Physics Letters A 372 (2008) 5968–5972

Contents lists available at ScienceDirect

Physics Letters A

www.elsevier.com/locate/pla

Fractional diffusion-wave problem in cylindrical coordinates

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ARTICLE INFO

Article history:
Received 12 May 2008
Received in revised form 22 July 2008
Accepted 23 July 2008
Available online 31 July 2008
Communicated by R. Wu

Keywords:
Fractional derivative
Fractional diffusion-wave system
Axial symmetry
Grünwald–Letnikov approach
Cylindrical coordinates

ABSTRACT

In this Letter, we present analytical and numerical solutions for an axis-symmetric diffusion-wave equation. For problem formulation, the fractional time derivative is described in the sense of Riemann–Liouville. The analytical solution of the problem is determined by using the method of separation of variables. Eigenfunctions whose linear combination constitute the closed form of the solution are obtained. For numerical computation, the fractional derivative is approximated using the Grünwald–Letnikov scheme. Simulation results are given for different values of order of fractional derivative. We indicate the effectiveness of numerical scheme by comparing the numerical and the analytical results for $\alpha = 1$ which represents the order of derivative.

1. Introduction

The awareness of the Fractional Diffusion-Wave Equation (FDWE) has grown during the last decades. These equations provide more accurate models of systems and processes under consideration. For this reason, there has been an increasing interest to investigate, in general, the response of the systems, and in particular, the analytical and numerical solutions of FDWE.

A FDWE is a linear partial integro-differential equation obtained from the classical diffusion or wave equation by replacing the first or second-order time derivative by a fractional derivative of order $\alpha > 0$, see Mainardi [1].

Mainardi [2] presented the fundamental solutions of the basic Cauchy and Signalling problems for the evolution of FDWE. The solutions of central-symmetric signalling, source and Cauchy problems for fractional diffusion equation in a spatially three-dimensional sphere were studied by Povstenko [3]. Wyss [4] derived the solution of the Cauchy and Signalling problems in terms of H-functions using the Mellin transform.

Agrawal [5,6] obtained the fundamental solutions of a FDWE which contains a fourth order space derivative and a fractional order time derivative. The solution of a FDWE defined in a bounded space domain was also considered by Agrawal [7]. Mainardi [8] obtained fundamental solutions for a FDWE and the solutions for fractional relaxation oscillations by using the Laplace transform method. The Green's function and propagator functions in multi-dimensions which are obtained for the solution of a general initial value problem for the time-fractional diffusion-wave equation with source term and for the anisotropic space–time fractional diffusion equation were researched by Hanyga [9,10]. Mainardi, Luchko and Pagnini [11] dealt with the fundamental solution of the space–time fractional diffusion equation.

Agrawal [12] presented stochastic analysis of FDWEs defined in one dimension whereas very little work has been done in the area of stochastic analysis of fractional order engineering systems.

In this Letter, the analytical and numerical solutions of an axis-symmetric FDWE in cylindrical coordinates are studied. More recently, the solution of an axis-symmetric fractional diffusion-wave equation in polar coordinates has been presented in [13]. El-Shahed [14] considered the motion of an electrically conducting, incompressible and non-Newtonian fluid in the presence of a magnetic field acting along the radius of a circular pipe. Furthermore, El-Shahed selected a cylindrical polar coordinate system with z-axis in the direction of motion and considered the flow as axially symmetric. Several axial-symmetric problems for a plane in cylindrical coordinates and central-symmetric problems for an infinite space in spherical coordinates were presented in [15–17]. Radial diffusion in a cylinder of radius R was considered by Narahari Achar and Hanneken [18]. Povstenko [19] developed the results of Narahari Achar and Hanneken. The main problem considered in [19] is similar to our work. However, the formulation of problem here differs with [19] in some respects. Firstly, Povstenko [19] formulates the problem by using polar coordinates in terms of Caputo fractional derivative and finds only the closed form analytic solution, whereas this Letter considers the problem with cylindrical coordinates in Riemann–Liouville (RL) sense, and also presents nu-

