J. Math. Anal. Appl. 346 (2008) 489–500

Contents lists available at ScienceDirect

J. Math. Anal. Appl.

www.elsevier.com/locate/jmaa

Approximation by polynomials and rational functions in weighted rearrangement invariant spaces

Daniyal M. Israfilov *, Ramazan Akgün

Balikesir University, Faculty of Arts and Sciences, Department of Mathematics, 10145 Balikesir, Turkey

ARTICLE INFO

Article history:
Received 17 August 2006
Available online 20 May 2008
Submitted by Richard M. Aron

Keywords:
Dini-smooth curve
Banach function spaces
Direct theorems
Inverse theorems
Weighted rearrangement invariant space
Modulus of smoothness
Constructive characterization

ABSTRACT

Let Γ be a Dini-smooth curve in the complex plane, and let $G := \operatorname{Int}\Gamma$. We prove some direct and inverse theorems of approximation theory by algebraic polynomials and rational functions in the weighted rearrangement invariant Smirnov spaces $E_X(G,\omega)$ defined on G.

1. Preliminaries and the main results

Let $\Gamma \subset \mathbb{C}$ be a closed rectifiable Jordan curve with the Lebesgue length measure $|d\tau|$ and let $X(\Gamma)$ be a rearrangement invariant (r.i.) space over Γ, generated by a r.i. function norm ρ, with associate space $X'(\Gamma)$. For each $f \in X(\Gamma)$ we define

$$\|f\|_{X(\Gamma)} := \rho(|f|), \quad f \in X(\Gamma).$$

A r.i. space $X(\Gamma)$ equipped with norm $\|\cdot\|_{X(\Gamma)}$ is a Banach space [4, Theorems 1.4 and 1.6, pp. 3, 5].

It is well known that

$$\|f\|_{X(\Gamma)} = \sup\left\{\int_\Gamma |fg||d\tau|:\ g \in X'(\Gamma),\ \|g\|_{X'(\Gamma)} \leqslant 1\right\},$$

$$\|g\|_{X'(\Gamma)} = \sup\left\{\int_\Gamma |fg||d\tau|:\ f \in X(\Gamma),\ \|f\|_{X(\Gamma)} \leqslant 1\right\} \tag{1}$$

hold.

If $f \in X(\Gamma)$ and $g \in X'(\Gamma)$, then fg is summable [4, Theorem 2.4, p. 9] and

$$\int_\Gamma |fg||d\tau| \leqslant \|f\|_{X(\Gamma)}\|g\|_{X'(\Gamma)}. \tag{2}$$

* Corresponding author.
E-mail addresses: mdaniyal@balikesir.edu.tr (D.M. Israfilov), rakgun@balikesir.edu.tr (R. Akgün).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2008.05.040

A function $\omega:\Gamma\to[0,\infty]$ is referred to as a *weight* if ω is measurable and the preimage $\omega^{-1}(\{0,\infty\})$ has measure zero. Following [20], we set

$$X(\Gamma,\omega):=\{f \text{ measurable: } f\omega\in X(\Gamma)\},$$

which is equipped with the norm

$$\|f\|_{X(\Gamma,\omega)}:=\|f\omega\|_{X(\Gamma)}.$$

A normed space $X(\Gamma,\omega)$ is called a *weighted r.i.* space.

For definitions and fundamental properties of general r.i. spaces we refer to [4].

If $\omega\in X(\Gamma)$ and $1/\omega\in X'(\Gamma)$, then $X(\Gamma,\omega)$ is a Banach function space and from the Hölder's inequality we have

$$L^\infty(\Gamma)\subset X(\Gamma,\omega)\subset L^1(\Gamma).$$

By the Luxemburg representation theorem [4, Theorem 4.10, p. 62], there is a unique r.i. function norm $\overline{\rho}$ over Lebesgue measure space $([0,|\Gamma|],m)$, where $|\Gamma|$ is the Lebesgue length of Γ, such that $\rho(f)=\overline{\rho}(f^*)$ for all non-negative and almost everywhere (a.e.) finite measurable functions f defined on Γ. Here f^* denotes the *non-increasing rearrangement* of f [4, p. 39]. The r.i. space over $([0,|\Gamma|],m)$ generated by $\overline{\rho}$ is called the *Luxemburg representation* of $X(\Gamma)$ and is denoted by $\overline{X}$.

Let g be a non-negative, almost everywhere finite and measurable function on $[0,|\Gamma|]$. For each $x>0$ we set

$$(H_xg)(t):=\begin{cases} g(xt), & xt\in[0,|\Gamma|],\\ 0, & xt\notin[0,|\Gamma|],\end{cases}\qquad t\in\big[0,|\Gamma|\big].$$

Then the operator $H_{1/x}$ is bounded on $\overline{X}$ [4, p. 165] with the operator norm

$$h_X(x):=\|H_{1/x}\|_{\mathcal{B}(\overline{X})},$$

where $\mathcal{B}(\overline{X})$ is the Banach algebra of bounded linear operators on $\overline{X}$.

The functions

$$\alpha_X:=\lim_{x\to 0}\frac{\log h_X(x)}{\log x},\qquad \beta_X:=\lim_{x\to\infty}\frac{\log h_X(x)}{\log x}$$

are called *lower* and *upper Boyd indices* [5] of r.i. space $X(\Gamma)$. The indices α_X, β_X are called *nontrivial* if $0<\alpha_X$ and $\beta_X<1$.

For $z\in\Gamma$ and $\epsilon>0$, let $\Gamma(z,\epsilon)$ denotes the portion of Γ contained in the open disc of radius ϵ and centered at z, i.e. $\Gamma(z,\epsilon):=\{t\in\Gamma:\ |t-z|<\epsilon\}$.

For fixed $p\in(1,\infty)$, we define $q\in(1,\infty)$ by $p^{-1}+q^{-1}=1$. The set of all weights $\omega:\Gamma\to[0,\infty]$ satisfying Muckenhoupt's A_p condition

$$\sup_{z\in\Gamma}\sup_{\epsilon>0}\Bigg(\frac{1}{\epsilon}\int\limits_{\Gamma(z,\epsilon)}\omega(\tau)^p|d\tau|\Bigg)^{1/p}\Bigg(\frac{1}{\epsilon}\int\limits_{\Gamma(z,\epsilon)}\omega(\tau)^{-q}|d\tau|\Bigg)^{1/q}<\infty$$

is denoted by $A_p(\Gamma)$.

We denote by $L^p(\Gamma,\omega)$ the set of all measurable functions $f:\Gamma\to\mathbb{C}$ such that $|f|\omega\in L^p(\Gamma)$.

Let Γ be a closed rectifiable Jordan curve and let $G:=\operatorname{int}\Gamma$, $G^-:=\operatorname{ext}\Gamma$, $\mathbb{D}:=\{w\in\mathbb{C}:\ |w|<1\}$, $\mathbb{T}:=\partial\mathbb{D}$, $\mathbb{D}^-:=\operatorname{ext}\mathbb{T}$. Without loss of generality we may assume $0\in G$.

Let $w=\varphi(z)$ and $w=\varphi_1(z)$ be the conformal mapping of G^- and G onto $\mathbb{D}^-$ normalized by the conditions

$$\varphi(\infty)=\infty,\qquad \lim_{z\to\infty}\varphi(z)/z>0,$$

and

$$\varphi_1(0)=\infty,\qquad \lim_{z\to 0}z\varphi_1(z)>0,$$

respectively. We denote by ψ and ψ_1, the inverse of φ and φ_1, respectively.

By $E^p(G)$ and $E^p(G^-)$, $0<p<\infty$, we denote the *Smirnov classes* of analytic functions in G and G^-, respectively. It is well known that every function $f\in E^1(G)$ or $f\in E^1(G^-)$ has a nontangential boundary values a.e. on Γ and if we use the same notation for the nontangential boundary value of f, then $f\in L^1(\Gamma)$.

