$$(\sigma_h f)(x,u) := \frac{1}{2h}\int\limits_{-h}^{h} f(x+tu)dt, \quad 0 < h < \pi,\ x \in \mathbb{T},\ -\infty < u < \infty.$$

Whenever $\omega \in A_p$, $1 < p,q < \infty$, the Hardy-Littlewood maximal function of $f \in L_\omega^{pq}(\mathbb{T})$ belongs to $L_\omega^{pq}(\mathbb{T})$ [3, Theorem 3]. Therefore the operator $\sigma_h f$ belongs to $L_\omega^{pq}(\mathbb{T})$.

Since $L_\omega^{pq}(\mathbb{T}) \subset L^1(\mathbb{T})$ when $\omega \in A_p$, $1 < p,q < \infty$ (see [5, the proof of Prop. 3.3]), we can define the Fourier series of $f \in L_\omega^{pq}(\mathbb{T})$. By not loosing of generalization suppose that Fourier series of f is

$$\sum_{r=1}^{\infty} c_r e^{irx} = \sum_{r=1}^{\infty} A_r(x). \tag{1.2}$$

Let $S_n(f,x)$, $(n = 0,1,2,...)$ be the nth partial sum of the series (1.2) at the point x, that is,

$$S_n(x,f) := \sum_{k=1}^{n} A_k(x),$$

By $E_n(f)_{pq,\omega}$ we denote the best approximation of $f \in L_\omega^{pq}(\mathbb{T})$ by polynomials in $\mathcal{T}_n$ i.e.,

$$E_n(f)_{pq,\omega} = \inf_{T_n \in \mathcal{T}_n} \|f - T_n\|_{pq,\omega}$$

where $\mathcal{T}_n$ is the set of trigonometric polynomials of degree $\leq n$.

Since the weighted Lorentz spaces are noninvariant with respect to the usual shift $f(x-hu)$, we define the convolution type transforms by using the mean value function $(\sigma_h f)(x,u)$.

For $f \in L^{pq}_{\omega}(\mathbb{T})$ we denote the norm of the convolution type transform by $D(f,\mu,h,pq)$:

$$D(f,\mu,h,pq) := \left\| \int_{-\infty}^{\infty} (\sigma_h f)(x,u) d\mu(u) \right\|_{pq,\omega}$$

where $\mu(u)$ is a real function of bounded variation on the real axis.

Throughout this paper, the constant c denotes a generic constant, i.e. a constant whose values can change even between different occurrences in a chain of inequalities. In this paper, we will use the following notation

$$A(x) \preceq B(x) \Leftrightarrow \exists c > 0 : A(x) \leq cB(x).$$

The following theorem estimates the quantity $D(f,\mu,h,pq)$ in terms of the best trigonometric approximation of the function f in the weighted Lorentz spaces.

Theorem 1. *If $1 < p,q < \infty$, $\omega \in A_p$ and $f \in L^{pq}_{\omega}(\mathbb{T})$. Then for every natural number m*

$$D(f,\mu,h,pq) \preceq \sum_{r=0}^{m} E_{2^r-1}(f)_{pq,\omega} \cdot \delta_{2^r,h} + E_{2^{m+1}}(f)_{pq,\omega}$$

where

$$\begin{aligned} \delta_{2^r,h} \quad &:= \sum_{l=2^r}^{2^{r+1}-1} \left| \hat{\mu}(lh) - \hat{\mu}((l+1)h) \right| + \left| \hat{\mu}(2^r h) \right|, \\ \hat{\mu}(x) \quad &:= \int_{-\infty}^{\infty} \frac{\sin ux}{ux} d\mu(u), \qquad 0 < h \leq \pi. \end{aligned}$$

Theorem 2. *If $1 < p,q < \infty$, $\omega \in A_p$ and $f \in L^{pq}_{\omega}(\mathbb{T})$. Assume that the function $F(x)$ satisfies the conditions*

$$\|F(x)\| \leq c_1, \ \sum_{k=2^{\mu}}^{2^{\mu+1}-1} \left| F(kh) - F((k+1)h) \right| \leq c_2, \ h \leq 2^{-m-1}.$$

with some constants c_1, c_2. If μ_1 and μ_2 are the functions satisfying the condition

$$\hat{\mu}_1(x) = \hat{\mu}_2(x) F(x), \quad |x| < 1$$

then

$$D(f,\mu_1,h,pq) \preceq D(f,\mu_2,h,pq) + E_{2^{m+1}}(f)_{pq,\omega}.$$

The similar theorems were proved in [7] for the functions in the Orlicz spaces. Then the theorems obtained in [7] were generalized to the weighted Orlicz spaces in [8]. In this paper, we obtain these theorems in the weighted Lorentz spaces with the more simple proofs.

