Astrophys Space Sci (2012) 342:365–370
DOI 10.1007/s10509-012-1182-7

ORIGINAL ARTICLE

# Structure coefficients for use in stellar analysis

**Gülay İnlek · Edwin Budding**

Received: 13 March 2012 / Accepted: 17 July 2012 / Published online: 27 July 2012


**Abstract** We present new values of the structural coefficients $\eta_j$, and related quantities, for realistic models of distorted stars in close binary systems. Our procedure involves numerical integration of Radau's equation for detailed structural data and we verified our technique by referring to the 8-digit results of Brooker & Olle (Mon. Not. R. Astron. Soc. 115:101, 1955) for purely mathematical models. We provide tables of representative values of $\eta_j$, and related quantities, for $j = 2, 3, \dots, 7$ for a selection of Zero Age Stellar Main Sequence (ZAMS) stellar models taken from the EZWeb compilation of the Dept. of Astronomy, University of Wisconsin-Madison. We include also some preliminary comparisons of our findings with the results of Claret and Gimenez (Astron. Astrophys. 519:A57 2010) for some observed stars.

**Keywords** Stellar structure · Structural coefficients · Close binary systems

## 1 Introduction

Kopal (1959) discusses a potential for unit mass located at a point $M$, external to a spherical shell of matter where a typical point is labelled $M'$, by an expression of the form

$$V = G \int \frac{dm'}{R}; \tag{1}$$

$G$ being the gravitation constant, $R$ the separation of $M$ and $M'$; the mass element $dm'$, given, in a naturally applicable spherical polar co-ordinate system, by

$$dm' = \int\int\int \rho r'^2 dr' \sin\theta' d\theta' d\phi' \tag{2}$$

with

$$R^2 = r^2 + r'^2 - 2rr'\cos\gamma, \tag{3}$$

$r, \theta, \phi$, being the co-ordinates of the point $M$, and then

$$\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi - \phi'). \tag{4}$$

This potential is understood to mean that which when differentiated gives the gravitational force on unit mass, although this meaning differs slightly from that of normal potential energy, which is higher for less tightly bound matter. Such a potential would require a minus sign before the right side expression in (1), and the corresponding derivative also requires a minus sign for an attractive force. The end result being the same for the force, there is some convenience in retaining the notation of (1).

The denominator $R$ lends itself readily to expansion in Legendre polynomials, so that the integral (1) can be expressed as the sum of a series in $n$ of terms, thus:

$$V = \sum_0^\infty r^{-(n+1)} V_n, \tag{5}$$

where each term $V_n$ is an integral of the form

$$V_n = G \int r'^n P_n(\cos\gamma) dm'. \tag{6}$$

A closely comparable form exists also when $M$ is internal to $M'$, except for a difference in the integral limits and that the powers of $r$ increase and those in $r'$ decrease in corresponding successive terms. The surviving power of $-1$

G. İnlek (✉)
Department of Physics, Faculty of Science and Arts, Balıkesir University, Cagis Campus, 10145 Balıkesir, Turkey
e-mail: inlek@balikesir.edu.tr

E. Budding
Carter Observatory, P.O. Box 2909, Wellington, New Zealand
e-mail: budding@xtra.co.nz

in the explicit unit of distance, given the surrounding factor $G\int dm' = Gm_1$, say, ensures that each term has the dimensions of energy per unit mass. Considerations are often aimed toward the surface distortion of a component in a close binary system, where the internal form for the potential disappears, so the external form tends to assume a more overt role.

