CONVOLUTIONS AND BEST APPROXIMATIONS IN VARIABLE EXPONENT LEBESGUE SPACES

DANIYAL M. ISRAFILOV and ELIFE YIRTICI

Communicated by Vasile Brînzănescu

In the variable exponent Lebesgue spaces a convolution is defined and its estimations in the variable exponent Lebesgue spaces by the best approximation numbers are obtained.

AMS 2010 Subject Classification: 41A10, 42A10.

Key words: convolution, variable exponent Lebesgue space, best approximation.

1. INTRODUCTION AND MAIN RESULTS

Let $p(\cdot):[0,2\pi]\to[1,\infty)$ be a Lebesgue measurable function. We define the modular functional

$$\rho_{p(\cdot)}(f):=\int_0^{2\pi}|f(x)|^{p(x)}\,\mathrm{d}x$$

on the Lebesgue measurable functions f on $[0,2\pi]$. By $L_{2\pi}^{p(\cdot)}$ we denote the class of 2π periodic Lebesgue measurable functions f, such that for a constant $\lambda=\lambda(f)>0$

$$\rho_{p(\cdot)}(f/\lambda)<\infty.$$

Equipped with the norm

$$\|f\|_{p(\cdot)}:=\inf\left\{\lambda>0:\rho_{p(\cdot)}(f/\lambda)\le 1\right\}$$

the class $L_{2\pi}^{p(\cdot)}$ creates a Banach space.

The variable exponent Lebesgue spaces are a generalization of the classical Lebesgue spaces, replacing the constant exponent p with a variable exponent function $p(\cdot)$. Interest in the variable exponent Lebesgue spaces has increased since 1990s, because of their use in the different applications to problems in mechanic, especially in fluid dynamic for the modeling of electrorheological fluids and also in the study of image processing and some problems in physics (see, for example the monographs [4, 5, 14] and the references cited therein). Nowadays there are sufficiently wide investigations relating to the fundamental problems of these spaces, in view of potential theory, maximal and singular integral operator theory and others. The sufficiently wide presentation of the corresponding results can be found in the monographs [4, 5, 14].

In the variable exponent Lebesgue spaces some fundamental problems of approximation theory were investigated also. Some necessary and sufficient conditions in term of the variable exponent for the basicity of the well known classical systems of functions were obtained [15, 16], the different modulus of smoothness were defined [7, 8, 17] and the direct and inverse theorems of approximation theory in these spaces defined on the intervals of the real line and on the domains of the complex plane were proved [1–3, 7–9, 17]. The detailed information on these results and also on the general aspects of approximation theory in the variable exponent Lebesgue spaces can be found in the monograph [18].

Note that in the variable exponent Lebesgue space theory some convolution operators were commonly used. This type of operators have some applications also in the approximation theory, in particular for the construction of the approximation polynomials. Therefore, the estimation problem of convolution operators by using the best approximation numbers is an actual problem of approximation theory. In this work, we investigate this problem in the variable exponent Lebesgue spaces. For the formulation of the main results obtained in this work we give some notations.

By $\beta_{2\pi}$ we denote the class of the exponents $p(\cdot)$, for which $1 < p_- \leq p_+ < \infty$, where

$$p_- := \underset{x\in[0,2\pi]}{essinf}\, p(x), \quad \text{and} \quad p_+ := \underset{x\in[0,2\pi]}{essup}\, p(x).$$

Definition 1. Let $p(\cdot) : [0, 2\pi] \to [1, \infty)$ be a 2π periodic, measurable function. We say that $p(\cdot)$ is a log-Holder continuous function on $[0, 2\pi]$ if

$$|p(x) - p(y)| \leq \frac{c_0}{-\log(|x-y|)}, \quad x, y \in [0, 2\pi] \quad \text{with} \quad |x-y| \leq 1/2 \tag{1.1}$$

with some constant $c_0 > 0$.

If $p(\cdot) \in \beta_{2\pi}$ and satisfies the condition (1.1), then we say that $p(\cdot) \in \widehat{\beta}_{2\pi}$.

In the space $L_{2\pi}^{p(\cdot)}$ with $p(\cdot) \in \widehat{\beta}_{2\pi}$ we define a mean value operator σ_h

$$(\sigma_h f)(x, u) := \frac{1}{2h}\int_{-h}^{h} f(x + tu)\,\mathrm{d}t, \ 0 < h < \pi, \ x \in [0, \pi], \ -\infty < u < \infty,$$

which is linear and bounded by [6], moreover $\|(\sigma_h f)\|_{p(\cdot)} \leq c_1 \|f\|_{p(\cdot)}$ for some constant $c_1 > 0$.