Definition 1. Let ω be a weight on Γ and let $E_X(G,\omega):=\{f\in E^1(G):\ f\in X(\Gamma,\omega)\}$, $E_X(G^-,\omega):=\{f\in E^1(G^-):\ f\in X(\Gamma,\omega)\}$, $\widetilde{E}_X(G^-,\omega):=\{f\in E_X(G^-,\omega):\ f(\infty)=0\}$. The classes of functions $E_X(G,\omega)$ and $E_X(G^-,\omega)$ will be called weighted r.i. Smirnov spaces with respect to domains G and G^-, respectively.

Since the Luxemburg norm of Orlicz space is itself a r.i. function norm, every Orlicz space is a r.i. space and therefore every weighted Smirnov–Orlicz space is a weighted r.i. Smirnov space.

Each function $f \in E_X(G,\omega)$ or $E_X(G^-,\omega)$ has a nontangential boundary values a.e. on Γ.

Let $f \in L^1(\Gamma)$. Then, the functions f^+ and f^- defined by

$$f^+(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\varsigma)}{\varsigma - z}\,d\varsigma, \quad z \in G, \qquad f^-(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\varsigma)}{\varsigma - z}\,d\varsigma, \quad z \in G^-,$$

are analytic in G and G^-, respectively and $f^-(\infty) = 0$.

For $g \in X(\mathbb{T},\omega)$, we set

$$\sigma_h(g)(w) := \frac{1}{2h}\int_{-h}^{h} g\big(we^{it}\big)\,dt, \quad 0 < h < \pi, \ \ w \in \mathbb{T}.$$

If α_X and β_X are nontrivial Boyd indices of the space $X(\mathbb{T},\omega)$ and $\omega \in A_{1/\alpha_X}(\mathbb{T}) \cap A_{1/\beta_X}(\mathbb{T})$, then by [13] we have

$$\big\|\sigma_h(g)\big\|_{X(\mathbb{T},\omega)} \leqslant c_1 \|g\|_{X(\mathbb{T},\omega)},$$

and consequently $\sigma_h(g) \in X(\mathbb{T},\omega)$ for any $g \in X(\mathbb{T},\omega)$.

Definition 2. Let α_X and β_X be nontrivial and $\omega \in A_{1/\alpha_X}(\mathbb{T}) \cap A_{1/\beta_X}(\mathbb{T})$. The function

$$\Omega^r_{X,\omega}(g,\delta) := \sup_{\substack{i=1,2,\ldots,r\\ 0<h_i\leqslant\delta}} \left\| \prod_{i=1}^{r}(I - \sigma_{h_i})g \right\|_{X(\mathbb{T},\omega)}, \quad \delta > 0, \ r = 1,2,\ldots,$$

is called rth modulus of smoothness of $g \in X(\mathbb{T},\omega)$, where I is the identity operator.

In this definition we use as shift the mean value operator σ_h, because the usual shift $g(\cdot) \to g(\cdot + h)$ is, in general, noninvariant in the weighted r.i. space. It can easily be verified that the function $\Omega^r_{X,\omega}(g,\cdot)$ is continuous, non-negative and satisfy

$$\lim_{\delta\to 0} \Omega^r_{X,\omega}(g,\delta) = 0, \quad \Omega^r_{X,\omega}(g+g_1,\cdot) \leqslant \Omega^r_{X,\omega}(g,\cdot) + \Omega^r_{X,\omega}(g_1,\cdot)$$

for $g, g_1 \in X(\mathbb{T},\omega)$.

A smooth Jordan curve Γ will be called *Dini-smooth*, if the function $\theta(s)$, the angle between the tangent line and the positive real axis expressed as a function of arclength s, has modulus of continuity $\Omega(\theta,s)$ satisfying the Dini condition

$$\int_0^\delta \frac{\Omega(\theta,s)}{s}\,ds < \infty, \quad \delta > 0.$$

If Γ is Dini-smooth, then [30]

$$\begin{aligned} &0 < c_2 < \big|\psi'(w)\big| < c_3 < \infty, \quad |w| \geqslant 1,\\ &0 < c_4 < \big|\varphi'(z)\big| < c_5 < \infty, \quad z \in \overline{G^-}, \end{aligned} \tag{3}$$

with some constants c_2, c_3, c_4 and c_5. Similar inequalities hold also for ψ_1' and φ_1', in case of $|w| = 1$ and $z \in \Gamma$, respectively.

Let Γ be a Dini-smooth curve and ω be a weight on Γ. We associate with ω, the following two weights defined on $\mathbb{T}$ by

$$\omega_0 := \omega \circ \psi, \qquad \omega_1 := \omega \circ \psi_1,$$

and let $f_0 := f \circ \psi$, $f_1 := f \circ \psi_1$ for $f \in X(\Gamma,\omega)$. Then from (3), we have $f_0 \in X(\mathbb{T},\omega_0)$ and $f_1 \in X(\mathbb{T},\omega_1)$ for $f \in X(\Gamma,\omega)$. Using the nontangential boundary values of f_0^+ and f_1^+ on $\mathbb{T}$, we define

$$\Omega^r_{\Gamma,X,\omega}(f,\delta) := \Omega^r_{X,\omega_0}\big(f_0^+,\delta\big), \quad \delta > 0, \qquad \widetilde{\Omega}^r_{\Gamma,X,\omega}(f,\delta) := \Omega^r_{X,\omega_1}\big(f_1^+,\delta\big), \quad \delta > 0,$$

for $r = 1,2,3,\ldots$.

We set

$$\mathcal{E}_n(f)_{X,\omega} := \inf_{P\in\mathcal{P}_n} \|f - P\|_{X(\mathbb{T},\omega)}, \qquad \widetilde{\mathcal{E}}_n(g)_{X,\omega} := \inf_{R\in\mathcal{R}_n} \|g - R\|_{X(\Gamma,\omega)},$$

where $f \in E_X(\mathbb{D}, \omega)$, $g \in E_X(G^-, \omega)$, $\mathcal{P}_n$ is the set of algebraic polynomials of degree not greater than n and $\mathcal{R}_n$ is the set of rational functions of the form

$$\sum_{k=0}^{n} \frac{a_k}{z^k}.$$

In this work we investigate the approximation problems in the spaces $X(\Gamma, \omega)$, $E_X(G, \omega)$ and $\widetilde{E}_X(G^-, \omega)$. First of all, we prove one general direct theorem of approximation theory by rational functions in the weighted r.i. space $X(\Gamma, \omega)$. Later we obtain the direct and inverse theorems of polynomial approximation in the spaces $E_X(G, \omega)$ and $\widetilde{E}_X(G^-, \omega)$. Using these results we give a constructive descriptions of the generalized Lipschitz classes defined in the spaces $E_X(G, \omega)$ and $\widetilde{E}_X(G^-, \omega)$. Note that our results are new also in the nonweighted cases.

These problems in the different subspaces of the r.i. space were investigated by several authors. The degree of polynomial approximation in the spaces $E^p(G)$ and $L^p(\Gamma)$ have been estimated in [2,3,7,14,15,24,29] under various restrictions on the boundary Γ of G. The similar problems in weighted Smirnov and Lebesgue spaces were studied in [16] and [17]. The appropriate inverse theorems and a constructive characterization of generalized Lipschitz class in the weighted Smirnov spaces were obtained in [18]. Some inverse theorems in Smirnov–Orlicz spaces were proved by V.M. Kokilashvili in [23]. In this space, some direct theorems of approximation theory by algebraic polynomials and by interpolating polynomials were obtained in [1,12,19].

Let us emphasize that in this work the Faber polynomials, Faber–Laurent rational functions and also the method, given by Dynkin in [9] and based on the boundedness of the Faber and Faber–Laurent operators were commonly used.

The main results of this work are the following.