The classical approach to finding the shape of a body distorted by forces associated with rotation and tides refers to equipotential surfaces, on which the potential associated with all forces in the problem is constant. This approach, coupled with the circumstance of a distinct ordering to the relative scale of pertinent forces, so that contributory effects can be regarded as additive perturbations upon simpler, more basic forms (e.g. having spherical symmetry), permits distinct inroads into the situation conforming *a priori* only to Poisson's Equation. Clauraut's theorem for bodies in equilibrium (cf. e.g. Pressly 2001) implies that the density $\rho$ is constant over an equipotential surface, which permits simplification of the integral formed by combining (2) and (6). Indeed, it becomes tractable if we can also express the equipotentials in terms of spherical harmonics $Y_j(a, \theta', \phi')$, that normally include Legendre polynomials $P(\cos\theta')$, due to the integrability of the relevant products, i.e. the orthogonality conditions applying to products of harmonics in an integral (cf. e.g. MacRobert 1927). The radius $r'$ is thus expressed as the series

$$r' = a\left\{1 + \sum_{j=2}^{\infty} Y_j^i\left(a, \theta', \phi'\right)\right\}, \tag{7}$$

where $a$ now represents a mean radius applying to any given equipotential, whose perturbation from sphericity is given in terms of the tesseral harmonics $Y_j^i$. This leads to (5) being expressible as a series of integrals involving only $a$, where the mixed products of different order harmonics vanish.

The potential considered thus far refers only to the body's own distribution of matter and its gravitational self attraction. For a body with no net motion of any constituent particle in a given frame of reference, this is regarded as balancing a 'disturbing potential' $V' = \sum_{i,j}^{\infty} c_{i,j} r^j P_j^i(\theta, \phi)$ that gives rise to forces acting in opposition to that of the self attraction, with the coefficients $c_{i,j}$ pertaining to given forms of disturbance at $a = a_1$. By balancing the coefficients in the expansion for the combined potential, since each equipotential surface is characterized by only one value of the total potential (independently of $\theta$ or $\phi$, i.e. regardless of whereabouts on the surface we may locate a test particle), we arrive, after a little manipulation (cf. Kopal 1959), at Clairaut's equation for the first order surface perturbation

$$\frac{G}{(2j+1)a_1^{j+1}} \int_0^{a_1} \left( j a^j Y_j^i + a^{j+1} \frac{\partial Y_j^i}{\partial a} \right) dm' = c_{i,j} a_1^j P_j^i(\theta, \phi). \tag{8}$$

The mass-shell weighted integral on the left side of this equation results from only the external form for the potential; the internal one disappearing at the surface ($a = a_1$).

Writing now

$$c_{i,j} a_1^j P_j^i = \frac{Gm_1}{a_1} \frac{Y_j^i}{\Delta_j}, \tag{9}$$

we expect the key coefficient $\Delta_j$ introduced here to be a purely numerical quantity of order unity. The forms of (7) and (8) imply the harmonic functions $Y_j^i$ are also numerical, with argument $a/a_1$. Clairaut's equation can then be rearranged as

$$\Delta_j = \frac{(2j+1)}{j + \eta_j(a_1)}, \tag{10}$$

where $\eta_j(a_1)$ is the surface value of the logarithmic derivative for the perturbation potential

$$\eta_j(a) = \frac{a}{Y_j^i} \frac{\partial Y_j^i}{\partial a}. \tag{11}$$

If the $a$-dependence of the harmonics $Y_j^i$ were simply as the powers $(a/a_1)^k$ then $\eta_j = k$. Notice that the reduction to only the index $j$ for $\Delta$ and $\eta$ anticipates that the relevant disturbing potentials can be expressed (by an appropriate co-ordinate choice) in terms of only (zonal) Legendre polynomials.

Kopal (1959) and others have studied the mathematical behaviour of the function $\eta_j$ in some detail. It has been shown to satisfy the differential equation, for $a$ on the range $0 < a < a_1$,

$$a\frac{d\eta_j}{da} + \frac{6\rho}{\overline{\rho}}(\eta_j + 1) + \eta_j(\eta_j - 1) = j(j+1), \tag{12}$$