2. Auxiliary result

We need the multiplier theorem and Littlewood-Paley theorem in $L_{\omega}^{pq}(\mathbb{T})$:

Lemma A. [1] *Let $\lambda_0, \lambda_1, ...$ be a sequence of real numbers such that*

$$|\lambda_l| \leq M, \quad \sum_{\nu=2^{l-1}}^{2^l-1} |\lambda_\nu - \lambda_{\nu+1}| \leq M$$

for all $\nu, l \in N$. If $1 < p, q < \infty$, $\omega \in A_p$ and $f \in L_{\omega}^{pq}(\mathbb{T})$ with Fourier series $\sum_{\nu=0}^{\infty} (a_\nu(f) \cos \nu x + b_\nu(f) \sin \nu x)$, then there is a function $h \in L_{\omega}^{pq}(\mathbb{T})$ such that the series $\sum_{\nu=0}^{\infty} \lambda_\nu (a_\nu(f) \cos \nu x + b_\nu(f) \sin \nu x)$ is the Fourier series of h and

$$\|h\|_{pq,\omega} \leq C \|f\|_{pq,\omega}$$

where C does not depend on f.

Lemma B. [1] *Let $1 < p, q < \infty$, $\omega \in A_p$ and $f \in L_{\omega}^{pq}(\mathbb{T})$ with Fourier series $\sum_{\nu=0}^{\infty} (a_\nu(f) \cos \nu x + b_\nu(f) \sin \nu x)$, then there exist constants c_1, c_2 independent of f such that*

$$c_1 \left\| \left(\sum_{\mu=\nu}^{\infty} |\Delta_\mu|^2 \right)^{1/2} \right\|_{pq,\omega} \leq \|f\|_{pq,\omega} \leq c_2 \left\| \left(\sum_{\mu=\nu}^{\infty} |\Delta_\mu|^2 \right)^{1/2} \right\|_{pq,\omega}$$

where

$$\Delta_\mu := \Delta_\mu(x, f) := \sum_{\nu=2^{\mu-1}}^{2^\mu-1} (a_\nu(f) \cos \nu x + b_\nu(f) \sin \nu x).$$

3. Proofs of main results

Proof of Theorem 1. Let $f(x) \in L_{\omega}^{pq}(\mathbb{T})$ and $S_{2^{m+1}}$ be the partial sum of its Fourier series and $h \leq 2^{-m-1}$. By virtue of the definition of the number $D(f, \mu, h, pq)$ and the properties of the norm

$$\begin{aligned} D(f, \mu, h, pq) &= \left\| \int_{-\infty}^{\infty} (\sigma_h f)(x, u) d\mu(u) \right\|_{pq,\omega} \\ &\leq \left\| \int_{-\infty}^{\infty} [(\sigma_h f)(x, u) - (\sigma_h S_{2^{m+1}})(x, u)] \, d\mu(u) \right\|_{pq,\omega} + \\ &\quad + \left\| \int_{-\infty}^{\infty} (\sigma_h S_{2^{m+1}})(x, u) d\mu(u) \right\|_{pq,\omega}. \end{aligned}$$

Since [5]

$$\|f(x) - S_n(f, x)\|_{pq,\omega} \leq c E_n(f)_{pq,\omega} \tag{3.1}$$

considering the boundedness of the operator σ_h,

$$D\left(f,\mu,h,pq\right)\leq\left\|\int_{-\infty}^{\infty}\left(\sigma_h S_{2^{m+1}}\right)(x,u)d\mu(u)\right\|_{pq,\omega}+cE_{2^{m+1}}(f)_{pq,\omega}.$$

Then

$$\begin{aligned}\int_{-\infty}^{\infty}\left(\sigma_h S_{2^{m+1}}\right)(x,u)d\mu(u) &= \int_{-\infty}^{\infty}\left(\frac{1}{2h}\int_{-h}^{h}S_{2^{m+1}}(x+tu)dt\right)d\mu(u)\\ &= \int_{-\infty}^{\infty}\left(\frac{1}{2h}\int_{-h}^{h}\sum_{r=1}^{2^{m+1}-1}c_r e^{ir(x+tu)}dt\right)d\mu(u)\\ &= \int_{-\infty}^{\infty}\left(\frac{1}{2h}\sum_{r=1}^{2^{m+1}-1}c_r e^{irx}\int_{-h}^{h}e^{irtu}dt\right)d\mu(u)\\ &= \sum_{r=1}^{2^{m+1}-1}A_r(x)\int_{-\infty}^{\infty}\frac{e^{irhu}-e^{-irhu}}{2irhu}d\mu(u)\\ &= \sum_{r=1}^{2^{m+1}-1}A_r(x)\hat{\mu}(rh). \qquad (3.2)\end{aligned}$$