For $f \in L_{2\pi}^{p(\cdot)}$ we define also the best approximation number

$$E_n(f)_{p(\cdot)} := \inf_{T_n} \|f - T_n\|_{p(\cdot)}$$

by trigonometric polynomials

$$T_n(x) = \sum_{k=0}^{n} c_k e^{ikx}$$

of degree at most n and by T_n^* the best approximation trigonometric polynomial of degree $\leq n$, such that

$$\|f - T_n^*\|_{p(\cdot)} = E_n(f)_{p(\cdot)}.$$

Let $f \in L_{2\pi}^{p(\cdot)}$. We define a convolution type operator

$$\int_{-\infty}^{\infty} (\sigma_h f)(\cdot, u)\, d\sigma(u)$$

with a bounded variation function $\sigma(u)$ on the real line $\mathbb{R}$ and denote

$$D(f, \sigma, h, p(\cdot)) := \left\| \int_{-\infty}^{\infty} (\sigma_h f)(\cdot, u)\, d\sigma(u) \right\|_{p(\cdot)}.$$

In this work, we estimate the quantity $D(f, \sigma, h, p(\cdot))$ using the best approximation number $E_n(f)_{p(\cdot)}$.

Our new results are following:

THEOREM 1. *If $f \in L_{2\pi}^{p(\cdot)}$ with $p(\cdot) \in \hat{\beta}_{2\pi}$, then*

$$D(f, \sigma, h, p(\cdot)) \leq c \sum_{k=0}^{m} E_{2^k - 1}(f)_{p(\cdot)}\, \delta_{2^k, h} + c_{p(\cdot)} E_{2^{m+1}}(f)_{p(\cdot)}$$

for every $m \in \mathbb{N}$, where

$$\delta_{2^k,h} \ := \sum_{l=2^k}^{2^{k+1}-1} |\hat{\sigma}(lh) - \hat{\sigma}((l+1)h)| + \left|\hat{\sigma}\left(2^k h\right)\right|,$$

$$\hat{\sigma}(x) \ := \int_{-\infty}^{\infty} \frac{\sin(ux)}{ux} d\sigma(u), \quad 0 < h < \pi.$$

THEOREM 2. *Let $f \in L_{2\pi}^{p(\cdot)}$ with $p(\cdot) \in \hat{\beta}_{2\pi} p(\cdot)$ and let $F(x)$ be a function with bounded variation, i.e.*

$$\|F(x)\| \leq c_1, \quad \sum_{\theta=2^\mu}^{2^{\mu+1}-1} |F(\theta h) - F((\theta+1)h)| \leq c_2, \quad h \leq 2^{-m-1}.$$

If σ_1 and σ_2 are two functions satisfying the condition

$$\hat{\sigma}_1(x) = \hat{\sigma}_2(x) F(x), \quad |x| < 1,$$

then

$$D(f, \sigma_1, h, p(\cdot)) = c \left[D(f, \sigma_2, h, p(\cdot)) + E_{2^{m+1}}(f)_{p(\cdot)} \right].$$

Using the usual shift $f(x+t)$ in the definition of the convolution operator Theorems 1 and 2 in the Orlicz spaces were proved in [13]. In the weighted Orlicz spaces in term of the mean value operator $(\sigma_h f)(x,u)$, Theorems 1 and 2 were obtained in [10]. We also use this operator in the variable exponent spaces $L_{2\pi}^{p(\cdot)}$, because $L_{2\pi}^{p(\cdot)}$ in general is non invariant with respect to the usual shift $f(x+t)$.

Throughout this paper, the constant c denotes a generic constant, *i.e.* a constant whose values can change even between different occurrences in a chain of inequalities.

2. AUXILIARY RESULTS

As was proved in [15], if $p(\cdot) \in \hat{ß}_{2\pi}$, then the system $\{e^{ikx}\}_{k\in\mathbb{Z}}$ of exponents greats a base in the space $L_{2\pi}^{p(\cdot)}$, which is equivalent to the inequality

$$\|S_n(f)\|_{p(\cdot)} \leq c_{p(\cdot)} \|f\|_{p(\cdot)} \quad (n = 0, 1, 2, ...),$$

where

$$S_n(f)(x) := \sum_{k=-n}^{n} \hat{f}_k(x) e^{ikx}$$

is the nth partial sum of the Fourier series

$$\sum_{k=1}^{\infty} \hat{f}_k e^{ikx} \quad ; \quad \hat{f}_k(x) = \frac{1}{2\pi}\int_0^{2\pi} f(y) e^{-iky} \mathrm{d}y$$

of f. This inequality implies the following

LEMMA 1. *If $f \in L_{2\pi}^{p(\cdot)}$ with $p(\cdot) \in \hat{ß}_{2\pi}$, then*

$$\|f - S_n f\|_{p(\cdot)} \leq c E_n(f)_{p(\cdot)}.$$

Proof.