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0375-9601/$ – see front matter © 2008 Published by Elsevier B.V.
doi:10.1016/j.physleta.2008.07.054

merical solutions by using Grünwald–Letnikov (GL) approach. The comparison of analytic and numeric solutions is analyzed by using simulation results. Therefore, the effectiveness of GL numerical approach for such a kind of problem is obtained. Secondly, Povstenko [19] obtains the plots for the solution with respect to distance and changes the values of fractional order of derivative. However, we create two and three dimension figures and also obtain the solutions with respect to not only the order of fractional derivative α but also step size h (the length of subintervals which is mentioned in GL numerical algorithm section), the number of Bessel function's zeros, time and length of cylinder. Therefore, we analyze the contribution of number of Bessel function's zeros to the solution of the problem and clarify the dependency of the solution to the step size h. In addition, we explain the behaviour of the system when α is changed.

This Letter is organized as follows. In Section 2, some basic definitions used for formulation of the problem are reviewed. The axis-symmetric FDW problem in cylindrical coordinates is defined and its analytical solution is obtained in Section 3. Section 4 explains the numerical approach. The analytical and the numerical simulation results are compared in Section 5. Finally, Section 6 presents conclusions.

2. Mathematical preliminaries

We begin with the definitions and identities which are necessary for our formulation. Here, we give Riemann–Liouville Fractional Derivative (RLFD) definition of a function $f(t)$ for an arbitrary fractional order $\alpha > 0$:

$$ {}_aD_t^\alpha f(t) = \begin{cases} (\frac{d}{dt})^n f(t), & \alpha = n, \\ \frac{1}{\Gamma(n-\alpha)} (\frac{d}{dt})^n \int_a^t (t-\tau)^{n-\alpha-1} f(\tau)\, d\tau, & n-1 \leqslant \alpha < n, \end{cases} \tag{1} $$

where $n \in \mathbb{Z}$ and $\Gamma(.)$ represents the well-known Euler's gamma function. In pure mathematics, RLFD is more commonly used than Caputo fractional derivative. Two definitions have some differences from the viewpoint of their application in mathematics, physics and engineering. However, it is well known that these two definitions coincide for zero initial condition assumptions. We prefer RLFD to formulate the problem. The main reason of our preference is the relation between RL and GL definitions. Because, for a wide class of functions RL and GL definitions are equivalent. This class of functions is very important for applications, because the character of the majority of dynamical processes is smooth enough and does not allow discontinuities [20]. For this propose, we use RL definition during the analytic solution of our problem and then turn to GL definition for numerical solution.

The formula of the Laplace transform method of the Mittag–Leffler function in two-parameters is the basis of the most effective and easy analytic methods for the solution of the fractional differential equations. A two-parameter Mittag–Leffler function is defined in [23] as:

$$ E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)} \quad (\alpha, \beta > 0). \tag{2} $$

In this Letter, the response of the FDW system is described as a linear combination of the eigenfunctions which are derived by using the method of separation of variables. We obtain eigenfunctions as the zero-order Bessel function of the first kind given in [24] as:

$$ J_0(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m}(m!)^2}. \tag{3} $$

3. The axis-symmetric FDW problem

In this section, we present an axis-symmetric FDWE in terms of the RLFD and use cylindrical coordinates to formulate the problem. However, several definitions of a fractional derivative can also be applied such as Grünwald–Letnikov, Weyl, Caputo, Marchaud and Riesz fractional derivatives [20–22].

An axis-symmetric FDWE can be defined as follows:

$$ \frac{\partial^\alpha w}{\partial t^\alpha} = c^2\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial z^2}\right) + u(r,z,t), \tag{4} $$

where r and z are cylindrical coordinates, c is a constant which depends on the physical properties of the system and $u(r,z,t)$ is the external source term.