Therefore, we have

$$D\left(f,\mu,h,pq\right)\leq\left\|\sum_{r=1}^{2^{m+1}-1}A_r(x)\hat{\mu}(rh)\right\|_{pq,\omega}+cE_{2^{m+1}}(f)_{pq,\omega}.$$

From Lemma B, we obtain

$$\begin{aligned}\left\|\sum_{r=1}^{2^{m+1}-1}A_r(x)\hat{\mu}(rh)\right\|_{pq,\omega} &\leq c\left\|\left(\sum_{r=0}^{m}\left|\sum_{l=2^r}^{2^{r+1}-1}A_l(x)\hat{\mu}(lh)\right|^2\right)^{1/2}\right\|_{pq,\omega}\\ &:= c\left\|\left(\sum_{r=0}^{m}\Delta_{r,\mu}^2\right)^{1/2}\right\|_{pq,\omega}\leq c\left\|\sum_{r=0}^{m}\left(\Delta_{r,\mu}^2\right)^{1/2}\right\|_{pq,\omega}\\ &= c\left\|\sum_{r=0}^{m}\Delta_{r,\mu}\right\|_{pq,\omega}\leq c\sum_{r=0}^{m}\left\|\Delta_{r,\mu}\right\|_{pq,\omega}.\end{aligned}$$

If we apply the Abel transform to $\Delta_{r,\mu}$

$$\begin{aligned}\Delta_{r,\mu} &= \sum_{l=2^r}^{2^{r+1}-1}\left[S_l(f,x)-S_{2^{r+1}-1}(f,x)\right]\left[\hat{\mu}(lh)-\hat{\mu}((l+1)h)\right]+\\ &\quad+\left[S_{2^{r+1}-1}(f,x)-S_{2^r-1}(f,x)\right]\hat{\mu}(2^r h).\end{aligned}$$

From (3.1)

$$\begin{aligned}
\|\Delta_{r,\mu}\|_{pq,\omega} &\leq \sum_{l=2^r}^{2^{r+1}-1} \|S_l(f,x) - S_{2^{r+1}-1}(f,x)\|_{pq,\omega} |\hat{\mu}(lh) - \hat{\mu}((l+1)h)| \\
&\quad + \|S_{2^{r+1}-1}(f,x) - S_{2^r-1}(f,x)\|_{pq,\omega} |\hat{\mu}(2^r h)| \\
&\leq E_{2^r-1}(f)_{pq,\omega}\delta_{2^r,h}.
\end{aligned}$$

Then

$$\left\| \sum_{r=1}^{2^{m+1}-1} A_r(x)\hat{\mu}(rh) \right\|_{pq,\omega} \leq c\sum_{r=0}^{m} E_{2^r-1}(f)_{pq,\omega}\delta_{2^r,h}.$$

This completes the proof.

Proof of Theorem 2. For $f \in L^{pq}_{\omega}(\mathbb{T})$, from the properties of the norm and (3.1)

$$D(f,\mu_1,h,pq) \leq \left\| \int_{-\infty}^{\infty} (\sigma_h S_{2^{m+1}})(x)d\mu_1(u) \right\|_{pq,\omega} + cE_{2^{m+1}}(f)_{pq,\omega}.$$

Using the properties of the function $F(x) = \hat{\mu}_1(x)(\hat{\mu}_2(x))^{-1}$, (3.2), Lemma A and the boundedness of the operator $S_n(f,x)$ in $L^{pq}_{\omega}(\mathbb{T})$ [5] we obtain

$$\begin{aligned}
\left\| \int_{-\infty}^{\infty} (\sigma_h S_{2^{m+1}})(x)d\mu_1(u) \right\|_{pq,\omega} &= \left\| \sum_{r=1}^{2^{m+1}-1} c_r e^{irx}\hat{\mu}_2(rh)F(rh) \right\|_{pq,\omega} \preceq \\
&\preceq \left\| \sum_{r=1}^{2^{m+1}-1} c_r e^{irx}\hat{\mu}_2(rh) \right\|_{pq,\omega} = \\
&= \left\| \int_{-\infty}^{\infty} (\sigma_h S_{2^{m+1}}(f))(x)d\mu_2(u) \right\|_{pq,\omega} \leq \\
&\leq \left\| \int_{-\infty}^{\infty} f(x)d\mu_2(u) \right\|_{pq,\omega}.
\end{aligned}$$

This completes the proof.

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