$$\begin{aligned} \|f - S_n f\|_{p(\cdot)} &\leq \|f - T_n\|_{p(\cdot)} + \|S_n f - T_n\|_{p(\cdot)} \\ &\leq E_n(f)_{p(\cdot)} + \|S_n(f - T_n)\|_{p(\cdot)} \\ &\leq E_n(f)_{p(\cdot)} + c\|(f - T_n)\|_{p(\cdot)} \\ &\leq c_1 E_n(f)_{p(\cdot)} \end{aligned}$$

with some constant $c_1 = c_1(p)$¿ 0. □

The proofs of the following two Theorems A and B can be found in [4, p. 27, Theorem 2.26] and [18, p. 39, Theorem 1.6.5], respectively.

Theorem A. *Let* $p : [0, 2\pi] \to [1, \infty)$ *be a* 2π *periodic, measurable function. If* $f \in L_{2\pi}^{p(\cdot)}$ *and* $g \in L_{2\pi}^{p'(\cdot)}$, *where* $1/p(\cdot) + 1/p'(\cdot) = 1$, *then* $fg \in L_{2\pi}^{1}$ *and*

$$\int_0^{2\pi} |f(x)\, g(x)|\, \mathrm{d}x \le K_{p(\cdot)} \|f\|_{p(\cdot)} \|g\|_{p'(\cdot)}$$

with

$$K_{p(\cdot)} = 1/p_- - 1/p_+ + 1.$$

Theorem B. *Let* $p(\cdot) : [0, 2\pi] \to [1, \infty)$ *and let* $f : [0, 2\pi] \times [0, 2\pi] \to \mathbb{R}$ *be a measurable function and* $f(\cdot, y) \in L_{2\pi}^{p(\cdot)}$ *for every* $y \in [0, 2\pi]$. *Then*

$$\left\| \int_0^{2\pi} f(\cdot, y)\, \mathrm{d}y \right\|_{p(\cdot)} \le c \int_0^{2\pi} \|f(\cdot, y)\|_{p(\cdot)}\, \mathrm{d}y$$

with some positive constant $c = c_{p(\cdot)}$.

Theorem C ([6]). *If* $f \in L_{2\pi}^{p(\cdot)}$ *with* $p(\cdot) \in \hat{\beta}_{2\pi}$, *then the maximal operator*

$$M(f)(x) := \sup_{I \ni x} \frac{1}{|I|} \int_I |f(t)|\, \mathrm{d}t$$

is bounded in $L_{2\pi}^{p(\cdot)}$ *and*

$$\|Mf\|_{p(\cdot)} \le c \|f\|_{p(\cdot)}$$

with some constant $c = c_{p(\cdot)} > 0$.

Theorem CF ([4, p. 212]). *Let* $\Im$ *be a family of pairs* (f, g) *of non-negative, measurable function* f *and* g *defined on* $(0, 2\pi)$ *such that for all* $\omega \in A_{p_0}(0, 2\pi)$ *with some* $p_0 \ge 1$

$$\int_0^{2\pi} f^{p_0}(x)\, \omega(x)\, \mathrm{d}x \le c \int_0^{2\pi} g^{p_0}(x)\, \omega(x)\, \mathrm{d}x, \qquad (f, g) \in \Im,$$

where a constant c *independent of* (f, g). *If* $p(\cdot) \in \hat{\beta}_{2\pi}$, *then for every* r, $1 < r < \infty$, *and sequence* $\{(f_i, g_i)\} \subset \Im$

$$\left\| \left(\sum_i f_i^r \right)^{1/r} \right\|_{p(\cdot)} \le c_{p(\cdot)} \left\| \left(\sum_i g_i^r \right)^{1/r} \right\|_{p(\cdot)}.$$

LEMMA 2 ([11]). *Let* $f \in L_{2\pi}^{p(\cdot)}$ *with* $p(\cdot) \in \hat{\beta}_{2\pi}$. *If* $A_l(x) = e^{ilx}\hat{f}_l$ *and* $A_{2^{-1}}(x) = 0$, $l = 0, 1, 2, ...$, *then*

$$c_1 \|f\|_{p(\cdot)} \leq \left\| \left(\sum_{j=0}^{\infty} \left| \sum_{l=2^{j-1}}^{2^j - 1} A_l(x) \right|^2 \right)^{1/2} \right\|_{p(\cdot)} \leq c_2 \|f\|_{p(\cdot)} .$$

