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SOME CONVOLUTION INEQUALITIES IN MUSIELAK ORLICZ SPACES

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Abstract. Uniform boundedness of some family of convolution-type operators with kernels, such as Steklov, Poisson, Cesàro, De la Vallée-Poussin, Fejér, Jackson, having some properties are investigated in Musielak Orlicz spaces. As an application we obtained approximate identities in these spaces.

1. Introduction

Approximate identities are very useful tool ([4, p.31, Def. 1.1.4], [19, p.62], [20, Ch.9]) in Fourier and Harmonic Analysis. In these books there are two approaches. For the approach defined in the books [19, p.62] and [20, Ch.9] approximate identities are investigated by Benkirane, Douieb, Val ([3]); Cruz-Uribe, Fiorenza ([5]); Hudzik ([8]); Maeda, Ohno, Mizuta, Shimomura ([10, 11]) and Samko ([13]) in generalized Lebesgue spaces with variable exponent and Musielak Orlicz spaces. Some convolution type inequalities were investigated by R. A. Bandaliev, A. H. Isayev in [2] and F. I. Mamedov, S. H. Ismailova in [12].

For the approach similar to definition in [4, p.31, Def. 1.1.4] some results are obtained by Sharapudinov ([15]) and Shah-Emirov ([14]) in (weighted) generalized Lebesgue spaces with variable exponent. Continuing this fact our work mainly focus on to obtain approximate identities in Musielak Orlicz spaces. To do this we will consider $\lambda \geq 1$ and 2π-periodic, essentially bounded kernels $k_\lambda = k_\lambda(x)$ on $T := [-\pi, \pi)$ such that

$$\int_T |k_\lambda(x)|dx \leq C_1; \tag{1.1}$$

$$\sup_{x\in T} |k_\lambda(x)| \leq C_2\lambda^\nu; \tag{1.2}$$

$$|k_\lambda(x)| \leq C_3; \quad \lambda^{-\gamma} \leq |x| \leq \pi \tag{1.3}$$

for some constants $C_{1,2,3}, \nu, \gamma > 0$, which are independent of λ. We define the operator

$$K_\lambda f(x) = \int_T f(t)k_\lambda(t-x)dt, \quad 1 \leq \lambda < \infty, \quad x \in T.$$

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Then we prove that sequence of operators $\{K_\lambda f\}_{1\leq\lambda<\infty}$ is uniformly bounded (in λ) in Musielak Orlicz spaces L^φ for some conditions on φ. For example Steklov, Poisson, Cesàro, De la Vallée-Poussin, Fejér, Jackson's and some other kernels satisfy (1.1-1.3). As a result we can obtain several approximate identities in Musielak Orlicz spaces L^φ. Note that we will use a Dini-Lipschitz type condition on φ. Also we obtain that the family $\{S_{\lambda,\tau}f\}_{1\leq\lambda<\infty}$ formed with translation of Steklov-type means in L^φ, is uniformly bounded for $\gamma>0$, $|\tau|\leq\pi\lambda^{-\gamma}$, where $S_{\lambda,\tau}f$ is defined ([16]) by

$$S_{\lambda,\tau}f(x):=S_\lambda f(x+\tau):=\lambda\int_{\tau-1/(2\lambda)}^{\tau+1/(2\lambda)}f(x+u)du.$$

In §2 we give preliminary notations and definitions. In §3 we consider uniform boundedness of the family $\{S_{\lambda,\tau}f\}_{1\leq\lambda<\infty}$. In §4 we consider the uniform boundedness of some family of convolution-type operators with kernels, such as Steklov, Poisson, Cesàro, De la Vallée-Poussin, Fejér, Jackson, having properties (1.1-1.3) in Musielak Orlicz spaces L^φ. In the last section §5 we obtain approximate identities in Musielak Orlicz spaces L^φ.

In what follows, $A\lesssim B$ will mean that, there exists a positive constant $C_{u,v,\ldots}$, dependent only on the parameters $u,v,\ldots$ and can be different in different places, such that the inequality $A\leq CB$ is hold. If $A\lesssim B$ and $B\lesssim A$ then we will write $B\approx A$.

2. Preliminaries

A function $\varphi:[0,\infty)\to[0,\infty]$ is called Φ-*function* (briefly $\varphi\in\Phi$) if Φ is convex, left continuous and

$$\varphi(0):=\lim_{t\to0^+}\varphi(t)=0,\quad \varphi(\infty):=\lim_{x\to\infty}\varphi(x)=\infty.$$

A Φ-function φ is said to be an N-function if it is continuous, positive and satisfies

$$\lim_{t\to0^+}\frac{\varphi(t)}{t}=0,\quad \lim_{t\to\infty}\frac{\varphi(t)}{t}=\infty.$$

Let $\Phi(T)$ be the collection of functions $\varphi:T\times[0,\infty)\to[0,\infty]$ such that

(i) $\varphi(x,\cdot)\in\Phi$ for every $x\in T$,

(ii) $\varphi(x,u)$ is in $L^0(T)$, the set of measurable functions, for every $u\geq0$.