that Kopal called Radau's equation. If the envelope density falls away, i.e. $\rho \to 0$, this equation could clearly be solved by $\eta_j = j+1$, in accordance with $Y_j^i$ having the form $c'_j(a/a_1)^{j+1}$. The coefficient $\Delta_j$ would then revert to unity, which accords with an intuitive expectation that, in the absence of matter, the disturbing and balancing potentials directly match; $c'_0 \equiv c_0 a_1/Gm_1$ in the case of self-attraction, for instance. A finite density $\rho > 0$ has the effect of reducing $\eta_j$ in (12) in order to balance the left side with the constant right, then entailing a diminution of the denominator in (10) and an amplification of the surface distortion through corresponding increase of the coefficient $\Delta_j$. The same effect can be seen in (8) when a decrease of the second, gradient term in the integrand would require a compensating increase in the coefficient of the $Y_j^i$ to entail constancy to the right side of the equation. For a body of uniform density, (12) can easily be seen to be satisfied by $\eta_j = j - 2$, so that $\Delta_j = (2j+1)/(2j-2)$. But this would be the maximum amplification of $\Delta_j$ feasible for a regular astrophysical body in equilibrium. For bodies with some degree

**Table 1** Apsidal-motion constants $k_2$. We have listed values of the coefficients $k_2$ corresponding to the procedure given in the text, interpolating to the mean masses adopted by Claret and Gimenez (2010)

| Star | Mass ($M_\odot$) | $k_2$ (Present work) | $k_2$ (Claret and Gimenez) |
|---|---|---|---|
| $V636$ $Cen$ | 1.051 | 0.02314 | 0.01920 |
| $EK$ $Cep$ | 2.025 | 0.00409 | 0.00765 |
| $PV$ $Cas$ | 2.816 | 0.00526 | 0.00435 |
| $GG$ $Lup$ | 4.106 | 0.00710 | 0.00594 |
| $V760$ $Sco$ | 4.969 | 0.00825 | 0.00629 |
| $QX$ $Car$ | 9.250 | 0.01261 | 0.00810 |

of central condensation, like stars, $\eta_j$ tends rather quickly towards $j+1$, so that $\Delta_j \to 1$ similarly. $\Delta_j = 1$ should thus hold for the centrally condensed 'Roche' approximation.

Brooker and Olle (1955) tabulated values of the solutions $\eta_j(a_1)$, to 8 decimal places accuracy, for polytropic models of stellar structure, with $j = 2, 3, \ldots, 7$; and 14 values of the polytropic index $n$ in the range $0 \le n \le 5$. Their data clearly show rapid increases of $\eta$ towards $j+1$ with increasing polytropic index $n$, i.e. central condensation. These results, cited as Table 2-1 by Kopal (1959), were used in many subsequent modellings of rotationally and tidally distorted stars and form a useful basis of comparison for the present compilation.

Kushwaha (1957) and Schwarzschild (1958) calculated theoretical apsidal constants ($k_2 = (\Delta_2 - 1)/2$) for homogeneous and evolved stars. Petty (1973) looked for an explanation of the discrepancy between observational and theoretical ($k_2$) values, using homogeneous stellar models. Hejlesen (1987) computed the structure constants $k_j$ for ZAMS models, and discussed their evolutionary variation for the $j$-values 2, 3 and 4 using the theoretical models of Jeffery (1984). He pointed out the uncertainty in the calculations arising from the use of different opacity tables in theoretical models. More recently, Torres et al. (2010) presented logarithms of $k_2$-values for 18 binaries between the ZAMS and TAMS (Terminal Age Main Sequence). Their results were related to the theoretical values of Claret (1995) in dependence on the surface gravity and mass.

Claret and Gimenez (2010) have also checked structural coefficients against data from double-lined eclipsing binaries. They used stellar models generated from the Granada evolutionary code of Claret (2004), and integrated the Radau equation as discussed in the foregoing. They paid particular attention to apsidal-motion rates, which can be related to mean systemic values of $k_2$. We have sought to check our results against observational data and will report more about this in subsequent work. In the interim, we present some preliminary findings in Table 1 that can be compared with corresponding values from Claret and Gimenez (2010).