LEMMA 3. *Let* $\{f_n\}_1^{\infty}$ *be a sequence of functions* $f_n \in L_{2\pi}^{p(\cdot)}$ *with* $p(\cdot) \in \hat{\beta}_{2\pi}$ *and let* S_{n,k_n} *be the kth partial sum of* f_n *with* $k = k_n$. *Then*

$$\left\| \left(\sum_{n=1}^{\infty} |S_{n,k_n}(x)|^2 \right)^{1/2} \right\|_{p(\cdot)} \leq c \left\| \left(\sum_{n=1}^{\infty} |f_n(x)|^2 \right)^{1/2} \right\|_{p(\cdot)} \tag{2.1}$$

with some constant $c > 0$ *independent of* $f_n.p$

Proof. The inequality (2.1) is a consequence of the extrapolation Theorem CF (for the case of $r = 2$, $g_i := f_i$ and $f_i := |S_{i,k_i}(x)|$, $i = 1, 2, ...,$) and of the norm inequality, proved by Kurtz in [12] in the weighted Lebesgue spaces L_{ω}^{p}.

Let L_{comp}^{∞} be the set of all bounded functions with compact support on $[0, 2\pi]$and let $\{f_n\}_1^{\infty}$ be a sequence in L_{comp}^{∞}. If

$$f(x) := \left(\sum_{n=1}^{\infty} |f_n(x)|^2 \right)^{1/2} \text{ and } Tf(x) := \left(\sum_{n=1}^{\infty} |S_{n,k_n}(x)|^2 \right)^{1/2},$$

then by [12] for a number p_0 with $1 < p_0 < \infty$ there is a constant c such that for all weight $\omega \in A_{p_0}$

$$\|Tf\|_{p_0,\omega} \leq c \|f\|_{p_0,\omega} .$$

Hence the conditions of the above cited extrapolation Theorem CF are fulfilled which implies the inequality (2.1) for the sequence in $\{f_n\}_1^{\infty}$ from L_{comp}^{∞}. Since the set L_{comp}^{∞} is dense in $L_{2\pi}^{p(\cdot)}$ the inequality (2.1) is also valid for all sequences $\{f_n\}_1^{\infty}$ from $L_{2\pi}^{p(\cdot)}$. □

THEOREM D. *Let* $\lambda_0, \lambda_1, ...$ *be a sequence of the numbers such that*

$$|\lambda_l| \leq M \ , \ \sum_{\upsilon=2^l}^{2^{l+1}-1} |\lambda_{\upsilon} - \lambda_{\upsilon+1}| \leq M \quad (l = 0, 1, 2, ...) . \tag{2.2}$$

If $p(\cdot) \in \hat{\beta}_{2\pi}$ *and* $a_{\upsilon}, b_{\upsilon}$ *be the Fourier coefficients of* $f \in L_{2\pi}^{p(\cdot)}$, *then the series*

$$a_0\lambda_0/2 + \sum_{\upsilon=1}^{\infty} \lambda_{\upsilon} (a_{\upsilon} \cos \upsilon x + b_{\upsilon} \sin \upsilon x)$$

is the Fourier series of function $F \in L_{2\pi}^{p(\cdot)}$ *and*

$$\|F\|_{p(\cdot)} \leq c\,\|f\|_{p(\cdot)}$$

with some constant $c = c_{p(\cdot)} > 0$ *independent of* f.

Proof. Let for $s \geqslant 2^{\mu-1}, \mu = 1, 2, ...,$

$$\begin{aligned} \Delta_{\mu,s} \; &: = \sum_{\upsilon=2^{\mu-1}}^{s} A_{\upsilon}(x), \quad A_{\upsilon}(x) := a_{\upsilon}\cos \upsilon x + b_{\upsilon}\sin \upsilon x \\ \Delta_{\mu} \; &: = \sum_{\upsilon=2^{\mu-1}}^{2^{\mu}-1} A_{\upsilon}(x) \quad \text{and} \quad \Delta_{\mu}^{'} := \sum_{\upsilon=2^{\mu-1}}^{2^{\mu}-1} \lambda_{\upsilon} A_{\upsilon}(x). \end{aligned}$$

As in [19, p. 347] we have the estimation

$$\left|\Delta_{\mu}^{'}\right|^{2} \leq 2M\left(\sum_{s=2^{\mu-1}}^{2^{\mu}-1} |\Delta_{\mu,s}|^{2}|\lambda_{s}-\lambda_{s+1}| + |\Delta_{\mu}|^{2}|\lambda_{2^{\mu}}|\right).$$

