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$N(k)$-quasi Einstein manifolds satisfying certain conditions

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ABSTRACT

We consider $N(k)$-quasi Einstein manifolds satisfying the conditions $\mathscr{R}(\xi,X)\cdot\mathscr{P}=0$, $\mathscr{P}(\xi,X)\cdot S=0$ and $\mathscr{P}(\xi,X)\cdot\mathscr{P}=0$. We construct physical examples of $N(k)$-quasi Einstein space–times.

1. Introduction

A non-flat n-dimensional Riemannian manifold (M,g) is said to be *quasi Einstein* [1] if its Ricci tensor S satisfies

$$S(X,Y)=ag(X,Y)+b\eta(X)\eta(Y),\quad X,\ Y\in TM \tag{1.1}$$

for some smooth functions a and $b\neq 0$, where η is a nonzero 1-form such that

$$g(X,\xi)=\eta(X),\quad g(\xi,\xi)=\eta(\xi)=1 \tag{1.2}$$

for the associated vector field ξ. The 1-form η is called the associated 1-form and the unit vector field ξ is called the generator of the manifold. For more details about quasi Einstein manifolds see also [2–7,9]. Quasi Einstein manifolds arose during the study of exact solutions of the Einstein field equations. There are many studies about Einstein field equations. For example, in [11], El Naschie turned the tables on the theory of elementary particles and showed that we could derive the expectation number of elementary particles of the standard model using Einstein's unified field equation or more precisely his somewhat forgotten strength criteria directly and without resorting to quantum field theory [8]. In [10], possible connections between Gödel's classical solution of Einstein's field equations and E-infinity were discussed.

If the generator ξ belongs to some k-nullity distribution $N(k)$ then the quasi Einstein manifold is called an $N(k)$-quasi Einstein manifold [15]. In [15], it was shown that an n-dimensional conformally flat quasi Einstein manifold is an $N(\frac{a+b}{n-1})$-quasi Einstein manifold and in particular a 3-dimensional quasi Einstein manifold is an $N(\frac{a+b}{2})$-quasi Einstein manifold. In [13], it was proved that in an n-dimensional $N(k)$-quasi Einstein manifold $k=\frac{a+b}{n-1}$. In [7], De, Sengupta and Saha studied conformally flat and semisymmetric quasi Einstein manifolds. Motivated by the above studies, in this study, we consider $N(k)$-quasi Einstein manifolds satisfying the conditions $\mathscr{R}(\xi,X)\cdot\mathscr{P}=0$, $\mathscr{P}(\xi,X)\cdot S=0$ and $\mathscr{P}(\xi,X)\cdot\mathscr{P}=0$, where $\mathscr{P}$ denotes the projective curvature tensor. We also present physical examples of $N(k)$-quasi Einstein manifolds. The paper is organized as follows: In Section 2, we give basic definitions and notions for an $N(k)$-quasi Einstein manifold. In Section 3, we construct examples of $N(k)$-quasi Einstein space–times. In Section 4, we consider $N(k)$-quasi Einstein manifolds satisfying the conditions $\mathscr{R}(\xi,X)\cdot\mathscr{P}=0$, $\mathscr{P}(\xi,X)\cdot S=0$ and $\mathscr{P}(\xi,X)\cdot\mathscr{P}=0$.

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2. $N(k)$-quasi Einstein manifolds

The Ricci operator Q of a Riemannian manifold (M, g) is defined by

$$S(X, Y) = g(QX, Y).$$

For a quasi Einstein manifold [1] the Ricci operator Q satisfies

$$Q = aI + b\eta \otimes \xi. \tag{2.1}$$

From (2.1) and (1.2) it follows that

$$S(X, \xi) = (a + b)\eta(X), \tag{2.2}$$

$$r = na + b, \tag{2.3}$$

where r is the scalar curvature of M.