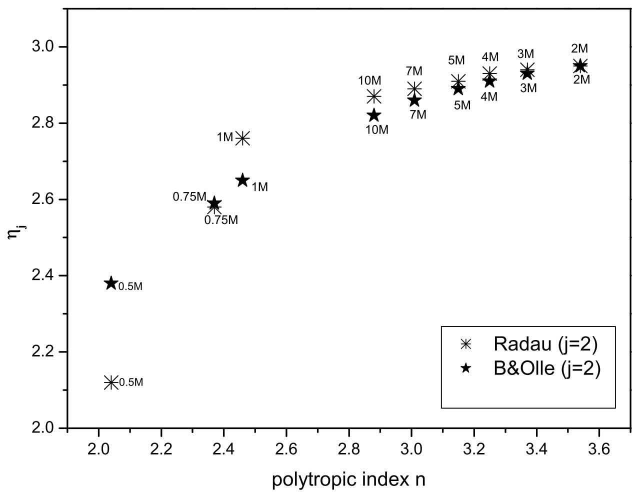


**Fig. 1** $\eta_j$ changing with representative polytropic index $n$. The diagram shows values of $\eta_j$ ($j = 2$) for different masses of stars as determined by integrating ZAMS models of these stars (taken from the EZWeb database) using the program RADAU. These values, listed also in Table 2, are shown as pentacles. They can be compared with corresponding values of $\eta_j$ ($j = 2$) shown as *asterisks* taken from the table of Brooker & Olle. These values are obtained by interpolation from the tabulated data of Brooker& Olle at the representative values of the polytropic index given on the abscissal scale. These values correspond to $n$ values derived in the manner shown in Fig. 2

## 2 Procedure

We first constructed a small piece of FORTRAN programming to carry out numerical integration of Radau's equation that was combined with a separate program used to integrate polytropic models of stars. The procedure was comparable to that of Brooker and Olle (1955), except that with the greater data processing speeds and capacities of modern computers, step sizes could easily be made suitably small to avoid the numerical problems mentioned by Brooker & Olle, and still return reliable results in a short time. The second-order Lane-Emden equation is rearranged as two simultaneous first-order difference equations for this, while Radau's equation becomes a first-order difference equation for the increment of $\eta_j$ at each layer. Referring to Brooker & Olle's results, we confirmed a numerical agreement to eight significant digits with our program (RADAU).