Since $p(x) \leq p_{+}$ by Lemma 3 and (2.2)

$$\rho_{p(\cdot)}\left(\left(\sum_{\mu=1}^{\infty}\left|\Delta_{\mu}^{'}\right|^{2}\right)^{1/2}\right) = \int_{0}^{2\pi}\left(\sum_{\mu=1}^{\infty}\left|\Delta_{\mu}^{'}\right|^{2}\right)^{p(x)/2} \mathrm{d}x$$

$$\leq \int_{0}^{2\pi}\left(\sum_{\mu=1}^{\infty} 2M\left(\sum_{s=2^{\mu-1}}^{2^{\mu}-1}|\Delta_{\mu,s}|^{2}|\lambda_{s}-\lambda_{s+1}| + |\Delta_{\mu}|^{2}|\lambda_{2^{\mu}}|\right)\right)^{p(x)/2} \mathrm{d}x$$

$$= \int_{0}^{2\pi}(2M)^{p(x)/2}\left(\sum_{\mu=1}^{\infty}\left(\sum_{s=2^{\mu-1}}^{2^{\mu}-1}|\Delta_{\mu,s}|^{2}|\lambda_{s}-\lambda_{s+1}| + |\Delta_{\mu}|^{2}|\lambda_{2^{\mu}}|\right)\right)^{p(x)/2} \mathrm{d}x$$

$$\leq c\int_{0}^{2\pi}(2M)^{p(x)/2}\left(\sum_{\mu=1}^{\infty}|\Delta_{\mu}|^{2}\left(\sum_{s=2^{\mu-1}}^{2^{\mu}-1}|\lambda_{s}-\lambda_{s+1}| + |\lambda_{2^{\mu}}|\right)\right)^{p(x)/2} \mathrm{d}x$$

$$\leq c\int_{0}^{2\pi}(2M)^{p_{+}}\left(\sum_{\mu=1}^{\infty}|\Delta_{\mu}|^{2}\right)^{p(x)/2} \mathrm{d}x \leq c_{3}\int_{0}^{2\pi}\left(\sum_{\mu=1}^{\infty}|\Delta_{\mu}|^{2}\right)^{p(x)/2} \mathrm{d}x$$

$$= c_{3}\rho_{p(\cdot)}\left(\left(\sum_{\mu=1}^{\infty}|\Delta_{\mu}|^{2}\right)^{1/2}\right),$$

which by Lemma 2 implies that

$$\begin{aligned} \|F\|_{p(\cdot)} &\leq \left\| \left(\sum_{\mu=1}^{\infty} \left| \Delta_{\mu}^{'} \right|^2 \right)^{1/2} \right\|_{p(\cdot)} \\ &\leq c_3 \left\| \left(\sum_{\mu=1}^{\infty} \left| \Delta_{\mu} \right|^2 \right)^{1/2} \right\|_{p(\cdot)} \leq c_4 \|f\|_{p(\cdot)} . \quad \square \end{aligned}$$

3. PROOFS OF MAIN RESULTS

Proof of Theorem 1. Let $f \in L_{2\pi}^{p(\cdot)}$, $m \in \mathbb{N}$ and let $S_{2^{m+1}}$ be the $2^{m+1}th$ partial sum of its Fourier series.

Let also $h \leq 2^{-m-1}$. Then

$$\begin{aligned} D(f, \sigma, h, p(\cdot)) &\leq \left\| \int_{-\infty}^{\infty} [(\sigma_h f)(x,u) - (\sigma_h S_{2^{m+1}} f)(x,u)] \mathrm{d}\sigma(u) \right\|_{p(\cdot)} \\ &\quad + \left\| \int_{-\infty}^{\infty} (\sigma_h S_{2^{m+1}} f)(x,u)] \mathrm{d}\sigma(u) \right\|_{p(\cdot)} . \end{aligned} \tag{3.1}$$

By applying Theorems B and C and Lemma 1 in the first term, we have

$$\begin{aligned} &\left\| \int_{-\infty}^{\infty} [(\sigma_h f)(x,u) - (\sigma_h S_{2^{m+1}} f)(x,u)] \mathrm{d}\sigma(u) \right\|_{p(\cdot)} \\ &\leq K_{p(\cdot)} \int_{-\infty}^{\infty} \|(\sigma_h f)(\cdot,u) - (\sigma_h S_{2^{m+1}} f)(\cdot,u)\|_{p(\cdot)} \mathrm{d}\sigma(u) \\ &= K_{p(\cdot)} \int_{-\infty}^{\infty} \|(\sigma_h (f - S_{2^{m+1}} f))(\cdot,u)\ \|_{p(\cdot)} \mathrm{d}\sigma(u) \\ &\leq c_5 K_{p(\cdot)} \int_{-\infty}^{\infty} \|f - S_{2^{m+1}} f\|_{p(\cdot)} \mathrm{d}\sigma(u) \\ &\leq c_5 K_{p(\cdot)} E_{2^{m+1}}(f)_{p(\cdot)} \int_{-\infty}^{\infty} \mathrm{d}\sigma(u) \leq c_6 E_{2^{m+1}}(f)_{p(\cdot)} . \end{aligned}$$