Theorem 1. *Let Γ be a Dini-smooth curve, α_X, β_X be the nontrivial indices and let $\omega \in A_{1/\alpha_X}(\Gamma) \cap A_{1/\beta_X}(\Gamma)$. If $f \in X(\Gamma, \omega)$, then there is a constant $c_6 > 0$ such that for any natural n,*

$$\left\| f - R_n(\cdot, f)\right\|_{X(\Gamma,\omega)} \leqslant c_6 \left\{ \Omega^r_{\Gamma,X,\omega}\big(f, 1/(n+1)\big) + \widetilde{\Omega}^r_{\Gamma,X,\omega}\big(f, 1/(n+1)\big)\right\}, \quad r = 1, 2, 3, \ldots,$$

where $R_n(\cdot, f)$ is the nth partial sum of the Faber–Laurent series of f.

Corollary 1. *Let Γ be a Dini-smooth curve, α_X, β_X be the nontrivial indices and let $\omega \in A_{1/\alpha_X}(\Gamma) \cap A_{1/\beta_X}(\Gamma)$. If $f \in E_X(G, \omega)$, then there is a constant $c_7 > 0$ such that for any natural n,*

$$\left\| f - P_n(\cdot, f)\right\|_{X(\Gamma,\omega)} \leqslant c_7 \Omega^r_{\Gamma,X,\omega}\big(f, 1/(n+1)\big), \quad r = 1, 2, 3, \ldots,$$

where $P_n(\cdot, f)$ is the nth partial sum of the Faber series of f.

Corollary 2. *Let Γ be a Dini-smooth curve, α_X, β_X be the nontrivial indices and let $\omega \in A_{1/\alpha_X}(\Gamma) \cap A_{1/\beta_X}(\Gamma)$. If $f \in \widetilde{E}_X(G^-, \omega)$, then there is a constant $c_8 > 0$ such that for any natural n,*

$$\left\| f - R_n(\cdot, f)\right\|_{X(\Gamma,\omega)} \leqslant c_8 \widetilde{\Omega}^r_{\Gamma,X,\omega}\big(f, 1/(n+1)\big), \quad r = 1, 2, 3, \ldots,$$

where $R_n(\cdot, f)$ as in Theorem 1.

The following inverse theorem holds.

Theorem 2. *Let Γ be a Dini-smooth curve and let $X(\mathbb{T})$ be a reflexive r.i. space with the nontrivial indices α_X and β_X. If $\omega \in A_{1/\alpha_X}(\Gamma) \cap A_{1/\beta_X}(\Gamma)$, then for $f \in E_X(G, \omega)$,*

$$\Omega^r_{\Gamma,X,\omega}(f, 1/n) \leqslant \frac{c_9}{n^{2r}} \left\{ \mathcal{E}_0(f, G)_{X,\omega} + \sum_{k=1}^{n} k^{2r-1} \mathcal{E}_k(f, G)_{X,\omega} \right\}, \quad r = 1, 2, 3, \ldots,$$

with a constant $c_9 > 0$.

Corollary 3. *Under the conditions of Theorem 2, if*

$$\mathcal{E}_n(f, G)_{X,\omega} = \mathcal{O}\big(n^{-\alpha}\big), \quad \alpha > 0,\ n = 1, 2, 3, \ldots,$$

then for $f \in E_X(G, \omega)$ and $r = 1, 2, 3, \ldots,$

$$\Omega^r_{\Gamma,X,\omega}(f, \delta) = \begin{cases} \mathcal{O}(\delta^{\alpha}), & r > \alpha/2; \\ \mathcal{O}(\delta^{\alpha} |\log \frac{1}{\delta}|), & r = \alpha/2; \\ \mathcal{O}(\delta^{2r}), & r < \alpha/2. \end{cases}$$

Definition 3. For $\alpha > 0$ and $r := [\alpha/2] + 1$ we set

$$Lip\,\alpha(X,\omega) := \{ f \in E_X(G,\omega) \colon \ \Omega^r_{\Gamma,X,\omega}(f,\delta) = \mathcal{O}(\delta^\alpha),\ \delta > 0 \},$$
$$\widetilde{Lip}\,\alpha(X,\omega) := \{ f \in \widetilde{E}_X(G^-,\omega) \colon \ \widetilde{\Omega}^r_{\Gamma,X,\omega}(f,\delta) = \mathcal{O}(\delta^\alpha),\ \delta > 0 \}.$$

Then, from Corollary 3 and Definition 3 we get the following.

Corollary 4. *Under the conditions of Theorem* 2, *if*

$$\mathcal{E}_n(f,G)_{X,\omega} = \mathcal{O}(n^{-\alpha}), \quad \alpha > 0,\ n = 1,2,3,\ldots,$$

then $f \in Lip\,\alpha(X,\omega)$.

By Corollaries 1 and 4 we have the constructive characterization of the classes $Lip\,\alpha(X,\omega)$.

Corollary 5. *Let* $\alpha > 0$ *and let the conditions of Theorem* 2 *be fulfilled. Then the following conditions are equivalent.*

(a) $f \in Lip\,\alpha(X,\omega)$;
(b) $\mathcal{E}_n(f,G)_{X,\omega} = \mathcal{O}(n^{-\alpha})$, $n = 1,2,3,\ldots.$

The inverse theorem for unbounded domains has the following form.

Theorem 3. *Let* Γ *be a Dini-smooth curve and* $X(\mathbb{T})$ *be a reflexive r.i. space with the nontrivial indices* α_X *and* β_X. *If* $\omega \in A_{1/\alpha_X}(\Gamma) \cap A_{1/\beta_X}(\Gamma)$, *then for* $f \in \widetilde{E}_X(G^-,\omega)$,

$$\widetilde{\Omega}^r_{\Gamma,X,\omega}(f,1/n) \leqslant \frac{c_{10}}{n^{2r}} \left\{ \widetilde{\mathcal{E}}_0(f)_{X,\omega} + \sum_{k=1}^{n} k^{2r-1} \widetilde{\mathcal{E}}_k(f)_{X,\omega} \right\}, \quad r = 1,2,3,\ldots,$$

with a constant $c_{10} > 0$.

By the similar way to that of the $E_X(G,\omega)$ we obtain the following corollaries.

Corollary 6. *Under the conditions of Theorem* 3, *if*

$$\widetilde{\mathcal{E}}_n(f)_{X,\omega} = \mathcal{O}(n^{-\alpha}), \quad \alpha > 0,\ n = 1,2,3,\ldots,$$

then for $f \in \widetilde{E}_X(G^-,\omega)$ *and* $r = 1,2,3,\ldots$:

$$\widetilde{\Omega}^r_{\Gamma,X,\omega}(f,\delta) = \begin{cases} \mathcal{O}(\delta^\alpha), & r > \alpha/2; \\ \mathcal{O}(\delta^\alpha |\log \frac{1}{\delta}|), & r = \alpha/2; \\ \mathcal{O}(\delta^{2r}), & r < \alpha/2. \end{cases}$$

Using Corollary 6 and Definition 3 we get the following.

Corollary 7. *Under the conditions of Theorem* 3, *if*

$$\widetilde{\mathcal{E}}_n(f)_{X,\omega} = \mathcal{O}(n^{-\alpha}), \quad \alpha > 0,\ n = 1,2,3,\ldots,$$

then $f \in \widetilde{Lip}\,\alpha(X,\omega)$.

By Corollaries 2 and 7 we have the following.

Corollary 8. *Let* $\alpha > 0$ *and the conditions of Theorem* 3 *be fulfilled. Then the following conditions are equivalent.*

(a) $f \in \widetilde{Lip}\,\alpha(X,\omega)$;
(b) $\widetilde{\mathcal{E}}_n(f)_{X,\omega} = \mathcal{O}(n^{-\alpha})$, $n = 1,2,3,\ldots.$

In the sequel, we denote by $c, c_1, c_2, \ldots,$ positive constants (possibly different at different occurrences) that either are absolute or depend on parameters not essential for the argument.