A $\varphi(\cdot,u)\in\Phi(T)$ said to satisfy Δ_2 condition ($\varphi\in\Delta_2$) with respect to u if $\varphi(x,2u)\leq K\varphi(x,u)$ holds for all $x\in T,u\geq0$, with some constant $K\geq2$.

Subclass $\Phi(N)$ consists of functions $\varphi\in\Phi(T)$ such that

(I) $\varphi(x,\cdot)$ is, for every $x\in T$, an N-function and $\varphi\in\Delta_2$;

(II) there exists a constant $c>0$ such that $\inf_{x\in T}\varphi(x,1)\geq c$;

(III) $\int_T\varphi(x,1)<\infty$ and $\psi(x,1)\leq c$ a.e. on T;

(IV) there exists a constant $A>0$ such that for all $x,y\in T$ we have

$$\frac{\varphi(x,u)}{\varphi(y,u)}\leq u^{-A\ln\frac{1}{|x-y|}},\quad u\geq1.$$

Some examples belonging to $\Phi(N)$: Let $p:T\to[1,\infty)$ be in $L^0(T)$ such that 2π-periodic, essentially bounded on T and, for all $x,y\in T$ it has Dini-Lipschitz property

$$|p(x) - p(y)| \ln \frac{1}{|x-y|} \leq c$$

with a constant $c > 0$. Then the functions

- $\varphi(x,u) = u^{p(x)}, \quad \sup_{x\in T} p(x) < \infty,$
- (ii) $\varphi(x,u) = u^{p(x)} \log(1+u), \quad \sup_{x\in T} p(x) < \infty,$
- (iii) $\varphi(x,u) = u(\log(1+u))^{p(x)}$

belong to the class $\Phi(N)$.

For $\varphi \in \Phi(N)$ we set $\varrho_\varphi(f) := \int_T \varphi(x, |f(x)|)\, dx$. Generalized Orlicz class L^φ (or Musielak Orlicz space) is the class of 2π periodic Lebesgue measurable functions $f : T \to \mathbb{R}$ satisfying the condition $\lim_{\lambda\to 0} \varrho_\varphi(\lambda f) = 0$. Equivalent condition for $f \in L^0(T)$ to belong to L^φ is that $\varrho_\varphi(\lambda f) < \infty$ for some $\lambda > 0$. L^φ becomes a normed space with the Orlicz norm

$$\|f\|_{[\varphi]} := \sup\left\{\int_T |f(x)\, g(x)|\, dx : \varrho_\psi(g) \leq 1\right\}$$

and with the Luxemburg norm

$$\|f\|_\varphi = \inf\left\{\lambda > 0 : \varrho_\varphi\left(\frac{f}{\lambda}\right) \leq 1\right\}$$

where $\psi(t,v) := \sup_{u\geq 0}(uv - \varphi(t,u))$, $v \geq 0$, $t \in T$, is the complementary function (with respect to variable v) of φ in the sense of Young. These two norms are equivalent:

$$\|f\|_\varphi \leq \|f\|_{[\varphi]} \leq 2\|f\|_\varphi .$$

Young's inequality holds for complementary functions $\varphi, \psi \in \Phi(N)$

$$us \leq \varphi(x,u) + \psi(x,s)$$

where $u, s \geq 0$, $x \in T$. From Young's inequality we have

$$\|f\|_{[\varphi]} \leq \varrho_\varphi(f) + 1.$$

Also $\|f\|_\varphi \leq \varrho_\varphi(f)$ if $\|f\|_\varphi > 1$ and $\|f\|_\varphi \geq \varrho_\varphi(f)$ if $\|f\|_\varphi \leq 1$. Hölder's inequality holds:

$$\int_T |f(x)\, g(x)|\, dx \leq \|f\|_\varphi \|f\|_{[\psi]} . \tag{2.1}$$

If φ is an N-function, $r(x)$ is nonnegative and $r(x) \not\equiv 0$, then Jensen's integral inequality holds:

$$\varphi\left(\frac{1}{\int_T r(x)\, dx}\int_T f(x)\, r(x)\, dx\right) \leq \frac{1}{\int_T r(x)\, dx}\int_T \varphi(f(x))\, r(x)\, dx. \tag{2.2}$$

3. Steklov operator

In this section we will consider the uniform boundedness of the family formed with translation of Steklov means.

Theorem 3.1. *If we take $\gamma > 0$, $1 \leq \lambda < \infty$, $|\tau| \leq \pi\lambda^{-\gamma}$, then the sequence of operators $\{S_{\lambda,\tau}\}_{1\leq\lambda<\infty}$ defined by*

$$S_{\lambda,\tau}f(x) := S_\lambda f(x+\tau) = \lambda \int_{x+\tau-1/(2\lambda)}^{x+\tau+1/(2\lambda)} f(u)du$$

is uniformly bounded in λ and τ, for functions f in L^φ with $\varphi \in \Phi(N)$.