Without loss of generality, we suppose that the Fourier series of f is

$$\sum_{k=1}^{\infty} \hat{A}_k(x) := \sum_{k=1}^{\infty} \hat{f}_k e^{ikx}.$$

Then

$$\int_{-\infty}^{\infty} (\sigma_h S_{2^{m+1}} f)(x,u) \mathrm{d}\sigma(u)$$

$$
\begin{aligned}
&= \int_{-\infty}^{\infty}[\frac{1}{2h}\int_{-h}^{h} S_{2^{m+1}}f\left(x+tu\right)\mathrm{d}t]\mathrm{d}\sigma\left(u\right) \\
&= \int_{-\infty}^{\infty}[\frac{1}{2h}\int_{-h}^{h}\sum_{k=1}^{2^{m+1}-1}\hat{f}_k e^{ik(x+tu)}\mathrm{d}t]\mathrm{d}\sigma\left(u\right) \\
&= \int_{-\infty}^{\infty}[\frac{1}{2h}\sum_{k=1}^{2^{m+1}-1}\hat{f}_k e^{ikx}\int_{-h}^{h} e^{iktu}\mathrm{d}t]\mathrm{d}\sigma\left(u\right) \\
&= \int_{-\infty}^{\infty}[\frac{1}{2h}\sum_{k=1}^{2^{m+1}-1}\hat{A}_k\left(x\right)\int_{-h}^{h} e^{iktu}\mathrm{d}t]\mathrm{d}\sigma\left(u\right) \\
&= \sum_{k=1}^{2^{m+1}-1}\hat{A}_k\left(x\right)\int_{-\infty}^{\infty}\frac{e^{ikhu}-e^{-ikhu}}{2ikhu}\mathrm{d}\sigma\left(u\right) \\
&= \sum_{k=1}^{2^{m+1}-1}\hat{A}_k\left(x\right)\hat{\sigma}\left(kh\right), \qquad (3.2)
\end{aligned}
$$

and hence by (3.2) and (3.1)

$$
D\left(f,\sigma,h,p(\cdot)\right) \leq \left\| \sum_{k=1}^{2^{m+1}-1}\hat{A}_k\left(x\right)\hat{\sigma}\left(kh\right)\right\|_{p(\cdot)} + c_6 E_{2^{m+1}}\left(f\right)_{p(\cdot)}. \tag{3.3}
$$

Now, by inequality $(a+b)^p < a^p + b^p$, which holds for every positive numbers a and b in the case of $0<p<1$ and by Lemma 2

$$
\begin{aligned}
&\left\| \sum_{k=1}^{2^{m+1}-1}\hat{A}_k\left(x\right)\hat{\sigma}\left(kh\right)\right\|_{p(\cdot)} \\
\leq\ & c\left\| \left(\sum_{k=0}^{m}\left|\sum_{l=2^k}^{2^{k+1}-1}\hat{A}_l\left(x\right)\hat{\sigma}\left(lh\right)\right|^2\right)^{1/2}\right\|_{p(\cdot)} \\
:\ &= c\left\| \left(\sum_{k=0}^{m}\Delta_{k,\sigma}^2\right)^{1/2}\right\|_{p(\cdot)} \\
<\ & c\left\| \sum_{k=0}^{m}\Delta_{k,\sigma}\right\|_{p(\cdot)} \leq c\sum_{k=0}^{m}\left\|\Delta_{k,\sigma}\right\|_{p(\cdot)}.
\end{aligned}
$$

On the other hand, applying the Abel transformation to the sum

$$\Delta_{k,\sigma} = \sum_{l=2^k}^{2^{k+1}-1} \hat{A}_l(x)\,\hat{\sigma}(lh)$$

we have

$$\begin{aligned}\Delta_{k,\sigma} &= \sum_{l=2^k}^{2^{k+1}-1} \left[S_l(f,x) - S_{2^{k+1}-1}(f,x)\right]\left[\hat{\sigma}(lh) - \hat{\sigma}((l+1)h)\right] \\ &\quad + \left[S_{2^{k+1}-1}(f,x) - S_{2^k-1}(f,x)\right]\hat{\sigma}\left(2^k h\right)\end{aligned}$$