2. Auxiliary results

Let Γ be a rectifiable Jordan curve, $f \in L^1(\Gamma)$ and let

$$(S_\Gamma f)(t) := \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int\limits_{\Gamma \setminus \Gamma(t,\epsilon)} \frac{f(\varsigma)}{\varsigma - t} d\varsigma, \quad t \in \Gamma,$$

be *Cauchy's singular integral* of f at the point t. The linear operator $S_\Gamma : f \to S_\Gamma\ f$ is called the *Cauchy singular operator*.

If one of the functions f^+ or f^- has the nontangential limits a.e. on Γ, then $S_\Gamma f(z)$ exists a.e. on Γ and also the other one has the nontangential limits a.e. on Γ. Conversely, if $S_\Gamma f(z)$ exists a.e. on Γ, then both functions f^+ and f^- have the nontangential limits a.e. on Γ. In both cases, the formulae

$$f^+(z) = (S_\Gamma f)(z) + f(z)/2, \qquad f^-(z) = (S_\Gamma f)(z) - f(z)/2, \tag{4}$$

and hence

$$f = f^+ - f^- \tag{5}$$

holds a.e. on Γ (*see, e.g.,* [11, p. 431]).

Lemma 1. *If $0 < \alpha_X, \beta_X < 1$, $\omega \in A_{1/\alpha_X}(\Gamma) \cap A_{1/\beta_X}(\Gamma)$, then $f^+ \in E_X(G,\omega)$ and $f^- \in \widetilde{E}_X(G^-,\omega)$ for every $f \in X(\Gamma,\omega)$.*

Proof. Using [6, Theorem 2.31, p. 58] we have that there are numbers $p, q \in (1,\infty)$ satisfying $1 < p < 1/\beta_X \leqslant 1/\alpha_X < q < \infty$, and $\omega \in A_p(\Gamma) \cap A_q(\Gamma)$. Then [25, Proposition 2.b.3, p. 132]

$$L^q(\Gamma) \subset X(\Gamma) \subset L^p(\Gamma),$$

where the inclusion maps being continuous. If $f \in X(\Gamma,\omega)$, then $f\omega \in X(\Gamma)$, and hence $f\omega \in L^p(\Gamma)$. The last relation is equivalent to the relation $f \in L^p(\Gamma,\omega)$, which by [16], implies that

$$f^+ \in E^1(G) \quad \text{and} \quad f^- \in E^1(G^-).$$

Since the operator S_Γ is bounded [21, Theorem 4.5] in $X(\Gamma,\omega)$, we obtain from (4)

$$f^+ \in X(\Gamma,\omega) \quad \text{and} \quad f^- \in X(\Gamma,\omega). \qquad \square$$

Lemma 2. *(See [13].) If α_X and β_X are nontrivial and $\omega \in A_{1/\alpha_X}(\mathbb{T}) \cap A_{1/\beta_X}(\mathbb{T})$, then there exists a constant $c_{11} > 0$ such that for every natural number n,*

$$\|g - T_n g\|_{X(\mathbb{T},\omega)} \leqslant c_{11} \Omega^r_{X,\omega}\big(g, 1/(n+1)\big), \quad g \in E_X(\mathbb{D},\omega),$$

where $r = 1, 2, 3, \ldots$ and $T_n g$ is nth partial sum of the Taylor series of g at the origin.

We know [28, pp. 52, 255] that

$$\frac{\psi'(w)}{\psi(w) - z} = \sum_{k=0}^{\infty} \frac{\Phi_k(z)}{w^{k+1}}, \quad z \in G,\ w \in \mathbb{D}^-,$$

and

$$\frac{\psi_1'(w)}{\psi_1(w) - z} = \sum_{k=1}^{\infty} \frac{F_k(1/z)}{w^{k+1}}, \quad z \in G^-,\ w \in \mathbb{D}^-,$$

where $\Phi_k(z)$ and $F_k(1/z)$ are the *Faber polynomials* of degree k with respect to z and $1/z$ for the continua $\overline{G}$ and $\overline{\mathbb{C}} \setminus G$, with the integral representations [28, pp. 35, 255]

$$\Phi_k(z) = \frac{1}{2\pi i} \int\limits_{|w|=R} \frac{w^k \psi'(w)}{\psi(w) - z} dw, \quad z \in G,\ R > 1,$$

$$F_k(1/z) = \frac{1}{2\pi i} \int\limits_{|w|=1} \frac{w^k \psi_1'(w)}{\psi_1(w) - z} dw, \quad z \in G^-,$$

and

$$\Phi_k(z) = \varphi^k(z) + \frac{1}{2\pi i}\int_\Gamma \frac{\varphi^k(\varsigma)}{\varsigma - z}\,d\varsigma, \quad z \in G^-,\ k = 0,1,2,\ldots, \tag{6}$$

$$F_k(1/z) = \varphi_1^k(z) - \frac{1}{2\pi i}\int_\Gamma \frac{\varphi_1^k(\varsigma)}{\varsigma - z}\,d\varsigma, \quad z \in G \setminus \{0\}. \tag{7}$$

We put

$$a_k := a_k(f) := \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{f_0(w)}{w^{k+1}}\,dw, \quad k = 0,1,2,\ldots,$$

$$\tilde a_k := \tilde a_k(f) := \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{f_1(w)}{w^{k+1}}\,dw, \quad k = 1,2,\ldots,$$

and correspond the series

$$\sum_{k=0}^{\infty} a_k \Phi_k(z) + \sum_{k=1}^{\infty} \tilde a_k F_k(1/z)$$

for the function $f \in L^1(\Gamma)$, i.e.,

$$f(z) \sim \sum_{k=0}^{\infty} a_k \Phi_k(z) + \sum_{k=1}^{\infty} \tilde a_k F_k(1/z).$$

This series is called the *Faber–Laurent* series of the function f and the coefficients a_k and $\tilde a_k$ are said to be the *Faber–Laurent coefficients* of f.

Let $\mathcal{P}$ be the set of all polynomials (with no restrictions on the degree), and let $\mathcal{P}(\mathbb{D})$ be the set of traces of members of $\mathcal{P}$ on $\mathbb{D}$.

We define the operators $T : \mathcal{P}(\mathbb{D}) \to E_X(G,\omega)$ and $\widetilde{T} : \mathcal{P}(\mathbb{D}) \to \widetilde{E}_X(G^-,\omega)$ defined on $\mathcal{P}(\mathbb{D})$ as

$$T(P)(z) := \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{P(w)\psi'(w)}{\psi(w) - z}\,dw, \quad z \in G,$$

$$\widetilde{T}(P)(z) := \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{P(w)\psi_1'(w)}{\psi_1(w) - z}\,dw, \quad z \in G^-.$$

Then, it is readily seen that

$$T\left(\sum_{k=0}^{n} b_k w^k\right) = \sum_{k=0}^{n} b_k \Phi_k(z) \quad \text{and} \quad \widetilde{T}\left(\sum_{k=0}^{n} d_k w^k\right) = \sum_{k=0}^{n} d_k F_k(1/z).$$

If $z' \in G$, then

$$T(P)(z') = \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{P(w)\psi'(w)}{\psi(w) - z'}\,dw = \frac{1}{2\pi i}\int_\Gamma \frac{(P\circ\varphi)(\varsigma)}{\varsigma - z'}\,d\varsigma = (P\circ\varphi)^+(z'),$$

which, by (4) implies that

$$T(P)(z) = S_\Gamma(P\circ\varphi)(z) + (1/2)(P\circ\varphi)(z)$$

a.e. on Γ.