**Table 2** Zero age solar composition models

| $j$ | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|
| $M = 0.5M_{\odot}; n_1 = 2.31, n_2 = 2.53$ | | | | | | |
| $\eta_j$ | 2.11988 | 3.49701 | 4.65909 | 5.74914 | 6.80576 | 7.84413 |
| $\Delta_j$ | 1.21363 | 1.07742 | 1.03937 | 1.02334 | 1.01517 | 1.01050 |
| $k_j$ | 0.10681 | 0.03871 | 0.01969 | 0.01167 | 0.00758 | 0.00525 |
| $M = 0.75M_{\odot}; n_1 = 2.37, n_2 = 2.62$ | | | | | | |
| $\eta_j$ | 2.58087 | 3.77719 | 4.85148 | 5.89118 | 6.91585 | 7.93253 |
| $\Delta_j$ | 1.09150 | 1.03288 | 1.01678 | 1.00999 | 1.00652 | 1.00452 |
| $k_j$ | 0.04575 | 0.01644 | 0.00839 | 0.00500 | 0.00326 | 0.00226 |
| $M = 1.0M_{\odot}; n_1 = 2.18, n_2 = 2.46$ | | | | | | |
| $\eta_j$ | 2.76930 | 3.88574 | 4.92550 | 5.94579 | 6.95816 | 7.96645 |
| $\Delta_j$ | 1.04837 | 1.01659 | 1.00835 | 1.00495 | 1.00323 | 1.00224 |
| $k_j$ | 0.02419 | 0.00830 | 0.00417 | 0.00248 | 0.00161 | 0.00112 |
| $M = 2.0M_{\odot}; n_1 = 2.42, n_2 = 2.82$ | | | | | | |
| $\eta_j$ | 2.95974 | 3.98990 | 4.99605 | 5.99809 | 6.99895 | 7.99937 |
| $\Delta_j$ | 1.00812 | 1.00145 | 1.00044 | 1.00017 | 1.00008 | 1.00004 |
| $k_j$ | 0.00406 | 0.00072 | 0.00022 | 0.00009 | 0.00004 | 0.00002 |
| $M = 3.0M_{\odot}; n_1 = 2.38, n_2 = 2.76$ | | | | | | |
| $\eta_j$ | 2.94512 | 3.98633 | 4.99466 | 5.99742 | 6.99858 | 7.99915 |
| $\Delta_j$ | 1.01110 | 1.00196 | 1.00059 | 1.00023 | 1.00011 | 1.00006 |
| $k_j$ | 0.00555 | 0.00098 | 0.00030 | 0.00012 | 0.00005 | 0.00003 |
| $M = 4.0M_{\odot}; n_1 = 2.36, n_2 = 2.72$ | | | | | | |
| $\eta_j$ | 2.93139 | 3.98293 | 4.99334 | 5.99677 | 6.99822 | 7.99893 |
| $\Delta_j$ | 1.01391 | 1.00244 | 1.00074 | 1.00029 | 1.00014 | 1.00007 |
| $k_j$ | 0.00696 | 0.00122 | 0.00037 | 0.00015 | 0.00007 | 0.00004 |
| $M = 5.0M_{\odot}; n_1 = 2.34, n_2 = 2.68$ | | | | | | |
| $\eta_j$ | 2.91837 | 3.97970 | 4.99209 | 5.99616 | 6.99788 | 7.99873 |
| $\Delta_j$ | 1.01660 | 1.00291 | 1.00088 | 1.00035 | 1.00016 | 1.00008 |
| $k_j$ | 0.00830 | 0.00145 | 0.00044 | 0.00017 | 0.00008 | 0.00004 |
| $M = 7.0M_{\odot}; n_1 = 2.35, n_2 = 2.66$ | | | | | | |
| $\eta_j$ | 2.89574 | 3.97402 | 4.98994 | 5.99513 | 6.99731 | 7.99838 |
| $\Delta_j$ | 1.02130 | 1.00372 | 1.00112 | 1.00044 | 1.00021 | 1.00011 |
| $k_j$ | 0.01065 | 0.00186 | 0.00056 | 0.00022 | 0.00010 | 0.00005 |
| $M = 10.0M_{\odot}; n_1 = 2.41, n_2 = 2.67$ | | | | | | |
| $\eta_j$ | 2.87080 | 3.96759 | 4.98754 | 5.99402 | 6.99671 | 7.99802 |
| $\Delta_j$ | 1.02653 | 1.00465 | 1.00139 | 1.00054 | 1.00025 | 1.00013 |
| $k_j$ | 0.01326 | 0.00233 | 0.00069 | 0.00027 | 0.00013 | 0.00007 |

We next considered what approach might be made that could find some mean or representative value of the index $n$ yielding the same value of $\eta_j$ as that for any given modern model obtained by detailed numerical integration of the structure equations: for example, a mass-shell weighted mean of the local polytropic index applying to any given mass-shell through the star. This could allow for suitable comparisons with historical treatments. This idea turned out not so directly applicable, however, since numerical integration of Radau's equation for a given structural model would need to be done anyway, in order to check the results of any alternative approach. However, in this way, we could show that representative values of $n$ for given numerically integrated stellar models correspond to $\eta_j$ values following