and then

$$\begin{aligned}\|\Delta_{k,\sigma}\|_{p(\cdot)} &\le \sum_{l=2^k}^{2^{k+1}-1} \|S_l(f) - S_{2^{k+1}}(f)\|_{p(\cdot)} \left|\hat{\sigma}(lh) - \hat{\sigma}((l+1)h)\right| \\ &\quad + \left\|S_{2^{k+1}-1}(f) - S_{2^k-1}(f)\right\|_{p(\cdot)} \left|\hat{\sigma}\left(2^k h\right)\right| \\ &= \|S_{2^k}(f) - S_{2^{k+1}}(f)\|_{p(\cdot)} \left|\hat{\sigma}\left(2^k h\right) - \hat{\sigma}\left(\left(2^k+1\right)h\right)\right| \\ &\quad + \ldots + \left\|S_{2^k+1}(f) - S_{2^{k+1}}(f)\right\|_{p(\cdot)} \left|\hat{\sigma}\left(\left(2^k+1\right)h\right) - \hat{\sigma}\left(\left(2^k+2\right)h\right)\right| \\ &\quad + \left\|S_{2^{k+1}-1}(f) - S_{2^k-1}(f)\right\|_{p(\cdot)} \left|\hat{\sigma}\left(2^k h\right)\right| \\ &\le \left[\|S_{2^k}(f) - f\|_{p(\cdot)} + \|S_{2^{k+1}}(f) - f\|_{p(\cdot)}\right] \left|\hat{\sigma}\left(2^k h\right) - \hat{\sigma}\left(\left(2^k+1\right)h\right)\right| + \ldots \\ &\quad + \left[\left\|S_{2^k+1}(f) - f\right\|_{p(\cdot)} + \|S_{2^{k+1}}(f) - f\|_{p(\cdot)}\right] \left|\hat{\sigma}\left(\left(2^{k+1}-1\right)h\right) - \hat{\sigma}\left(2^k h\right)\right| \\ &\quad + \left[\left\|S_{2^{k+1}-1}(f) - f\right\|_{p(\cdot)} + \left\|S_{2^k-1}(f) - f\right\|_{p(\cdot)}\right] \left|\hat{\sigma}\left(2^k h\right)\right| \\ &\le cE_{2^k-1}(f)_{p(\cdot)}\,\delta_{2^k,h}.\end{aligned}$$

Hence

$$\left\|\sum_{k=1}^{2^{m+1}-1} \hat{A}_k(x)\,\hat{\sigma}(kh)\right\|_{p(\cdot)} \le c\sum_{r=0}^{m} E_{2^k-1}(f)_{p(\cdot)}\delta_{2^k,h}$$

and by (3.3) we obtain the required inequality. □

Proof of Theorem 2. Let $f \in L_{2\pi}^{p(\cdot)}$. Repeating the techniques used for the estimation of the quantity $D(f,\sigma,h,p(\cdot))$ from Theorem 1, we have
(3.4)

$$D(f,\sigma_1,h,p(\cdot)) \le \left\|\int_{-\infty}^{\infty} (\sigma_h S_{2^{m+1}} f)(\cdot,u)\,d\sigma_1(u)\right\|_{p(\cdot)} + c_{p(\cdot)}E_{2^{m+1}}(f)_{p(\cdot)}.$$

On the other by (3.2) and Lemma 1

$$\left\|\int_{-\infty}^{\infty}\left(\sigma_h S_{2^{m+1}} f\right)(x,u)\,\mathrm{d}\sigma_1(u)\right\|_{p(\cdot)} = \left\|\sum_{k=1}^{2^{m+1}-1}\hat{A}_k(x)\,\hat{\sigma}_1(kh)\right\|_{p(\cdot)}$$
$$= \left\|\sum_{k=1}^{2^{m+1}-1}\hat{A}_k(x)\,\hat{\sigma}_2(kh)\,F(kh)\right\|_{p(\cdot)} = \left\|\sum_{k=1}^{2^{m+1}-1}\hat{f}_k e^{ikx}\hat{\sigma}_2(kh)\,F(kh)\right\|_{p(\cdot)}$$
$$\leq c\left\|\sum_{k=1}^{2^{m+1}-1}\hat{f}_k e^{ikx}\hat{\sigma}_2(kh)\right\|_{p(\cdot)} = c\left\|\int_{-\infty}^{\infty}\left(\sigma_h S_{2^{m+1}} f\right)(x,u)\,\mathrm{d}\sigma_2(u)\right\|_{p(\cdot)}$$
$$= c\left\|\int_{-\infty}^{\infty} S_{2^{m+1}}\left(\sigma_h f\right)(x,u)\,\mathrm{d}\sigma_2(u)\right\|_{p(\cdot)}$$
$$= c\left\|S_{2^{m+1}}\int_{-\infty}^{\infty}\left(\sigma_h f\right)(x,u)\,\mathrm{d}\sigma_2(u)\right\|_{p(\cdot)} \leq \left\|\int_{-\infty}^{\infty}\sigma_h f(x,u)\,\mathrm{d}\sigma_2(u)\right\|_{p(\cdot)}.$$