Similarly, taking the nontangential limit $z'' \to z \in \Gamma$, outside Γ, in the relation

$$\widetilde{T}(P)\big(z''\big) = \frac{1}{2\pi i}\int_\Gamma \frac{P(\varphi_1(\varsigma))}{\varsigma - z''}\,d\varsigma = \big[(P\circ\varphi_1)\big]^-\big(z''\big), \quad z'' \in G^-,$$

we get

$$\widetilde{T}(P)(z) = -(1/2)(P\circ\varphi_1)(z) + S_\Gamma(P\circ\varphi_1)(z)$$

a.e. on Γ.

Since S_Γ is bounded in $X(\Gamma,\omega)$, we have the following result.

Lemma 3. *Let Γ be a Dini-smooth curve and let the indices α_X, β_X be nontrivial. If $\omega \in A_{1/\alpha_X}(\Gamma) \cap A_{1/\beta_X}(\Gamma)$, then the linear operators*

$$T : \mathcal{P}(\mathbb{D}) \to E_X(G, \omega), \qquad \widetilde{T} : \mathcal{P}(\mathbb{D}) \to \widetilde{E}_X(G^-, \omega)$$

are bounded.

The set of trigonometric polynomials is dense [13] in $X([-\pi, \pi], \omega)$, which implies density of the algebraic polynomials in $E_X(\mathbb{D}, \omega)$. Consequently, from Lemma 3, using the Hahn–Banach theorem, we can extend the operators T and $\widetilde{T}$ from $\mathcal{P}(\mathbb{D})$ to the spaces $E_X(\mathbb{D}, \omega_0)$ and $E_X(\mathbb{D}, \omega_1)$ as linear and bounded operators, respectively, and for the extensions $T : E_X(\mathbb{D}, \omega_0) \to E_X(G, \omega)$ and $\widetilde{T} : E_X(\mathbb{D}, \omega_1) \to \widetilde{E}_X(G^-, \omega)$ we have the representations

$$T(g)(z) = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{g(w)\psi'(w)}{\psi(w) - z}\, dw, \quad z \in G, \; g \in E_X(\mathbb{D}, \omega_0),$$

$$\widetilde{T}(g)(z) = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{g(w)\psi_1'(w)}{\psi_1(w) - z}\, dw, \quad z \in G^-, \; g \in E_X(\mathbb{D}, \omega_1).$$

Lemma 4. *If $0 < \alpha_X, \beta_X < 1$, $\omega \in A_{1/\alpha_X}(\mathbb{T}) \cap A_{1/\beta_X}(\mathbb{T})$ and $X(\mathbb{T})$ is a reflexive r.i. space, then for any $f \in X(\mathbb{T}, \omega)$,*

$$\left\| P_r(f) - f \right\|_{X(\mathbb{T},\omega)} \to 0, \quad \text{as } r \to 1^-,$$

where

$$P_r(f)(w) := \frac{1}{2\pi} \int_0^{2\pi} P(r, \theta - t) f\left(e^{it}\right) dt, \quad w = re^{i\theta}, \; 0 < r < 1,$$

and $P(r, \theta - t)$ is the Poisson kernel.

Proof. Let $p, q \in (1, \infty)$ be the numbers such that

$$1 < p < 1/\beta_X \leqslant 1/\alpha_X < q < \infty \quad \text{and} \quad \omega \in A_p(\mathbb{T}) \cap A_q(\mathbb{T}).$$

Then [26, Theorem 10] P_r is bounded in $L^p(\mathbb{T}, \omega)$ and $L^q(\mathbb{T}, \omega)$. Consequently, the operator $W_r := \omega P_r \omega^{-1} I$ is bounded in $L^p(\mathbb{T})$ and $L^q(\mathbb{T})$. Now, the Boyd interpolation theorem [5] implies that W_r is bounded in $X(\mathbb{T})$. Therefore

$$\left\| P_r(f) \right\|_{X(\mathbb{T},\omega)} \leqslant c_{12} \| f \|_{X(\mathbb{T},\omega)}. \tag{8}$$

Since $X(\mathbb{T})$ is reflexive we have that $X(\mathbb{T}, \omega)$ is reflexive [22, Corollary 2.8] and therefore the set of continuous functions on $\mathbb{T}$ is dense [20, Lemmas 1.2 and 1.3] in $X(\mathbb{T}, \omega)$. Consequently, for a given $f \in X(\mathbb{T}, \omega)$ and $\epsilon > 0$ there is a continuous function f^* such that

$$\left\| f - f^* \right\|_{X(\mathbb{T},\omega)} < \epsilon. \tag{9}$$

On the other hand, since the Poisson integral of a continuous function converges to it uniformly on $\mathbb{T}$ [27, p. 239], from (1), we have

$$\begin{aligned} \left\| P_r\left(f^*\right) - f^* \right\|_{X(\mathbb{T},\omega)} &= \sup_{\|g\|_{X'} \leqslant 1} \int_{\mathbb{T}} \left| P_r\left(f^*\right)(w) - f^*(w) \right| \left| g(w) \right| \omega(w) |dw| \\ &< \epsilon \sup_{\|g\|_{X'} \leqslant 1} \int_{\mathbb{T}} \omega(w) \left| g(w) \right| |dw| = \epsilon \|\omega\|_{X(\mathbb{T})}, \end{aligned} \tag{10}$$

for $0 < 1 - r < \delta(\varepsilon)$. Then, from (8), (9) and (10), we conclude that

$$\begin{aligned} \left\| P_r(f) - f \right\|_{X(\mathbb{T},\omega)} &\leqslant \left\| P_r(f) - P_r\left(f^*\right) \right\|_{X(\mathbb{T},\omega)} + \left\| P_r\left(f^*\right) - f^* \right\|_{X(\mathbb{T},\omega)} + \left\| f^* - f \right\|_{X(\mathbb{T},\omega)} \\ &= \left\| P_r\left(f - f^*\right) \right\|_{X(\mathbb{T},\omega)} + \left\| P_r\left(f^*\right) - f^* \right\|_{X(\mathbb{T},\omega)} + \left\| f^* - f \right\|_{X(\mathbb{T},\omega)} \\ &\leqslant c_{13} \left\| f^* - f \right\|_{X(\mathbb{T},\omega)} + \left\| P_r\left(f^*\right) - f^* \right\|_{X(\mathbb{T},\omega)} < \left\{ c_{13} + \|\omega\|_{X(\mathbb{T})} \right\} \epsilon. \end{aligned}$$

Since $\omega \in A_{1/\alpha_X}(\mathbb{T}) \cap A_{1/\beta_X}(\mathbb{T})$, we have that $\omega \in X(\mathbb{T})$. This completes the proof. □

Theorem 4. *Let Γ be a Dini-smooth curve and $X(\mathbb{T})$ be a reflexive r.i. space with the nontrivial indices α_X and β_X. If $\omega \in A_{1/\alpha_X}(\Gamma) \cap A_{1/\beta_X}(\Gamma)$, then the operators*

$$T : E_X(\mathbb{D}, \omega_0) \to E_X(G, \omega) \quad \textit{and} \quad \widetilde{T} : E_X(\mathbb{D}\widetilde{T} : E_X(\mathbb{D}, \omega_1) \to \widetilde{E}_X(G^-, \omega)$$

are one-to-one and onto.

Proof. The proof we give only for the operator T. For the operator $\widetilde{T}$ the proof goes similarly. Let $g \in E_X(\mathbb{D}, \omega_0)$ with the Taylor expansion

$$g(w) := \sum_{k=0}^{\infty} \alpha_k w^k, \quad w \in \mathbb{D}.$$

If Γ is a Dini-smooth curve, then via (3), the conditions $\omega \in A_{1/\alpha_X}(\Gamma) \cap A_{1/\beta_X}(\Gamma)$, $\omega_0 \in A_{1/\alpha_X}(\mathbb{T}) \cap A_{1/\beta_X}(\mathbb{T})$ and $\omega_1 \in A_{1/\alpha_X}(\mathbb{T}) \cap A_{1/\beta_X}(\mathbb{T})$ are equivalent. Since $\omega_0 \in A_{1/\alpha_X}(\mathbb{T}) \cap A_{1/\beta_X}(\mathbb{T})$, there exist $p, q \in (1, \infty)$ such that $1 < p < 1/\beta_X \leqslant 1/\alpha_X < q < \infty$, $\omega_0 \in A_p(\mathbb{T}) \cap A_q(\mathbb{T})$ and $L^q(\mathbb{T}) \subset X(\mathbb{T}) \subset L^p(\mathbb{T})$.