Let $\mathscr{R}$ denote the Riemannian curvature tensor of a Riemannian manifold M. The *k-nullity distribution* $N(k)$ [14] of a Riemannian manifold M is defined by

$$N(k) : p \to N_p(k) = \{U \in T_pM | \mathscr{R}(X, Y)U = k(g(Y, U)X - g(X, U)Y)\}$$

for all $X, Y \in TM$, where k is some smooth function. In a quasi Einstein manifold M, if the generator ξ belongs to some k-nullity distribution $N(k)$, then M is said to be an $N(k)$-quasi Einstein manifold [15]. In fact, k is not arbitrary as the following:

Lemma 2.1 [13]. *In an n-dimensional $N(k)$-quasi Einstein manifold it follows that*

$$k = \frac{a + b}{n - 1}. \tag{2.4}$$

Now, it is immediate to note that in an n-dimensional $N(k)$-quasi Einstein manifold (see [13])

$$\mathscr{R}(X, Y)\xi = \frac{a + b}{n - 1}\{\eta(Y)X - \eta(X)Y\}, \tag{2.5}$$

which is equivalent to

$$\mathscr{R}(X, \xi)Y = \frac{a + b}{n - 1}\{\eta(Y)X - g(X, Y)\xi\} = -\mathscr{R}(\xi, X)Y. \tag{2.6}$$

From (2.5) we get

$$\mathscr{R}(\xi, X)\xi = \frac{a + b}{n - 1}\{\eta(X)\xi - X\}. \tag{2.7}$$

3. Physical examples of $N(k)$-quasi Einstein manifolds

In [15], Tripathi and Kim proved that an n-dimensional conformally flat quasi Einstein manifold is an $N(k)$-quasi Einstein manifold. Now we consider a conformally flat perfect fluid space–time (M^4, g) satisfying Einstein's equation without cosmological constant. Further, let ξ be the unit time-like velocity vector of the fluid. It is known that Einstein's equation without cosmological constant can be written as (see [12])

$$S(X, Y) - \frac{1}{2}rg(X, Y) = \kappa T(X, Y), \tag{3.1}$$

where κ is the gravitational constant and T is the energy momentum tensor of type (0,2). In the present case (3.1) can be written as follows:

$$S(X, Y) - \frac{1}{2}rg(X, Y) = \kappa[(\sigma + p)\eta(X)\eta(Y) + pg(X, Y)],$$

where σ is the energy density and p is the isotropic pressure of the fluid. Then we have (see [12])

$$S(X, Y) = \left(\kappa p + \frac{1}{2}r\right)g(X, Y) + \kappa(\sigma + p)\eta(X)\eta(Y). \tag{3.2}$$

Since the space–time is conformally flat, by [15], it is $N(k)$-quasi Einstein. From (3.2), by a contraction we get

$$r = \kappa(\sigma - 3p).$$

Hence the Eq. (3.2) can be written as

$$S(X, Y) = \left(\frac{\kappa}{2}(\sigma - p)\right)g(X, Y) + \kappa(\sigma + p)\eta(X)\eta(Y).$$

So from (1.1) we have

$$a = \frac{\kappa}{2}(\sigma - p)$$

and

$$b = \kappa(\sigma + p).$$

In view of (2.4), since $k = \frac{a+b}{3}$ we obtain

$$k = \frac{\kappa(3\sigma + p)}{6}.$$

Hence we can state the following example:

Example 3.1. A conformally flat perfect fluid space–time (M^4, g) satisfying Einstein's equation without cosmological constant is an $N\left(\frac{\kappa(3\sigma+p)}{6}\right)$-quasi Einstein manifold.

Now we consider a perfect fluid space–time (M^4, g) satisfying Einstein's equation with cosmological constant. Then Einstein's equation can be written as

$$S(X,Y) - \frac{1}{2}rg(X,Y) + \lambda g(X,Y) = \kappa[(\sigma + p)\eta(X)\eta(Y) + pg(X,Y)],$$

which gives us

$$S(X,Y) = \left(\frac{1}{2}r - \lambda + p\kappa\right)g(X,Y) + \kappa(\sigma + p)\eta(X)\eta(Y). \tag{3.3}$$

So from (3.3), by a contraction, we get

$$r = 4\lambda + \kappa(\sigma - 3p).$$

Hence the Eq. (3.3) turns into

$$S(X,Y) = \left(\lambda + \frac{\kappa}{2}(\sigma - p)\right)g(X,Y) + \kappa(\sigma + p)\eta(X)\eta(Y).$$

Then from (1.1) we have

$$a = \lambda + \frac{\kappa}{2}(\sigma - p)$$

and

$$b = \kappa(\sigma + p).$$

In view of (2.4), since $k = \frac{a+b}{3}$ we obtain

$$k = \frac{\lambda}{3} + \frac{\kappa(3\sigma + p)}{6}.$$

So as a generalization of Example 3.1, we obtain the following example.