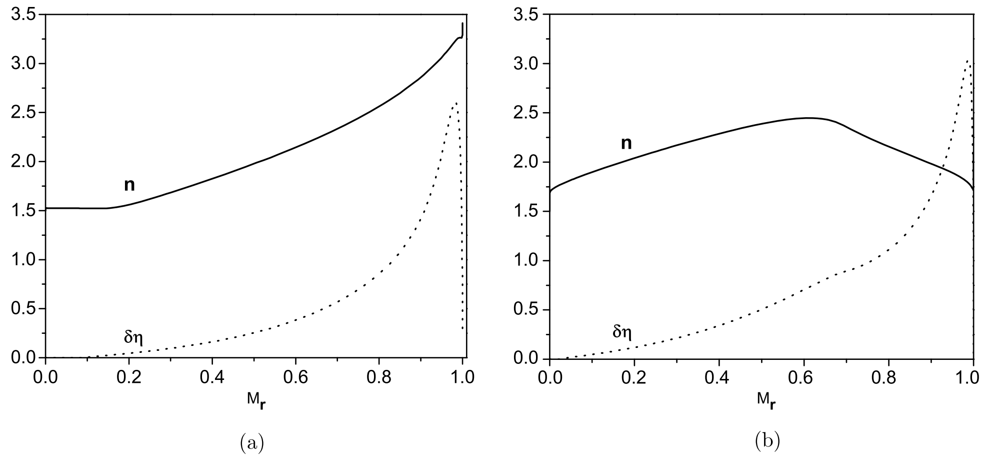


**Fig. 2** Integration of $\eta$ against mass distribution $M_r$ (**a**) $2M_\odot$ model (**b**) $0.5M_\odot$ model. The ordinate scale for the increment $\delta\eta$ is 1/50 of that shown, which directly applies to the local polytropic index value $n$ at the corresponding mass $M_r$. Note that the $n$-value (continuous line) corresponding to the peak value of $\delta\eta$, which maximizes towards the outermost mass-containing layers of the star, gives a good representation for equivalent polytropic stellar models compared with alternatives mentioned in the text. The low-mass star (convective envelope) thus has a representative value of $n$ less than 2, compared with a little $>3$ for the higher mass (radiative envelope) star

general expectations regarding the degree of central condensation. Mass-shell weighted means of local $n$ values ($n_1$ in Table 2) resulted in $\eta_j$ values that were typically accordant with the corresponding numerically integrated detailed stellar models to 2 or 3 significant digits. Another estimate ($n_2$), giving a comparable indication, comes from simply averaging the slope of the $\log\rho$ versus $\log T$ from the centre to each layer. But, an estimate having a closer reflection of the effects of the changing proportions of convective and radiative heat transports through the stars and the consequences of this on the mass distribution and corresponding deformations comes from the local value of the polytropic index in that layer of the star where the incremental contribution to the integration of $\eta$ maximises (Fig. 1).

Of course, no such averaging gainsays the desirability of the relatively simple evaluation of $\eta_j$ and the derivative structural coefficients $\Delta_j$ and $k_j$ for any given stellar model. In the present report we address, for this purpose, the models of stars, as directly available from the EZWeb website maintained by R. Townsend and associates at the University of Wisconsin-Madison (2011).[1] EZWeb models are based on Eggleton's (1971) evolution program.

## 3 Results and discussion

We applied the foregoing procedures to the Zero Age Main Sequence models for composition $Z = 0.02$ from the EZWeb website. Out results are listed in Table 2, which lists values of $\eta_j$, $\Delta_j$, $k_j$ for values of $j = 2$–7 for representative solar-like composition models in the mass range 0.5–10 $M_\odot$.[2]

Also, indications coming from the representative polytropic indices $n$ are borne out by the general trend of increasing $n$, associated with the radiative envelopes of the outer parts of more massive stars. For the low mass stars the opposite holds (see also Fig. 2). However, there is some reversal of this trend at the highest masses. This is shown in Fig. 1, and it is also reflected in the comparisons of Table 1.