Let $g_r(w) := g(rw)$, $0 < r < 1$. Since $g \in E^1(\mathbb{D})$ is the Poisson integral of its boundary function [8, p. 41], we have

$$\|g_r - g\|_{X(\mathbb{T},\omega_0)} = \big\| P_r(g) - g \big\|_{X(\mathbb{T},\omega_0)}$$

and using Lemma 4, we get $\|g_r - g\|_{X(\mathbb{T},\omega_0)} \to 0$, as $r \to 1^-$.

Therefore, the boundedness of the operator T implies that

$$\big\| T(g_r) - T(g) \big\|_{X(\Gamma,\omega)} \to 0, \quad \text{as } r \to 1^-. \tag{11}$$

Since $\sum_{k=0}^{\infty} \alpha_k w^k$ is uniformly convergent for $|w| = r < 1$, $\sum_{k=0}^{\infty} \alpha_k r^k w^k$ is uniformly convergent on $\mathbb{T}$, and hence

$$T(g_r)(z') = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{g_r(w)\psi'(w)}{\psi(w) - z'} \, dw = \sum_{m=0}^{\infty} \alpha_m r^m \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{w^m \psi'(w)}{\psi(w) - z'} \, dw = \sum_{m=0}^{\infty} \alpha_m r^m \Phi_m(z'), \quad z' \in G.$$

From the last equality and Lemma 3 of [10, p. 43], we have

$$a_k\big(T(g_r)\big) = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{T(g_r)(\psi(w))}{w^{k+1}} \, dw = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{\sum_{m=0}^{\infty} \alpha_m r^m \Phi_m(\psi(w))}{w^{k+1}} \, dw = \sum_{m=0}^{\infty} \alpha_m r^m \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{\Phi_m(\psi(w))}{w^{k+1}} \, dw = \alpha_k r^k$$

and therefore

$$a_k\big(T(g_r)\big) \to \alpha_k, \quad \text{as } r \to 1^-. \tag{12}$$

On the other hand, applying (3) and Hölder's inequality (2), we obtain

$$\begin{aligned}
\big|a_k\big(T(g_r)\big) - a_k\big(T(g)\big)\big| &= \left| \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{[T(g_r) - T(g)](\psi(w))}{w^{k+1}} \, dw \right| \\
&\leqslant \frac{1}{2\pi} \int_{\mathbb{T}} \big|\big[T(g_r) - T(g)\big]\big(\psi(w)\big)\big| |dw| = \frac{1}{2\pi} \int_{\Gamma} \big|\big[T(g_r) - T(g)\big](z)\big| \big|\varphi'(z)\big| |dz| \\
&\leqslant \frac{c_{14}}{2\pi} \int_{\Gamma} \big|\big[T(g_r) - T(g)\big](z)\big| |dz| = \frac{c_{14}}{2\pi} \int_{\Gamma} \big|\big[T(g_r) - T(g)\big](z)\big| \omega(z) \omega^{-1}(z) |dz| \\
&\leqslant \frac{c_{14}}{2\pi} \big\| \big(T(g_r) - T(g)\big)\omega(z) \big\|_{X(\Gamma)} \big\| \omega^{-1}(\cdot) \big\|_{X'(\Gamma)} \leqslant \frac{c_{15}}{2\pi} \big\| T(g_r) - T(g) \big\|_{X(\Gamma,\omega)},
\end{aligned}$$

because $\|\omega^{-1}(\cdot)\|_{X'(\Gamma)} < \infty$ by Theorem 2.1 of [21].

Using here the relation (11), we get

$$a_k\big(T(g_r)\big) \to a_k\big(T(g)\big), \quad \text{as } r \to 1^-,$$

and then by (12), $a_k(T(g)) = \alpha_k$ for $k = 0, 1, 2, \ldots$. If $T(g) = 0$, then $\alpha_k = a_k(T(g)) = 0$, $k = 0, 1, 2, \ldots$, and therefore $g = 0$. This means that the operator T is one-to-one.

Now we take a function $f \in E_X(G, \omega)$ and consider the function $f_0 = f \circ \psi \in X(\mathbb{T}, \omega_0)$. The Cauchy type integral

$$\frac{1}{2\pi i} \int_{\mathbb{T}} \frac{f_0(\tau)}{\tau - w} \, d\tau$$

represents analytic functions f_0^+ and f_0^- in $\mathbb{D}$ and $\mathbb{D}^-$, respectively. Since $\omega_0 \in A_{1/\alpha_X}(\mathbb{T}) \cap A_{1/\beta_X}(\mathbb{T})$, by Lemma 1, we have

$$f_0^+ \in E_X(\mathbb{D}, \omega_0) \quad \text{and} \quad f_0^- \in \widetilde{E}_X\big(\mathbb{D}^-, \omega_0\big),$$

and moreover

$$f_0(w) = f_0^+(w) - f_0^-(w) \tag{13}$$

a.e. on $\mathbb{T}$. Since $f_0^- \in E^1(\mathbb{D}^-)$ and $f_0^-(\infty) = 0$, we have

$$a_k = \frac{1}{2\pi i}\int\limits_{\mathbb{T}} \frac{f_0(w)}{w^{k+1}}\,dw = \frac{1}{2\pi i}\int\limits_{\mathbb{T}} \frac{f_0^+(w)}{w^{k+1}}\,dw - \frac{1}{2\pi i}\int\limits_{\mathbb{T}} \frac{f_0^-(w)}{w^{k+1}}\,dw = \frac{1}{2\pi i}\int\limits_{\mathbb{T}} \frac{f_0^+(w)}{w^{k+1}}\,dw,$$

which proves that the coefficients a_k, $k = 0, 1, 2, \ldots$, also become the Taylor coefficients of the function f_0^+ at the origin, i.e.,

$$f_0^+(w) = \sum_{k=0}^{\infty} a_k w^k, \quad w \in \mathbb{D},$$

and also

$$T\big(f_0^+\big) \backsim \sum_{k=0}^{\infty} a_k \Phi_k.$$

Hence the functions $T(f_0^+)$ and f have the same Faber coefficients a_k, $k = 0, 1, 2, \ldots$, and therefore $T(f_0^+) = f$. This proves that the operator T is onto. □

3. Proofs of main results

Proof of Theorem 1. We prove that the rational function

$$R_n(z, f) := \sum_{k=0}^{n} a_k \Phi_k(z) + \sum_{k=1}^{n} \tilde{a}_k F_k(1/z)$$

satisfies the required inequality of Theorem 1. This inequality is true if we can show that

$$\left\| f^-(z) + \sum_{k=1}^{n} \tilde{a}_k F_k(1/z) \right\|_{X(\Gamma,\omega)} \leqslant c_{16} \widetilde{\Omega}^r_{\Gamma,X,\omega}\big(f, 1/(n+1)\big) \tag{14}$$

and

$$\left\| f^+(z) - \sum_{k=0}^{n} a_k \Phi_k(z) \right\|_{X(\Gamma,\omega)} \leqslant c_{17} \Omega^r_{\Gamma,X,\omega}\big(f, 1/(n+1)\big), \tag{15}$$

because $f(z) = f^+(z) - f^-(z)$ a.e. on Γ.