Here, we consider $0 < \alpha \leqslant 2$, whereas α can be any positive number. We further consider the following boundary and initial conditions:

$$ \begin{cases} w = 0 & (z = 0,\ 0 < r < R), \\ w = 0 & (r = R,\ 0 < z < L), \\ w = 0 & (z = L,\ 0 < r < R), \\ w \text{ is finite} & (0 < r < R,\ 0 < z < L), \end{cases} \tag{5} $$

and

$$ w(r,z,0) = \frac{\partial w}{\partial r}(r,z,0) = \frac{\partial w}{\partial z}(r,z,0) = 0, \tag{6} $$

where R is the radius and L is length of the domain.

To find the response of this system, we use the method of separation of variables and obtain the eigenfunctions

$$ \Phi_{ij}(r,z) = J_0\left(\frac{\psi_j}{R}r\right)\sin\left(\frac{i\pi}{L}z\right), \quad i,j = 1,2,\ldots,\infty, \tag{7} $$

where $J_0(.)$ is the zero-order Bessel function of the first kind and ψ_j, $j = 1,2,\ldots,\infty$, are the positive zeros of the equation

$$ J_0(\psi_j) = 0. \tag{8} $$

We assume the solution of Eq. (4) as the following series

$$ w(r,z,t) = \sum_{i,j=1}^{\infty} J_0\left(\frac{\psi_j}{R}r\right)\sin\left(\frac{i\pi}{L}z\right)q_{ij}(t). \tag{9} $$

By substituting Eq. (9) into Eq. (4), multiplying both sides of the resulting equation by $rJ_0(\frac{\psi_k}{R}r)$ and integrating the result from 0 to R, respectively, we obtain

$$ \frac{d^\alpha q_{ij}(t)}{dt^\alpha} = -c^2\left\{\left(\frac{\psi_j}{R}\right)^2 + \left(\frac{i\pi}{L}\right)^2\right\}q_{ij}(t) + f_{ij}(t), \quad i,j = 1,2,\ldots,\infty, \tag{10} $$

with initial conditions

$$ q_{ij}(0) = \dot{q}_{ij}(0) = 0 \tag{11} $$

and

$$ f_{ij}(t) = \frac{2}{R^2 J_1^2(\psi_j/R)\sin((i\pi/L)z)} \int_0^R rJ_0\left(\frac{\psi_j}{R}r\right)u(r,z,t)\,dr, \tag{12} $$

where $J_1(.)$ is the first-order Bessel function of the first kind. The second condition in Eq. (11) is considered when $\alpha > 1$.

By applying the Laplace transform to Eq. (10), using Eq. (11) and then taking the inverse Laplace transform, we get

$$ q_{ij}(t) = \int_0^t Q_{ij}(t-\tau)f_{ij}(\tau)\,d\tau, \tag{13} $$

where

$$Q_{ij}(t) = L^{-1}\left\{\frac{1}{s^{\alpha} + c^2[(\psi_j/R)^2 + (i\pi/L)^2]}\right\}, \tag{14}$$

is the fractional Green's function, which can be written in the closed form as

$$Q_{ij}(t) = t^{\alpha-1}E_{\alpha,\alpha}\left\{-c^2\left[\left(\frac{\psi_j}{R}\right)^2 + \left(\frac{i\pi}{L}\right)^2\right]t^{\alpha}\right\}. \tag{15}$$

Here, L^{-1} represents the inverse Laplace transform operator and $E_{\alpha,\beta}$ is the two-parameter Mittag–Leffler function. Substituting Eq. (13) into Eq. (9), we take the closed form solution of the axis-symmetric FDWE defined by Eqs. (4)–(6) as

$$w(r,z,t) = \sum_{i,j=1}^{\infty} J_0\left(\frac{\psi_j}{R}r\right)\sin\left(\frac{i\pi}{L}z\right)\int_0^t Q_{ij}(t-\tau)f_{ij}(\tau)\,d\tau. \tag{16}$$

Therefore, $w(r,z,t)$ can be obtained provided $u(r,z,t)$ is known.

In the next section, we explain the Grünwald–Letnikov algorithm to obtain the numerical solution of the fractional diffusion-wave equations which are defined by Eqs. (10) and (11).