The constants $k_j$, often considered in the context of studies of the apsidal motion of eccentric close binary systems, decrease with increasing $j$ values (Claret and Gimenez 2010). In fact, practical comparisons, in such studies, can usually only be directed to some mean value (of both components) for the second harmonic coefficient $\bar{k}_2$. Our preliminary results support the trend of values of $\bar{k}_2$ of Claret and Gimenez (2010), although we have not studied the effects of evolution, composition or internal structural variations associated with more detailed modeling. Some small apparent discrepancies between our results and those of Claret and Gimenez (2010) are of interest and should be checked further, as well as taking into account more critical recent appraisals of the solar composition.

On the other hand, all $\eta_j$ are involved separately, for either component, in the specification of the main tidal and rotational distortions through the coefficients $\Delta_j$ appearing in the formulae for the photometric effects of proximity for

[1] http://www.astro.wisc.edu/~townsend/static.php?ref=ez-web (2011).

[2] We are grateful to a referee for pointing out that the solar metallicity value has been substantially revised in the last decade. According to the critical compilation of Asplund et al. (2009) and also Grevesse et al. (2010) $Z_\odot = 0.0134$ gives a much better representation.

close binary stars (cf. e.g. Kopal 1959). Such formulae specify photometric variations arising from 'ellipticity' (tidal and rotational) effects that depend on, in addition to a number of separate parameters ($a_i$), factored by the relative luminosity of either component $L_k$, the coefficients $\Delta_j$. Typical treatment proceeds to the fifth order in the relative radii $r_{1,2}$. Although the effect of tides on tides is neglected in such 'first-order' approximations (as with the Roche models), the main contributions from finite density envelope structure are self-consistently included (unlike with the Roche models). Fast and robust curve-fitting programs that analyze for such effect are discussed elsewhere (cf. Budding and Demircan 2007). These are likely to have increasing importance with the growth of significantly improved photometric accuracies in the post-Kepler Mission era when light curves of mmag accuracy or better are expected. The proximity effects considered here are typically of order 0.1 mag in the majority of normal eclipsing binary light curves. The above table shows that stellar type dependent structural variations affecting the principle terms of the ellipticity variation become significant at the 1 % level, i.e. ∼0.001 mag, and therefore will require attention in this context.

## References


Asplund, M., Grevesse, N., Sauval, A.J., Scott, P.: Annu. Rev. Astron. Astrophys. **47**, 481 (2009)

Brooker, R.A., Olle, T.W.: Mon. Not. R. Astron. Soc. **115**, 101 (1955)

Budding, E., Demircan, O.: An Introduction to Astronomical Photometry. Cambridge Univ. Press, Cambridge (2007)

Claret, A.: Astron. Astrophys. Suppl. Ser. **109**, 441 (1995)

Claret, A.: Astron. Astrophys. **424**, 919 (2004)

Claret, A., Gimenez, A.: Astron. Astrophys. **519**, A57 (2010)

Eggleton, P.P.: Mon. Not. R. Astron. Soc. **151**, 351 (1971)

Grevesse, N., Asplund, M., Sauval, A.J., Scott, P.: Astrophys. and Space. Science **328**, 179 (2010)

Hejlesen, P.M.: Astron. Astrophys. Suppl. Ser. **69**, 251 (1987)

Jeffery, C.M.: Mon. Not. R. Astron. Soc. **207**, 323 (1984)

Kopal, Z.: Close Binary Systems, Chap. 2. Chapman and Hall, London (1959)

Kushwaha, R.S.: Astrophys. J. **125**, 242 (1957)

MacRobert, T.M.: Spherical Harmonics. E.P. Dutton, London (1927)

Petty, A.F.: Astrophys. and Space. Science **21**, 189 (1973)

Pressly, E.C.: US. Naval Research Laboratory. Washington, Report 3715 (2001)

Schwarzschild, M.: Evolution of the Stars. Dover, New York (1958)

Torres, G., Andersen, J., Gimenez, A.: Astron. Astrophys. Rev. **18**, 67 (2010)