First we prove (14). Let $f \in X(\Gamma, \omega)$. Then $f_1 \in X(\mathbb{T}, \omega_1)$, $f_0 \in X(\mathbb{T}, \omega_0)$. According to (13)

$$f(\varsigma) = f_0^+\big(\varphi(\varsigma)\big) - f_0^-\big(\varphi(\varsigma)\big) \tag{16}$$

a.e. on Γ. On the other hand, from Lemma 1, we find that

$$f_1(w) = f_1^+(w) - f_1^-(w),$$

which implies the inequality

$$f(\varsigma) = f_1^+\big(\varphi_1(\varsigma)\big) - f_1^-\big(\varphi_1(\varsigma)\big) \tag{17}$$

a.e. on Γ.

Let $z' \in G \setminus \{0\}$. Using (7) and (17), we have

$$\begin{aligned}
\sum_{k=1}^{n} \tilde{a}_k F_k(1/z') &= \sum_{k=1}^{n} \tilde{a}_k \varphi_1^k(z') - \frac{1}{2\pi i} \int_{\Gamma} \frac{\sum_{k=1}^{n} \tilde{a}_k \varphi_1^k(\varsigma)}{\varsigma - z'}\, d\varsigma \\
&= \sum_{k=1}^{n} \tilde{a}_k \varphi_1^k(z') - \frac{1}{2\pi i} \int_{\Gamma} \frac{\sum_{k=1}^{n} (\tilde{a}_k \varphi_1^k(\varsigma) - f_1^+(\varphi_1(\varsigma)))}{\varsigma - z'}\, d\varsigma - \frac{1}{2\pi i} \int_{\Gamma} \frac{f_1^-(\varphi_1(\varsigma))}{\varsigma - z'}\, d\varsigma - \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\varsigma)}{\varsigma - z'}\, d\varsigma \\
&= \sum_{k=1}^{n} \tilde{a}_k \varphi_1^k(z') - \frac{1}{2\pi i} \int_{\Gamma} \frac{\sum_{k=1}^{n} (\tilde{a}_k \varphi_1^k(\varsigma) - f_1^+(\varphi_1(\varsigma)))}{\varsigma - z'}\, d\varsigma - f_1^-\big(\varphi_1(z')\big) - f^-(z').
\end{aligned}$$

Hence, taking the nontangential limit $z' \to z \in \Gamma$, inside Γ, we obtain

$$\sum_{k=1}^{n} \tilde{a}_k F_k(1/z) = \sum_{k=1}^{n} \tilde{a}_k \varphi_1^k(z) - \frac{1}{2}\left(\sum_{k=1}^{n} \tilde{a}_k \varphi_1^k(z) - f_1^+\big(\varphi_1(z)\big) \right) - S_\Gamma \left[\sum_{k=1}^{n} \tilde{a}_k \varphi_1^k - \big(f_1^+ \circ \varphi_1\big) \right] - f_1^-\big(\varphi_1(z)\big) - f^+(z)$$

a.e. on Γ.

Using (5), (17), Minkowski's inequality and the boundedness of S_Γ, we get

$$\begin{aligned}
\left\| f^-(z) + \sum_{k=1}^{n} \tilde{a}_k F_k(1/z') \right\|_{X(\Gamma,\omega)} &= \left\| \frac{1}{2}\left(\sum_{k=1}^{n} \tilde{a}_k \varphi_1^k(z) - f_1^+\big(\varphi_1(z)\big) \right) - S_\Gamma \left[\sum_{k=1}^{n} \tilde{a}_k \varphi_1^k - \big(f_1^+ \circ \varphi_1\big) \right](z) \right\|_{X(\Gamma,\omega)} \\
&\leqslant c_{18} \left\| \sum_{k=1}^{n} \tilde{a}_k \varphi_1^k(z) - f_1^+\big(\varphi_1(z)\big) \right\|_{X(\Gamma,\omega)} \leqslant c_{19} \left\| f_1^+(w) - \sum_{k=1}^{n} \tilde{a}_k w^k \right\|_{X(\mathbb{T},\omega_1)}.
\end{aligned}$$

On the other hand, from the proof of Theorem 4 we know that the Faber–Laurent coefficients $\tilde{a}_k$ of the function f and the Taylor coefficients of the function f_1^+ at the origin are the same. Then taking Lemma 2 into account, we conclude that

$$\left\| f^- + \sum_{k=1}^{n} \tilde{a}_k F_k(1/z) \right\|_{X(\Gamma,\omega)} \leqslant c_{20} \Omega_{X,\omega_1}^r\big(f_1^+, 1/(n+1)\big) = c_{20} \widetilde{\Omega}_{\Gamma,X,\omega}^r\big(f, 1/(n+1)\big),$$

and (14) is proved.

The proof of relation (15) goes similarly; we use the relations (6) and (16) instead of (7) and (17), respectively. Hence (5), (14) and (15) complete the proof. □

Proof of Theorem 2. Let $f \in E_X(G,\omega)$. Then we have $T(f_0^+) = f$. Since by Theorem 4 the operator $T : E_X(\mathbb{D},\omega_0) \to E_X(G,\omega)$ is linear, bounded, one-to-one and onto, the operator $T^{-1} : E_X(G,\omega) \to E_X(\mathbb{D},\omega_0)$ is also linear and bounded. We take $p_n^* \in \mathcal{P}_n$ as the best approximating algebraic polynomial to f in $E_X(G,\omega)$, i.e.,

$$\mathcal{E}_n(f,G)_{X,\omega} = \|f - p_n^*\|_{X(\Gamma,\omega)}.$$

Then $T^{-1}(p_n^*) \in \mathcal{P}_n(\mathbb{D})$ and therefore

$$\begin{aligned}
\mathcal{E}_n\big(f_0^+\big)_{X,\omega_0} &\leqslant \big\|f_0^+ - T^{-1}\big(p_n^*\big)\big\|_{X(\mathbb{T},\omega_0)} = \big\|T^{-1}(f) - T^{-1}\big(p_n^*\big)\big\|_{X(\mathbb{T},\omega_0)} = \big\|T^{-1}\big(f - p_n^*\big)\big\|_{X(\mathbb{T},\omega_0)} \\
&\leqslant \big\|T^{-1}\big\| \|f - p_n^*\|_{X(\Gamma,\omega)} = \big\|T^{-1}\big\| \mathcal{E}_n(f,G)_{X,\omega},
\end{aligned} \tag{18}$$

because the operator T^{-1} is bounded.

On the other hand, from [13] we have

$$\Omega_{X,\omega_0}^r\big(f_0^+, 1/n\big) \leqslant \frac{c_{24}}{n^{2r}} \left\{ \mathcal{E}_0\big(f_0^+\big)_{X,\omega_0} + \sum_{k=1}^{n} k^{2r-1} \mathcal{E}_k\big(f_0^+\big)_{X,\omega_0} \right\}, \quad r = 1, 2, \ldots.$$

The last inequality and (18) imply that

$$\begin{aligned}
\Omega_{\Gamma,X,\omega}^r(f, 1/n) = \Omega_{X,\omega_0}^r\big(f_0^+, 1/n\big) &\leqslant \frac{c_{25}}{n^{2r}} \left\{ \mathcal{E}_0\big(f_0^+\big)_{X,\omega_0} + \sum_{k=1}^{n} k^{2r-1} \mathcal{E}_k\big(f_0^+\big)_{X,\omega_0} \right\} \\
&\leqslant \frac{c_{26}\|T^{-1}\|}{n^{2r}} \left\{ \mathcal{E}_0(f,G)_{X,\omega} + \sum_{k=1}^{n} k^{2r-1} \mathcal{E}_k(f,G)_{X,\omega} \right\}, \quad r = 1, 2, \ldots. \qquad \square
\end{aligned}$$