Proof. Let $N := \lfloor\lambda^\gamma\rfloor$, $h := 1/N$, $x \in T$, $x_k := (kh-1)\pi$, $U_k := [x_k, x_{k+1})$. Then $T = \bigcup_{k=0}^{2N-1} U_k$ where the length of U_k is $l(U_k) = |x_{k+1} - x_k| = \pi/\lfloor\lambda^\gamma\rfloor$.

Assume that $\|f\|_\varphi \leq 1$. We need to show that

$$\rho_\varphi(S_{\lambda,\tau}f) = \int_T \varphi(x, |(S_{\lambda,\tau}f)(x)|)\, dx \leq c$$

with $c > 0$ independent of f. Then

$$\begin{aligned}\rho_\varphi(S_{\lambda,\tau}f) &= \rho_\varphi\left(\lambda \int_{x+\tau-1/(2\lambda)}^{x+\tau+1/(2\lambda)} f(t)dt\right)\\ &= \int_T \varphi\left(x, \left|\lambda \int_{x+\tau-1/(2\lambda)}^{x+\tau+1/(2\lambda)} f(t)dt\right|\right) dx\\ &\leq \sum_{k=0}^{2N-1} \int_{x_k}^{x_{k+1}} \varphi\left(x, 1 + \lambda\left|\int_{x+\tau-1/(2\lambda)}^{x+\tau+1/(2\lambda)} f(t)dt\right|\right) dx.\end{aligned}$$

We set

$$\varphi_k(u) := \inf\left\{\varphi(x,u) : x \in \Xi^k\right\} \leq \inf\{\varphi(x,u) : x \in U_k\} =: \check{\varphi}(u)$$

for some larger set $\Xi^k \supset U_k$, which will be chosen later with the property

$$l\left(\Xi^k\right) \leq m\pi/\lfloor\lambda^\gamma\rfloor \tag{3.1}$$

for some $m > 1$. On the other hand

$$\rho_\varphi(S_{\lambda,\tau}f) \lesssim \sum_{k=0}^{2N-1} \int_{x_k}^{x_{k+1}} A_k(x,\lambda)\, \varphi_k\left(1 + \lambda\left|\int_{x+\tau-1/(2\lambda)}^{x+\tau+1/(2\lambda)} f(t)dt\right|\right) dx$$

where

$$A_k(x,\lambda) := \frac{\varphi\left(x, 1 + \lambda\left|\int_{x+\tau-1/(2\lambda)}^{x+\tau+1/(2\lambda)} f(t)dt\right|\right)}{\varphi_k\left(1 + \lambda\left|\int_{x+\tau-1/(2\lambda)}^{x+\tau+1/(2\lambda)} f(t)dt\right|\right)} := \frac{\varphi(x, \alpha(x,\lambda))}{\varphi_k(\alpha(x,\lambda))}.$$

We prove the uniform estimate $A_k(x,\lambda) \leq c$ for $x \in U_k$ where $c > 0$ is independent of x, k and λ. Indeed, since

$$\frac{\varphi(x,t)}{\varphi_k(t)} = \frac{\varphi(x,t)}{\varphi_k(\varsigma_k, t)} \leq t^{\frac{A}{\ln\left(\frac{1}{|x-\varsigma_k|}\right)}}, \quad x \in U_k, \varsigma_k \in \Xi^k$$

we have

$$A_k(x,\lambda) = \frac{\varphi(x, \alpha(x,\lambda))}{\varphi_k(\alpha(x,\lambda))} \leq \alpha(x,\lambda)^{\frac{A}{\ln\left(\frac{1}{|x-\varsigma_k|}\right)}}.$$

Also $|x-\varsigma_k| \leq l\left(\Xi^k\right) \leq m\pi/\lfloor\lambda^\gamma\rfloor$ and

$$\lambda^{\frac{A}{\ln\left(\frac{1}{|x-\varsigma_k|}\right)}} \leq \lambda^{\frac{A}{\ln\left(\frac{\lambda^\gamma}{6m}\right)}} \leq c\,(m,A)\,,$$

$$\left(\int_{x+\tau-1/(2\lambda)}^{x+\tau+1/(2\lambda)} |f(t)|\,dt\right) \leq C\,\|f\|_\varphi \leq C,$$

$$\alpha\,(x,\lambda)^{\frac{A}{\ln\left(\frac{1}{|x-\varsigma_k|}\right)}} \leq (\lambda\,(C+2))^{\frac{A}{\ln\left(\frac{\lambda^\gamma}{6m}\right)}} \leq C\,(m,A)\,.$$