Example 3.2. A conformally flat perfect fluid space–time (M^4, g) satisfying Einstein's equation with cosmological constant is an $N\left(\frac{\lambda}{3} + \frac{\kappa(3\sigma+p)}{6}\right)$-quasi Einstein manifold.

4. The projective curvature tensor of an $N(k)$-quasi Einstein manifold

The *projective curvature tensor* $\mathscr{P}$ in an n-dimensional Riemannian manifold (M, g) is defined by (see [16])

$$\mathscr{P}(X,Y)Z = \mathscr{R}(X,Y)Z - \frac{1}{n-1}\{S(Y,Z)X - S(X,Z)Y\} \tag{4.1}$$

for all vector fields X, Y, Z on M.

Now, we prove the following Proposition for later use.

Proposition 4.1. *In an n-dimensional $N(k)$-quasi Einstein manifold M, the projective curvature tensor $\mathscr{P}$ satisfies*

$$\mathscr{P}(X,Y)\xi = 0, \tag{4.2}$$

$$\mathscr{P}(\xi,X)Y = \frac{b}{n-1}\{g(X,Y)\xi - \eta(X)\eta(Y)\xi\}, \tag{4.3}$$

$$\eta(\mathscr{P}(X,Y)Z) = \frac{b}{n-1}\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\} \tag{4.4}$$

for all vector fields X, Y, Z on M.

Proof. From (2.2), (4.1), (2.5) and (2.6) the Eqs. (4.2)–(4.4) follow easily. □

Theorem 4.2. *Let M be an n-dimensional N(k)-quasi Einstein manifold. Then M satisfies the condition* $\mathscr{R}(\xi,X)\cdot\mathscr{P}=0$ *if and only if* $a+b=0$.

Proof. Let M be an n-dimensional $N(k)$-quasi Einstein manifold. Since M satisfies the condition $\mathscr{R}(\xi,X)\cdot\mathscr{P}=0$ we can write

$$0=\mathscr{R}(\xi,X)\mathscr{P}(Y,Z)W-\mathscr{P}(\mathscr{R}(\xi,X)Y,Z)W-\mathscr{P}(Y,\mathscr{R}(\xi,X)Z)W-\mathscr{P}(Y,Z)\mathscr{R}(\xi,X)W$$

for all vector fields X,Y,Z,W on M. So from (2.6) we get

$$0=\frac{a+b}{n-1}\{P(Y,Z,W,X)\xi-\eta(\mathscr{P}(Y,Z)W)X-g(X,Y)\mathscr{P}(\xi,Z)W+\eta(Y)\mathscr{P}(X,Z)W-g(X,Z)\mathscr{P}(Y,\xi)W+\eta(Z)\mathscr{P}(Y,X)W-g(X,W)\mathscr{P}(Y,Z)\xi+\eta(W)\mathscr{P}(Y,Z)X\},$$

which implies either $a+b=0$ or

$$0=P(Y,Z,W,X)\xi-\eta(\mathscr{P}(Y,Z)W)X-g(X,Y)\mathscr{P}(\xi,Z)W+\eta(Y)\mathscr{P}(X,Z)W-g(X,Z)\mathscr{P}(Y,\xi)W+\eta(Z)\mathscr{P}(Y,X)W-g(X,W)\mathscr{P}(Y,Z)\xi+\eta(W)\mathscr{P}(Y,Z)X \tag{4.5}$$

holds on M, where $P(Y,Z,W,X)=g(\mathscr{P}(Y,Z)W,X)$. Taking the inner product of both sides of (4.5) with ξ we have

$$0=P(Y,Z,W,X)-\eta(\mathscr{P}(Y,Z)W)\eta(X)-g(X,Y)\eta(\mathscr{P}(\xi,Z)W)+\eta(Y)\eta(\mathscr{P}(X,Z)W)-g(X,Z)\eta(\mathscr{P}(Y,\xi)W)+\eta(Z)\eta(\mathscr{P}(Y,X)W)-g(X,W)\eta(\mathscr{P}(Y,Z)\xi)+\eta(W)\eta(\mathscr{P}(Y,Z)X). \tag{4.6}$$