Proof of Theorem 3. Let $f \in \widetilde{E}_X(G^-, \omega)$. Then $\widetilde{T}(f_1^+) = f$. By Theorem 4 the operator $\widetilde{T}^{-1} : \widetilde{E}_X(G^-, \omega) \to E_X(\mathbb{D}, \omega_1)$ is linear and bounded. Let $r_n^* \in \mathcal{R}_n$ be a function such that

$$\widetilde{\mathcal{E}}_n(f)_{X,\omega} = \|f - r_n^*\|_{X(\Gamma,\omega)}.$$

Then $\widetilde{T}^{-1}(r_n^*) \in \mathcal{P}_n(\mathbb{D})$ and therefore

$$\begin{aligned}\mathcal{E}_n(f_1^+)_{X,\omega_1} &\leqslant \|f_1^+ - \widetilde{T}^{-1}(r_n^*)\|_{X(\mathbb{T},\omega_1)} = \|\widetilde{T}^{-1}(f) - \widetilde{T}^{-1}(r_n^*)\|_{X(\mathbb{T},\omega_1)} \\ &= \|\widetilde{T}^{-1}(f - r_n^*)\|_{X(\mathbb{T},\omega_1)} \leqslant \|\widetilde{T}^{-1}\| \|f - r_n^*\|_{X(\Gamma,\omega)} = \|\widetilde{T}^{-1}\| \widetilde{\mathcal{E}}_n(f)_{X,\omega}. \end{aligned} \tag{19}$$

It can be deduced from [13] that

$$\Omega^r_{X,\omega_1}(f_1^+, 1/n) \leqslant \frac{c_{27}}{n^{2r}} \left\{ \mathcal{E}_0(f_1^+)_{X,\omega_1} + \sum_{k=1}^{n} k^{2r-1} \mathcal{E}_k(f_1^+)_{X,\omega_1} \right\}, \quad r = 1, 2, \ldots.$$

From the last inequality and (19) we conclude that

$$\begin{aligned}\widetilde{\Omega}^r_{\Gamma,X,\omega}(f, 1/n) = \Omega^r_{X,\omega_1}(f_1^+, 1/n) &\leqslant \frac{c_{28}}{n^{2r}} \left\{ \mathcal{E}_0(f_1^+)_{X,\omega_1} + \sum_{k=1}^{n} k^{2r-1} \mathcal{E}_k(f_1^+)_{X,\omega_1} \right\} \\ &\leqslant \frac{c_{29} \|\widetilde{T}^{-1}\|}{n^{2r}} \left\{ \widetilde{\mathcal{E}}_0(f)_{X,\omega} + \sum_{k=1}^{n} k^{2r-1} \widetilde{\mathcal{E}}_k(f)_{X,\omega} \right\}, \quad r = 1, 2, \ldots. \qquad \square \end{aligned}$$

Acknowledgments

The authors are indebted to the referees for valuable comments and suggestions.

References

[1] R. Akgün, D.M. Israfilov, Approximation by interpolating polynomials in Smirnov–Orlicz class, J. Korean Math. Soc. 43 (2006) 412–424.
[2] S.Ya. Al'per, Approximation in the mean of analytic functions of class E^p, in: Investigations on the Modern Problems of the Function Theory of a Complex Variable, Gos. Izdat. Fiz.-Mat. Lit., Moscow, 1960, pp. 272–286 (in Russian).
[3] J.E. Andersson, On the degree of polynomial approximation in $E^p(D)$, J. Approx. Theory 19 (1977) 61–68.
[4] C. Benneth, R. Sharpley, Interpolation of Operators, Pure Appl. Math., vol. 129, Academic Press, Boston, 1988.
[5] D.W. Boyd, Spaces between a pair of reflexive Lebesgue spaces, Proc. Amer. Math. Soc. 18 (1967) 215–219.
[6] A. Böttcher, A.Yu. Karlovich, Carleson Curves, Muckenhoupt Weights, and Toeplitz Operators, Progr. Math., vol. 154, Birkhäuser, 1997.
[7] A. Çavuş, D.M. Israfilov, Approximation by Faber–Laurent rational functions in the mean of functions of the class $L_p(\Gamma)$ with $1 < p < \infty$, J. Approx. Theory Appl. 11 (1995) 105–118.
[8] P.L. Duren, Theory of H^p Spaces, Academic Press, 1970.
[9] E.M. Dynkin, Uniform approximation of functions in Jordan domains, Siberian Math. J. 7 (1977) 548–557.
[10] D. Gaier, Lectures on Complex Approximation, Birkhäuser, 1987.
[11] G.M. Goluzin, Geometric Theory of Functions of a Complex Variable, Transl. Math. Monogr., vol. 26, AMS, Providence, RI, 1969.
[12] A. Guven, D.M. Israfilov, Polynomial approximation in Smirnov–Orlicz classes, Comput. Methods Funct. Theory 2 (2002) 509–517.
[13] A. Guven, D.M. Israfilov, Approximation by trigonometric polynomials in weighted rearrangement invariant spaces, submitted for publication.
[14] I.I. Ibragimov, D.I. Mamedhanov, A constructive characterization of a certain class of functions, Dokl. Akad. Nauk SSSR 223 (1975) 35–37; Soviet Math. Dokl. 4 (1976) 820–823.
[15] D.M. Israfilov, Approximate properties of the generalized Faber series in an integral metric, Izv. Akad. Nauk Azerb. SSR Ser. Fiz.-Tekh. Math. Nauk 2 (1987) 10–14 (in Russian).
[16] D.M. Israfilov, Approximation by p-Faber polynomials in the weighted Smirnov class $E^p(G, \omega)$ and the Bieberbach polynomials, Constr. Approx. 17 (2001) 335–351.
[17] D.M. Israfilov, Approximation by p-Faber–Laurent rational functions in the weighted Lebesgue spaces, Czechoslovak Math. J. 54 (2004) 751–765.
[18] D.M. Israfilov, A. Guven, Approximation in weighted Smirnov classes, East J. Approx. 11 (2005) 91–102.
[19] D.M. Israfilov, B. Oktay, R. Akgun, Approximation in Smirnov–Orlicz classes, Glas. Mat. 40 (2005) 87–102.
[20] A.Yu. Karlovich, The essential norm of the Cauchy singular integral operator in weighted rearrangement invariant spaces, Integral Equations Operator Theory 38 (2000) 28–50.
[21] A.Yu. Karlovich, Algebras of singular integral operators with PC coefficients in rearrangement-invariant spaces with Muckenhoupt weights, J. Operator Theory 47 (2002) 303–323.
[22] A.Yu. Karlovich, Fredholmness of singular integral operators with piecewise continuous coefficients on weighted Banach function spaces, J. Integral Equations Appl. 15 (2003) 263–320.
[23] V.M. Kokilashvili, On analytic functions of Smirnov–Orlicz classes, Studia Math. 31 (1968) 43–59.
[24] V.M. Kokilashvili, A direct theorem on mean approximation of analytic functions by polynomials, Soviet Math. Dokl. 10 (1969) 411–414.
[25] J. Lindenstrauss, L. Tzafriri, Classical Banach Spaces I and II, Classics Math., Springer-Verlag, 1996.
[26] B. Muckenhoupt, Weighted norm inequalities for the Hardy maximal function, Trans. Amer. Math. Soc. 167 (1972) 207–226.
[27] W. Rudin, Real and Complex Analysis, third ed., McGraw–Hill, 1987.
[28] P.K. Suetin, Series of Faber Polynomials, vol. 1, Gordon and Breach, Reading, 1998.
[29] J.L. Walsh, H.G. Russel, Integrated continuity conditions and degree of approximation by polynomials or by bounded analytic functions, Trans. Amer. Math. Soc. 92 (1959) 355–370.
[30] S.E. Warschawski, Über das ranverhalten der Ableitung der Abildungsfunktion bei konformer Abbildung, Math. Z. 35 (1932) 321–456.