Since $\varphi\,(x,t)$ is convex with respect to t, φ_k is convex and

$$\rho_\varphi\,(S_{\lambda,\tau}f) \lesssim \sum_{k=0}^{2N-1}\int_{x_k}^{x_{k+1}}\frac{c}{2}\varphi_k\,(1)\,dx +$$

$$\sum_{k=0}^{2N-1}\int_{x_k}^{x_{k+1}}\frac{C}{2}\varphi_k\left(\lambda\int_{x+\tau-1/(2\lambda)}^{x+\tau+1/(2\lambda)}|f(t)|\,dt\right)dx$$

$$=\frac{c\;\check{\varphi}\,(2\pi)}{2}\int_T dx + \frac{C}{2}\sum_{k=0}^{2N-1}\int_{x_k}^{x_{k+1}}\varphi_k\left(\lambda\int_{x+\tau-1/(2\lambda)}^{x+\tau+1/(2\lambda)}|f(t)|\,dt\right)dx$$

$$=c\check{\varphi}\,(2\pi)\,\pi + \frac{C}{2}\sum_{k=0}^{2N-1}\int_{x_k}^{x_{k+1}}\varphi_k\left(\lambda\int_{x+\tau-1/(2\lambda)}^{x+\tau+1/(2\lambda)}|f(t)|\,dt\right)dx.$$

In the last integral we use the Jensen's integral inequality (2.2) and

$$\begin{aligned}
\rho_\varphi\,(S_{\lambda,\tau}f) &\lesssim c+\sum_{k=0}^{2N-1}\int_{x_k}^{x_{k+1}}\varphi_k\left(\lambda\int_{x+\tau-1/(2\lambda)}^{x+\tau+1/(2\lambda)}|f(t)|\,dt\right)dx\\
&\lesssim c+\sum_{k=0}^{2N-1}\int_{x_k}^{x_{k+1}}\lambda\int_{x+\tau-1/(2\lambda)}^{x+\tau+1/(2\lambda)}\varphi_k\,(|f(t)|)\,dtdx\\
&\lesssim c+\lambda\sum_{k=0}^{2N-1}\int_{x_k}^{x_{k+1}}\int_{\tau-1/(2\lambda)}^{\tau+1/(2\lambda)}\varphi_k\,(|f(x+t)|)\,dtdx\\
&\lesssim c+\lambda\int_{\tau-1/(2\lambda)}^{\tau+1/(2\lambda)}\sum_{k=0}^{2N-1}\int_{x_k}^{x_{k+1}}\varphi_k\,(|f(x+t)|)\,dxdt\\
&\lesssim c+\lambda\int_{\tau-1/(2\lambda)}^{\tau+1/(2\lambda)}\sum_{k=0}^{2N-1}\int_{x_k-t}^{x_{k+1}-t}\varphi_k\,(|f(x)|)\,dxdt
\end{aligned}$$

We take as Ξ^k the set

$$\bigcup_{t\in(-\tau-1/(2\lambda),\tau+1/(2\lambda))}\{x : x+t\in U_k\}\,.$$

Clearly $\Xi^k \supset U_k$ and $l\left(\Xi^k\right) \leq 5\pi/\lfloor\lambda^\gamma\rfloor$. Then (3.1) is satisfied with $m=5$. Since each point $x\in T$ belongs simultaneously not more than to a finite number n_0

of the sets U_k, taking maximum with respect to all the sets U_k containing x we obtain

$$\begin{aligned}\rho_{\varphi}\left(S_{\lambda,\tau}f\right) &\lesssim c+\lambda\int_{\tau-1/(2\lambda)}^{\tau+1/(2\lambda)}dt\int_{T}\tilde{\varphi}\left(x,\left|f(x)\right|\right)dx\\ &\lesssim c+\int_{T}\tilde{\varphi}\left(x,\left|f(x)\right|\right)dx\end{aligned}$$

with $\tilde{\varphi}\left(x,u\right):=\max_i\varphi_i\left(t\right)$. Now using

$$\tilde{\varphi}\left(x,u\right)\leq\varphi\left(x,u\right),\quad\forall x\in T,$$

we get

$$\rho_{\varphi}\left(S_{\lambda,\tau}f\right)\lesssim c+\int_{T}\varphi\left(x,\left|f(x)\right|\right)dx\lesssim c+\left\|f\right\|_{\varphi}\leq C.$$

These are give

$$\left\|S_{\lambda,\tau}f\right\|_{\varphi}\lesssim\left\|f\right\|_{\varphi}.$$

and the result follows. □

Let $\varphi\in\Phi\left(N\right)$, $f\in L^{\varphi}$, $0<h\leq1$ and define the Steklov operator

$$T_hf\left(x\right):=S_{1/h,h/2}f\left(x\right)=\frac{1}{h}\int_0^h f\left(x+t\right)dt,\quad x\in T.$$

For $0\leq\delta\in\mathbb{R}^+$ we define the modulus of continuity for $f\in L^{\varphi}$, $\varphi\in\Phi\left(N\right)$, as

$$\Omega\left(f,\delta\right)_{\varphi}:=\sup_{0\leq h\leq\delta}\left\|\left(I-T_h\right)f\right\|_{\varphi}$$

where I is the identity operator. We have that if $\varphi\in\Phi\left(N\right)$, $f\in L^{\varphi}$ and $\delta\geq0$, then

$$\Omega\left(f,\delta\right)_{\varphi}\lesssim\left\|f\right\|_{\varphi}$$

holds for some constant depending only on φ. In general, modulus of continuity $\Omega\left(f,\cdot\right)_{\varphi}$ is the main tool in Approximation Theory ([1, 9, 17]).