Hence in view of (4.2)–(4.4) the Eq. (4.6) is reduced to

$$0=P(Y,Z,W,X)+\frac{b}{n-1}\{g(X,Z)g(Y,W)-g(X,Y)g(Z,W)\}.$$

Then by the use of (4.1) we obtain

$$0=R(Y,Z,W,X)-\frac{1}{n-1}\{S(Z,W)g(X,Y)-S(Y,W)g(X,Z)\}+\frac{b}{n-1}\{g(X,Z)g(Y,W)-g(X,Y)g(Z,W)\}. \tag{4.7}$$

So by a suitable contraction of (4.7) we get

$$bg(Z,W)=0,$$

which gives us $b=0$. This contradicts to our assumption that M is an $N(k)$-quasi Einstein manifold. The converse statement is trivial. This completes the proof of the theorem. □

Next, we have the following theorem.

Theorem 4.3. *Let M be an n-dimensional N(k)-quasi Einstein manifold. Then M satisfies the condition* $\mathscr{P}(\xi,X)\cdot S=0$ *if and only if* $a+b=0$.

Proof. From the condition $\mathscr{P}(\xi,X)\cdot S=0$, we get

$$S(\mathscr{P}(\xi,X)Y,Z)+S(Y,\mathscr{P}(\xi,X)Z)=0,$$

which in view of (4.3) gives

$$0=\frac{b}{n-1}\{g(X,Y)S(\xi,Z)-\eta(X)\eta(Y)S(\xi,Z)+g(X,Z)S(Y,\xi)-\eta(X)\eta(Z)S(Y,\xi)\}.$$

Since $b\neq 0$, using (2.2) we have

$$0=(a+b)\{g(X,Y)\eta(Z)+g(X,Z)\eta(Y)-2\eta(X)\eta(Y)\eta(Z)\}. \tag{4.8}$$

From (4.8), by a contraction, we get

$$(n-1)(a+b)=0,$$

which gives us $a+b=0$. The converse statement is trivial. Our theorem is thus proved. □

So by Theorem 2 in [7], Theorem 3.3 in [15], Theorem 4.2 and Theorem 4.3 we state the following corollary.

Corollary 4.4. *Let M be an n-dimensional N(k)-quasi Einstein manifold. Then the following statements are equivalent:*

(i) $\mathscr{R}(\xi,X)\cdot\mathscr{R}=0$
(ii) $\mathscr{R}(\xi,X)\cdot\mathscr{P}=0$
(iii) $\mathscr{P}(\xi,X)\cdot S=0$
(iv) $a+b=0$

for every vector field X on M.

Theorem 4.5. *There is no $N(k)$-quasi Einstein manifold satisfying $\mathscr{P}(\xi,X)\cdot\mathscr{P}=0$.*

Proof. From the condition $\mathscr{P}(\xi,X)\cdot\mathscr{P}=0$ we can write

$$0=\mathscr{P}(\xi,X)\mathscr{P}(Y,Z)W-\mathscr{P}(\mathscr{P}(\xi,X)Y,Z)W-\mathscr{P}(Y,\mathscr{P}(\xi,X)Z)W-\mathscr{P}(Y,Z)\mathscr{P}(\xi,X)W.$$

So from (4.3) we have

$$\begin{aligned}0=\frac{b}{n-1}\{&P(Y,Z,W,X)\xi-\eta(X)\eta(\mathscr{P}(Y,Z)W)\xi-g(X,Y)\mathscr{P}(\xi,Z)W+\eta(X)\eta(Y)\mathscr{P}(\xi,Z)W-g(X,Z)\mathscr{P}(Y,\xi)W\\&+\eta(X)\eta(Z)\mathscr{P}(Y,\xi)W-g(X,W)\mathscr{P}(Y,Z)\xi+\eta(X)\eta(W)\mathscr{P}(Y,Z)\xi\}.\end{aligned}\tag{4.9}$$