4. Grünwald–Letnikov numerical algorithm

The numerical algorithm given here relies on the Grünwald–Letnikov approximation of the fractional derivative. We simply rewrite fractional differential equations and initial conditions defined in Eqs. (10) and (11) as follows

$$\frac{d^{\alpha}q(t)}{dt^{\alpha}} = -aq(t) + f(t) \tag{17}$$

and

$$q(0) = \dot{q}(0) = 0, \tag{18}$$

where $a = c^2\{(\psi_j/R)^2 + (i\pi/L)^2\}$. Note that, we drop the subscript i and j for simplicity.

Then, the algorithm can be explained in 4 steps:

1. Divide the time interval into subintervals of equal size h (also called step size).
2. Approximation of $\frac{d^{\alpha}q(t)}{dt^{\alpha}}$ at node m using the Grünwald–Letnikov formula as [20]

$$\frac{d^{\alpha}q(t)}{dt^{\alpha}} = \frac{1}{h^{\alpha}}\sum_{j=0}^{m} w_j^{(\alpha)}q_{m-j}, \tag{19}$$

where q_j is the numerically computed value of q at node j, and $w_j^{(\alpha)}$ are the coefficients defined as [20]

$$w_0^{(\alpha)} = 1, \qquad w_j^{(\alpha)} = \left(1 - \frac{\alpha+1}{j}\right)w_{j-1}^{(\alpha)}, \quad j = 1, 2, \ldots. \tag{20}$$

3. Using approximation (19), derive the following algorithm for obtaining the numerical solution:

$$h^{-\alpha}\sum_{j=0}^{m} w_j^{(\alpha)}q_{m-j} + aq_m = f_m \quad (m = 1, 2, \ldots),$$
$$q_0 = 0, \tag{21}$$

$$q_m = -ah^{\alpha}q_{m-1} - \sum_{j=1}^{m} w_j^{(\alpha)}q_{m-j} + h^{\alpha}f_m,$$
$$(m = n, n+1, \ldots), \tag{22}$$

where $n-1 < \alpha \leqslant n$, $n \in \mathbb{Z}$, and $q_k = 0$ $(k = 1, 2, \ldots, n-1)$.
4. Use Eqs. (21) and (22) to find q_m at all nodes m.

Therefore, we obtain the numerical solutions of the problem by applying these steps to fractional differential equation part of the system.

5. Numerical results

In this section, we give some simulation results for the axis-symmetric diffusion-wave system described by Eqs. (4) to (6) for $0 < \alpha \leqslant 2$, $z \in [0, L]$, $r \in [0, R]$ and $t > 0$. To obtain simulation

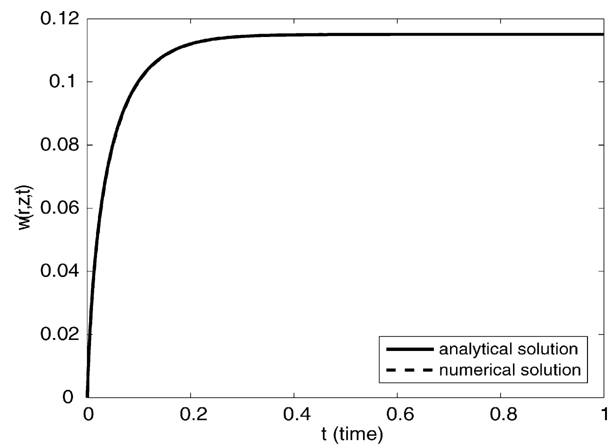

Fig. 1. Comparison of the analytical and the numerical solution of $w(r,z,t)$ for $\alpha = 1$, $r = 0.5$, $z = 0.3$, $M = 5$ and $h = 0.001$.