4. Some convolution inequalities

Let $\lambda\geq1$, $k_\lambda=k_\lambda(x)$ be 2π-periodic, essentially bounded function defined on T, such that (1.1-1.3) hold. We define the operator

$$K_\lambda f(x)=\int_T f(t)k_\lambda(t-x)dt,\quad 1\leq\lambda<\infty,\quad x\in T.\tag{4.1}$$

Such type conditions on kernel and operators (4.1) were investigated for variable exponent Lebesgue spaces in [15].

Theorem 4.1. *Let $\lambda\geq1$, $k_\lambda=k_\lambda(x)$ be 2π-periodic, essentially bounded function defined on T, such that (1.1)-(1.3) to hold. If f in L^{φ} with $\varphi\in\Phi\left(N\right)$, then there exist a constant, independent of λ and f, such that*

$$\left\|K_\lambda f\right\|_{\varphi}\lesssim\left\|f\right\|_{\varphi}$$

holds.

Proof. The proof is similar to the proof of Theorem 3.1. Let $N := \lfloor \lambda^{\gamma} \rfloor$, $h := 1/N$, $x \in T$, $x_k := (kh-1)\pi$, $U_k := [x_k, x_{k+1})$,

$$E_x := \begin{cases} T \setminus (x-\pi h, x+\pi h) & \text{, when } (x-\pi h, x+\pi h) \subset T, \\ T \setminus \{(-\pi, x+\pi h) \cup (x-\pi h+2\pi, \pi)\} & \text{, when } x-\pi h < -\pi, \\ T \setminus \{(x-\pi h, \pi) \cup (-\pi, x+\pi h-2\pi)\} & \text{, when } x+\pi h > \pi. \end{cases}$$

Then $T = \bigcup_{k=0}^{2N-1} U_k$ where the length of U_k is $l(U_k) = |x_{k+1} - x_k| = \pi / \lfloor \lambda^{\gamma} \rfloor$.

Assume that $\|f\|_{\varphi} = 1$. We need to show that

$$\rho_{\varphi}(K_{\lambda} f) = \int_T \varphi(x, |(K_{\lambda} f)(x)|)\, dx \le c$$

with $c > 0$ independent of f. Then convexity of φ implies

$$\begin{aligned} \rho_{\varphi}(K_{\lambda} f) &= \rho_{\varphi}\left(\int_T f(t) k_{\lambda}(t-x) dt\right) = \rho_{\varphi}\left(\left\{\int_{x-\pi h}^{x+\pi h} + \int_{E_x}\right\} f(t) k_{\lambda}(t-x) dt\right) \\ &\le \frac{K}{2} \rho_{\varphi}\left(\int_{x-\pi h}^{x+\pi h} f(t) k_{\lambda}(t-x) dt\right) + \frac{K}{2} \rho_{\varphi}\left(\int_{E_x} f(t) k_{\lambda}(t-x) dt\right) \\ &=: I_1 + I_2. \end{aligned}$$

If $x \in T$ and $t \in E_x$, then, from (1.3), we have

$$|k_{\lambda}(t-x)| \lesssim 1.$$

Using Hölder's inequality (2.1) and (III) we obtain

$$\begin{aligned} \left|\int_{E_x} f(t) k_{\lambda}(t-x) dt\right| &\lesssim \int_T |f(t)|\, dt \\ &\lesssim \|f\|_{\varphi} \|1\|_{[\psi]} \lesssim \|1\|_{[\psi]} \lesssim c+1 \end{aligned}$$

and hence

$$\begin{aligned} I_2 &\lesssim \rho_{\varphi}\left(2C \int_{E_x} f(t) k_{\lambda}(t-x) dt\right) \le K \int_T \varphi\left(x, \int_{E_x} f(t) k_{\lambda}(t-x) dt\right) dx \\ &\lesssim \int_T \varphi(x, c+1)\, dx \lesssim \int_T \varphi(x, 1)\, dx \le C. \end{aligned}$$

Now

$$\begin{aligned} I_1 &\lesssim \int_T \varphi\left(x, \int_{x-\pi h}^{x+\pi h} |f(t)|\, |k_{\lambda}(t-x)|\, dt\right) dx \\ &\le \sum_{k=0}^{2N-1} \int_{x_k}^{x_{k+1}} \varphi\left(x, 1 + \int_{x-\pi h}^{x+\pi h} |f(t)|\, |k_{\lambda}(t-x)|\, dt\right) dx. \end{aligned}$$

On the other hand

$$I_1 \lesssim \sum_{k=0}^{2N-1} \int_{x_k}^{x_{k+1}} A_k(x, \lambda)\, \varphi_k\left(1 + \int_{x-\pi h}^{x+\pi h} |f(t)|\, |k_{\lambda}(t-x)|\, dt\right) dx$$

where

$$A_k(x,\lambda) := \frac{\varphi\left(x, 1+\int_{x-\pi h}^{x+\pi h}|f(t)|\,|k_\lambda(t-x)|\,dt\right)}{\varphi_k\left(1+\int_{x-\pi h}^{x+\pi h}|f(t)|\,|k_\lambda(t-x)|\,dt\right)} := \frac{\varphi(x,\alpha(x,\lambda))}{\varphi_k(\alpha(x,\lambda))}.$$