Since $b\neq 0$ taking the inner product of (4.9) by ξ, in view of (4.2)–(4.4) we get

$$0=P(Y,Z,W,X)-\frac{b}{n-1}\{g(Z,W)g(X,Y)-g(X,Y)\eta(Z)\eta(W)+g(X,Z)\eta(Y)\eta(W)-g(X,Z)g(Y,W)\}.\tag{4.10}$$

So by the use of (4.1) the last equation turns into

$$\begin{aligned}0=R(Y,Z,W,X)-\frac{1}{n-1}\{S(Z,W)g(X,Y)-S(Y,W)g(X,Z)\}-\frac{b}{n-1}\{&g(Z,W)g(X,Y)-g(X,Y)\eta(Z)\eta(W)+g(X,Z)\eta(Y)\eta(W)\\&-g(X,Z)g(Y,W)\}.\end{aligned}$$

From the last equation by a contraction one can easily get

$$b(g(Z,W)-\eta(Z)\eta(W))=0.$$

Since M is not an Einstein manifold this is not possible. This completes the proof of the theorem. □

5. Conclusions

Quasi Einstein manifolds arose during the study of exact solutions of the Einstein field equations. In the present paper, we consider an $N(k)$-quasi Einstein manifold, which is a special class of a quasi Einstein manifold. Examples of $N(k)$-quasi Einstein manifolds are given as perfect fluid space–times. We have proved that if an $N(k)$-quasi Einstein manifold satisfies the condition $\mathscr{R}(\xi,X)\cdot\mathscr{P}=0$ or $\mathscr{P}(\xi,X)\cdot S=0$ then the sum of the associated scalars is zero. We also show that there is no $N(k)$-quasi Einstein manifold satisfying $\mathscr{P}(\xi,X)\cdot\mathscr{P}=0$.

References

[1] Chaki MC, Maity RK. On quasi Einstein manifolds. Publ Math Debrecen 2000;57(3–4):297–306.
[2] Chaki MC, Ghoshal PK. Some global properties of quasi Einstein manifolds. Publ Math Debrecen 2003;63(4):635–41.
[3] De UC, Ghosh GC. On quasi Einstein manifolds. Period Math Hungar 2004;48(1–2):223–31.
[4] De UC, Ghosh GC. On quasi Einstein manifolds. II. Bull Calcutta Math Soc 2004;96(2):135–8.
[5] De UC, Ghosh GC. On quasi Einstein and special quasi Einstein manifolds. In: Proceedings of the international conference on mathematics and its applications (ICMA 2004), Kuwait: Kuwait University Department of Mathematics Computer Science; 2005. p. 178–91.
[6] De UC, Ghosh GC. On conformally flat special quasi Einstein manifolds. Publ Math Debrecen 2005;66(1–2):129–36.
[7] De UC, Sengupta J, Saha D. Conformally flat quasi-Einstein spaces. Kyungpook Math J 2006;46(3):417–23.
[8] Einstein A. Grundzuge der relativitats theory. Berlin: Springer; 2002.
[9] Ghosh GC, De UC, Binh TQ. Certain curvature restrictions on a quasi Einstein manifold. Publ Math Debrecen 2006;69(1–2):209–17.
[10] El Naschie MS. Gödel universe, dualities and high energy particles in *E*-infinity. Chaos, Solitons & Fractals 2005;25(3):759–64.
[11] El Naschie MS. Is Einstein's general field equation more fundamental than quantum field theory and particle physics? Chaos, Solitons & Fractals 2006;30(3):525–31.
[12] O'Neill B. Semi-Riemannian geometry with applications to relativity. Pure and applied mathematics, vol. 103. New York: Academic Press Inc.; 1983.
[13] Özgür C, Tripathi MM. On the concircular curvature tensor of an $N(k)$-quasi Einstein manifold. Math Pannon 2007;18(1):95–100.
[14] Tanno S. Ricci curvatures of contact Riemannian manifolds. Tôhoku Math J 1988;40:441–8.
[15] Tripathi MM, Kim JS. On $N(k)$-quasi Einstein manifolds. Commun Korean Math Soc 2007;22(3):411–7.
[16] Yano K, Kon M. Structures on manifolds. Series in pure mathematics, vol. 3. Singapore: World Scientific Publishing Co.; 1984.