The last inequality together with (3.4) implies the required relation. □

Acknowledgments. The authors sincerely thank the anonymous reviewers for their careful reading and constructive comments. This work was supported by TUBITAK grant 114F422: *Approximation Problems in the Variable Exponent Lebesgue Spaces.*

REFERENCES

[1] R. Akgun, *Trigonometric approximation of functions in generalized Lebesgue spaces with variable exponent.* Ukrainian Math. J. **63** (2011), *1*, 3–23.

[2] R. Akgun, *Polynomial approximation of functions in weighted Lebesgue and Smirnov spaces with nonstandard growth.* Georgian Math. J. **18** (2011), 203–235.

[3] R. Akgun and V.M. Kokilashvili, *The refined direct and converse inequalities of trigonometric approximation in weighted variable exponent Lebesgue spaces.* Georgian Math. J. **18** (2011), 399–423.

[4] D.V. Cruz-Uribe and A. Fiorenza, *Variable Lebesgue Spaces Foundation and Harmonic Analysis.* Birkhäsuser, 2013.

[5] L. Diening, P. Harjulehto, P. Hästö and M. Ruzicka, *Lebesgue and Sobolev Spaces with Variable Exponents.* Springer, Heidelberg, Dordrecht, London, New York, 2011.

[6] L. Diening, *Maximal function on generalized Lebesgue spaces $L^{p(\cdot)}$.* Math. Inequal. Appl. **7** (2004), 245–253.

[7] A. Guven and D.M. Israfilov, *Trigonometric approximation in generalized Lebesgue spaces $L^{p(x)}$.* J. Math. Inequal. **4** (2010), *2*, 285–299.

[8] D.M. Israfilov, V.M. Kokilashvili and S. Samko, *Approximation in Weighted Lebesgue and Smirnov Spaces with Variable Exponents.* Proc. A. Razmadze Math. Inst. **143** (2007), 25–35.

[9] D.M. Israfilov and A. Testici, *Approximation in Smirnov classes with variable exponent.* Complex Var. Elliptic Equ. **60** (2015), *9*, 1243–1253.

[10] D.M. Israfilov and Y.E. Yildirir, *The properties of convolution type transforms in weighted Orlicz spaces*, Gla. Mat. **45** (2010), *2*, 461–474.

[11] T. Kopaliani, *Interpolation theorems in the variable exponent Lebesgue spaces.* J. Funct. Anal. **257** (2009), 3541–3551.

[12] D.S. Kurtz, *Littlewood-Paley and multiplier theorems in weighted L^p spaces.* Trans. Amer. Math. Soc. **259** (1980), 235–254.

[13] V.G. Ponomarenko and M.F. Timan, *The properties of convolution type transforms in the Orlicz spaces.* In: *Theory of Approximation of Functions*, Proc. Inst. Math. Mech. **3**, Donetsk, 1998.

[14] M. Růžička, *Elektrorheological Fluids: Modeling and Mathematical Theory.* Springer, 2000.

[15] I.I. Sharapudinov, *The basis property of the Haar system in the space* $L^{p(t)}([0,1])$ *and the principle of localization in the mean.* Mat. Sb. **130** (1986), *172*, 275–283.

[16] I.I. Sharapudinov, *The basis property of the Legendre polynomials in the variable exponent Lebesgue spaces system in the space* $L^{p(x)}(0,1)$, *and the principle of localization in the mean*. Sb. Math. **200** (2009), *1*, 130–156.

[17] I.I. Sharapudinov, *Approximation of functions in* $L_{2\pi}^{p(x)}$ *by trigonometric polynomials.* Izvestiya RAN: Ser. Math. **77** (2013), *2*, 197-224. English transl., Izvestiya: Mathematics **77** (2013), *2*, 407–434.

[18] I.I. Sharapudinov, *Some questions of approximation theory in the Lebesgue spaces with variable exponent.* Vladikavkaz. Mat. Zh., 2012.

[19] A. Zygmund, *Trigonometric Series.* Cambridge Univ. Press, Cambridge, 1968.

Received 6 March 2015

Balikesir University,
Faculty of Art and Science,
Department of Mathematics,
10145, Balikesir, Turkey
ANAS Institute of Mathematics
and Mechanics,
Baku, Azerbaijan
mdaniyal@balikesir.edu.tr

Balikesir University,
Faculty of Art and Science,
Department of Mathematics,
10145, Balikesir, Turkey
elife.yirtici@gmail.com