We prove the uniform estimate $A_k(x,\lambda) \le c$ for $x \in U_k$ where $c > 0$ is independent of x, k and λ. Indeed, since

$$\frac{\varphi(x,t)}{\varphi_k(t)} = \frac{\varphi(x,t)}{\varphi_k(\varsigma_k,t)} \le t^{\overline{\ln\left(\frac{1}{x-\varsigma_k}\right)}^{A}}, \quad x \in U_k, \varsigma_k \in \Xi^k$$

we have

$$A_k(x,\lambda) = \frac{\varphi(x,\alpha(x,\lambda))}{\varphi_k(\alpha(x,\lambda))} \le \alpha(x,\lambda)^{\overline{\ln\left(\frac{1}{x-\varsigma_k}\right)}^{A}}.$$

Also $|x-\varsigma_k| \le l\left(\Xi^k\right) \le m\pi/\lfloor\lambda^\gamma\rfloor$ and

$$|\alpha(x,\lambda)| \le \lambda^\upsilon\left(1+\int_{x-\pi h}^{x+\pi h}|f(t)|\,dt\right) \le c\lambda^\upsilon\|f\|_\varphi = c\lambda^\upsilon,$$

$$\begin{aligned}\alpha(x,\lambda)^{\overline{\ln\left(\frac{1}{x-\varsigma_k}\right)}^{A}} &\le \alpha(x,\lambda)^{\overline{\ln\left(\frac{\lambda^\gamma}{6m}\right)}^{A}} \le (C\lambda^\upsilon)^{\overline{\ln\left(\frac{\lambda^\gamma}{6m}\right)}^{A}} \\ &\le C(m,A)\left(\lambda^{1/\ln\left(\frac{\lambda}{6m}\right)}\right)^{\upsilon A} \le C(m,A,\upsilon).\end{aligned}$$

Let $\mu_\lambda = \int_{x-\pi h}^{x+\pi h}|k_\lambda(t-x)|\,dt = \int_{-\pi h}^{\pi h}|k_\lambda(t)|\,dt$.Then $\mu_\lambda \le C$. Without loss of generality we may assume that $\mu_\lambda > 0$, because the sequence of operators $\{K_\lambda f\}_{1\le\lambda<\infty}$ formed with with $\mu_\lambda = 0$ is uniformly bounded in L^φ, $\varphi \in \Phi(N)$.

As before, by Jensen's integral inequality (2.2)

$$\begin{aligned}I_1 &\lesssim \sum_{k=0}^{2N-1}\int_{x_k}^{x_{k+1}}\varphi_k\left(1+C\frac{1}{\mu_\lambda}\int_{x-\pi h}^{x+\pi h}|f(t)|\,|k_\lambda(t-x)|\,dt\right)dx \\ &\lesssim c + C\sum_{k=0}^{2N-1}\int_{x_k}^{x_{k+1}}\varphi_k\left(\frac{1}{\mu_\lambda}\int_{x-\pi h}^{x+\pi h}|f(t)|\,|k_\lambda(t-x)|\,dt\right)dx \\ &\lesssim c + \sum_{k=0}^{2N-1}\int_{x_k}^{x_{k+1}}\frac{1}{\mu_\lambda}\int_{x-\pi h}^{x+\pi h}\varphi_k(|f(t)|)\,|k_\lambda(t-x)|\,dtdx \\ &\lesssim c + \sum_{k=0}^{2N-1}\frac{1}{\mu_\lambda}\int_{-\pi h}^{\pi h}|k_\lambda(t)|\int_{x_k}^{x_{k+1}}\varphi_k(|f(x+t)|)\,dxdt \\ &\lesssim c + \frac{1}{\mu_\lambda}\int_{-\pi h}^{\pi h}|k_\lambda(t)|\sum_{k=0}^{2N-1}\int_{x_k}^{x_{k+1}}\varphi_k(|f(x+t)|)\,dxdt \\ &\lesssim c + \frac{1}{\mu_\lambda}\int_{-\pi h}^{\pi h}|k_\lambda(t)|\sum_{k=0}^{2N-1}\int_{x_k-t}^{x_{k+1}-t}\varphi_k(|f(x)|)\,dxdt.\end{aligned}$$

We take as Ξ^k the set

$$\bigcup_{t\in(-\pi h,\pi h)}\{x : x+t \in U_k\}.$$

Clearly $\Xi^k \supset U_k$ and $l\left(\Xi^k\right) \leq 3\pi/\lfloor\lambda^\gamma\rfloor$. Then (3.1) is satisfied with $m=3$. Since each point $x \in T$ belongs simultaneously not more than to a finite number n_0 of the sets U_k, taking maximum with respect to all the sets U_k containing x we obtain

$$\begin{aligned} I_1 &\lesssim c + \frac{1}{\mu_\lambda}\int_{-\pi h}^{\pi h} |k_\lambda(t)|\,dt \int_T \tilde{\varphi}\left(x,|f(x)|\right)dx \\ &\lesssim c + \int_T \tilde{\varphi}\left(x,|f(x)|\right)dx \end{aligned}$$

with $\tilde{\varphi}\left(x,u\right) := \max_i \varphi_i\left(t\right)$. Now using

$$\tilde{\varphi}\left(x,u\right) \leq \varphi\left(x,u\right), \quad \forall x \in T,$$

we get

$$\rho_\varphi\left(K_\lambda f\right) \lesssim c + \int_T \varphi\left(x,|f(x)|\right)dx \lesssim c + \|f\|_\varphi \leq C.$$

These are give

$$\|K_\lambda f\|_\varphi \lesssim \|f\|_\varphi$$

and the result follows. □

5. Approximate identities

Hölder's inequality (2.1) and (III) imply

$$\int_T |f(t)|\,dt \lesssim \|f\|_\varphi \|1\|_{[\psi]} \leq C\,\|f\|_\varphi$$

and hence $L^\varphi \subset L^1$. Let

$$f\left(x\right) \backsim \frac{a_0\left(f\right)}{2} + \sum_{k=1}^{\infty}\left(a_k\left(f\right)\cos kx + b_k\left(f\right)\sin kx\right) =: \sum_{k=0}^{\infty} A_k\left(x,f\right) \tag{5.1}$$

be the Fourier series of f in L^φ with $\varphi \in \Phi\left(N\right)$ and

$$S_n\left(x,f\right) := \sum\nolimits_{k=0}^{n} A_k\left(x,f\right), \quad n = 0,1,2,\ldots.$$

be the partial sum of the Fourier series (5.1). It is well known that

$$S_n(x,f) = \frac{1}{2\pi}\int_T f(t)D_n(t-x)dt \tag{5.2}$$

with Dirichlet kernel $D_n(u) := 1 + 2\sum\limits_{k=1}^{n}\cos ku$.

We define, for $n,m \in \mathbb{N}\cup\{0\}$, De la Vallée-Poussin mean

$$V_m^n(f,\cdot) = \frac{1}{m+1}\sum_{i=0}^{m} S_{n+i}(\cdot,f). \tag{5.3}$$

Note that we can give below examples of kernels satisfying the properties (1.1)-(1.3):

(a) Steklov Operator $\sigma_\lambda f$: Let $\Delta_\lambda := [-1/(2\lambda), 1/(2\lambda)]$, $\lambda \geq 1$ and

$$k_\lambda(x) := \begin{cases} \lambda & , x \in \Delta_\lambda, \\ 0 & , x \in T\setminus\Delta_\lambda. \end{cases}$$

We extend k_λ to $\mathbb{R} :=(-\infty,\infty)$ with period 2π. Steklov operator $\sigma_\lambda f$ is represented as

$$\sigma_\lambda f(x)=\lambda\int_{x-1/(2\lambda)}^{x+1/(2\lambda)} f(u)du=\int_T f(t)k_\lambda(t-x)dt.$$

kernel k_λ satisfies the properties (1.1)-(1.3) with $\nu=1=\gamma$.

(b) De la Vallée-Poussin Operator $\mathcal{V}_m^n f$: Based on (5.3)and (5.2) we define De la Vallée-Poussin Operator as

$$\mathcal{V}_m^n f\,(x)=\int_T f(t)K_m^n(t-x)dt$$

where

$$K_m^n(u):=\frac{\sin^2{(m+n+1)\,u/2}-\sin^2{(nu/2)}}{2\,(m+1)\sin^2{(nu/2)}}.$$

In this case kernels K_{n-1}^n and K_n^n are satisfy the conditions (1.1)-(1.3).

(c) Fejér Operator $\mathcal{F}_\lambda f$: Let $n\in\mathbb{N}$,

$$k_n\,(x)=\frac{1}{2\,(n+1)}\left[\frac{\sin{((n+1)\,x/2)}}{\sin{(x/2)}}\right]^2, \tag{5.4}$$

be the Fejér kernel and $k_\lambda\,(x):=k_n\,(x)$ for $n\leq\lambda<n+1$. The Fejér Operator is defined as $\mathcal{F}_\lambda f\,(x):=\frac{1}{\pi}\int_T f(t)k_\lambda(t-x)dt$. The Fejér kernel (5.4) satisfies the properties (1.1)-(1.3) with $\nu=1,\gamma=1/2$ since

$$k_n(t)\leq\frac{n+1}{2},\quad k_n(t)\leq\frac{C}{(n+1)\,t^2}$$

for $0<t<\pi$.

(d) Cesàro Operator $\mathcal{C}_\lambda f$: Let $\lambda\in\mathbb{N}$, $\alpha>0$ and

$$\mathcal{C}_\lambda f\,(x):=\frac{1}{\pi}\int_T f(t)k_\lambda^\alpha(t-x)dt$$

be the Cesàro Operator with Cesàro kernel

$$\begin{aligned} k_\lambda^\alpha(t) &= \sum_{k=0}^{\lambda}\frac{A_{\lambda-k}^{\alpha-1}\mathbf{D}_k\,(t)}{A_\lambda^\alpha},\quad \mathbf{D}_k\,(t)=\sum_{v=0}^{k}\frac{\sin{((v+1/2)\,t)}}{2\sin{(1/2)}\,t},\\ A_\lambda^\alpha &= \binom{\lambda+\alpha}{\alpha}\approx\frac{\lambda^\alpha}{\Gamma\,(1+\alpha)}\end{aligned}$$

satisfies the properties (1.1)-(1.3) with $\nu=1,\gamma=\alpha/\left(\alpha+1\right)$, because

$$k_\lambda^\alpha(t)\leq 2n,\quad k_\lambda^\alpha(t)\leq\frac{C_\alpha}{\lambda^\alpha\left|t\right|^{\alpha+1}}$$

for $0<|t|<\pi$.

(e) Poisson Operator $\mathcal{P}_\lambda f$: Let $0\leq r<1$ and $\lambda=1/\left(1-r\right)$. We define Poisson Operator

$$\mathcal{P}_\lambda\,(f,x):=\frac{1}{\pi}\int_T f(t)k_\lambda(t-x)dt$$

with the Poisson kernel

$$k_\lambda\,(x)=P\,(r,x)=\frac{1-r^2}{2\,(1-2r\cos x+r^2)}$$

which satisfies the properties (1.1)-(1.3) with $\nu = 1, \gamma = 1$ because $\int_T k_\lambda(x)dx = \pi$, $k_\lambda(x) \le (1+r)/(2(1-r))$, $k_\lambda(x) \le \pi$ $(\lambda \le x \le \pi)$.

(f) Jackson Operator $J_\lambda f$: We define the Jackson operator

$$J_\lambda f(x) := \frac{1}{\pi}\int_T f(t)k_\lambda(t-x)dt, \quad \lambda \in \mathbb{N},$$

where k_n is the Jackson kernel

$$k_\lambda(x) := \frac{3}{2\lambda(2\lambda^2+1)}\left(\frac{\sin(\lambda x/2)}{\sin(\lambda/2)}\right)^4$$

satisfy (1.1)-(1.3) with $\nu = 1, \gamma = 3/4$ as

$$\begin{aligned} \frac{1}{\pi}\int_T k_\lambda(t)dt &= 1, \\ |k_\lambda(u)| &\lesssim 1, \quad \lambda^{-3/4} \le u \le 2\pi - \lambda^{-3/4}, \\ \max_{t\in T}|k_\lambda(u)| &\lesssim \lambda. \end{aligned}$$

(g) Let $k_n(u) := \begin{cases} \frac{1}{n\left(2\sin\frac{\pi}{2n}\right)^2} & , |u| \le \frac{\pi}{2n} \\ n^{-1}\left(2\sin\frac{u}{2}\right)\sin nu & , \frac{\pi}{2n} < u \le 2\pi - \frac{\pi}{2n} \end{cases}$ and extend $k_n(u)$ to a 2π-periodic function ([18]) on the whole real axis. Then satisfy $k_n(u)$ (1.1)-(1.3) with $\nu = 1, \gamma = 1/2$.

Now, Theorem 4.1 gives that

Corollary 5.1. *The sequence of operators $\{O_\lambda f\}_{1\le\lambda<\infty}$, given in examples (a)-(g), is uniformly bounded (in λ) in L^φ with $\Phi(N)$.*

Theorem 5.1. *Let $\lambda \ge 1$, $k_\lambda = k_\lambda(x)$ be 2π-periodic, essentially bounded function defined on T, such that (1.1)-(1.3) and $\int_T k_\lambda(x)dx = 1$. If f in L^φ with $\varphi \in \Phi(N)$, then $K_\lambda f$ is an approximate identity, i.e.*

$$\|(K_\lambda - I)f\|_\varphi \to 0$$

as $\lambda \to \infty$.

Proof. Using Corollary 3.7 of [6] we have

$$L^1 \cap L^p \hookrightarrow L^\varphi, \quad \varphi(x, |f(x)|) \le \varphi(x,1)\max\{D|f(x)|^p, |f(x)|\}$$

where $D > 2$ is Δ_2 constant of φ and $p := \log_2 D$. Then

$$\|(K_\lambda - I)f\|_\varphi \le C\|K_\lambda f - f\|_p \to 0$$

as $\lambda \to \infty$. □

Note that Steklov Operator $\sigma_\lambda f$, Fejér Operator $\mathcal{F}_\lambda f$, Cesàro Operator $\mathcal{C}_\lambda f$, Poisson Operator $\mathcal{P}_\lambda f$, Jackson Operator $J_\lambda f$ is approximate identity in L^φ with $\Phi(N)$.

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