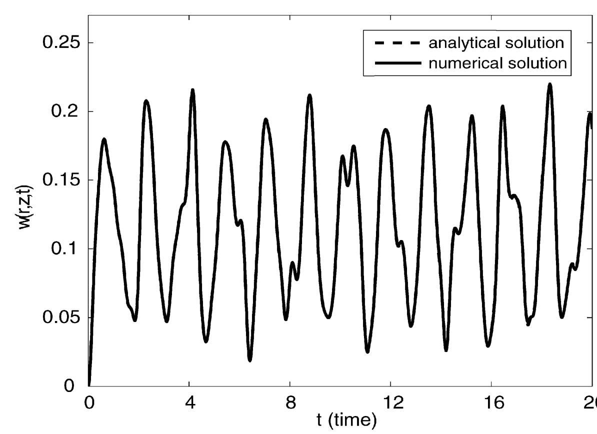

Fig. 2. Comparison of the analytical and the numerical solution of $w(r,z,t)$ for $\alpha = 2$, $r = 0.5$, $z = 0.3$, $M = 5$ and $h = 0.01$.

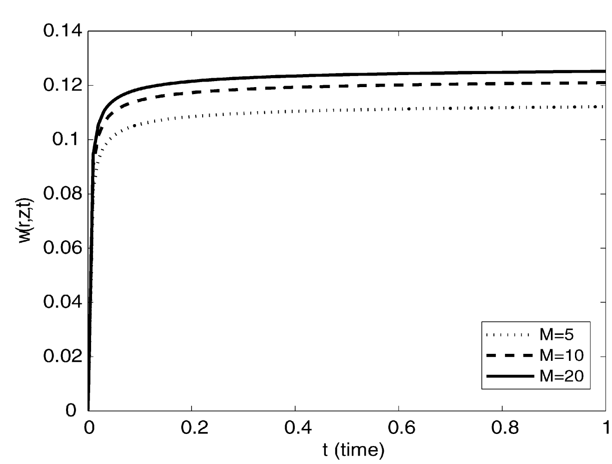

Fig. 3. The solution of $w(r,z,t)$ for $\alpha = 0.5$, $r = 0.5$, $z = 0.3$ and $M = 5, 10, 20$.

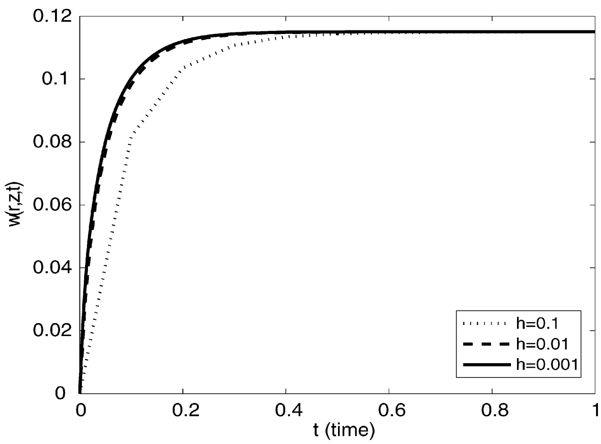

Fig. 4. Evolution of $w(r,z,t)$ for $\alpha = 1$, $r = 0.5$, $z = 0.3$, $M = 5$ and $h = 0.1, 0.01, 0.001$.

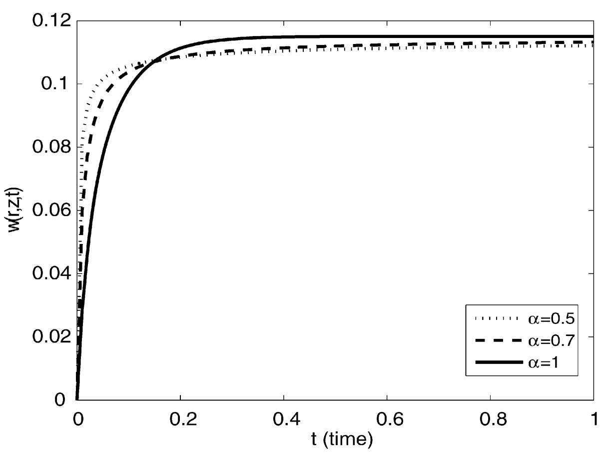

Fig. 5. Evolution of $w(r,z,t)$ for $r = 0.5$, $z = 0.3$, $h = 0.01$, $M = 5$ and $\alpha = 0.5, 0.7, 1$.

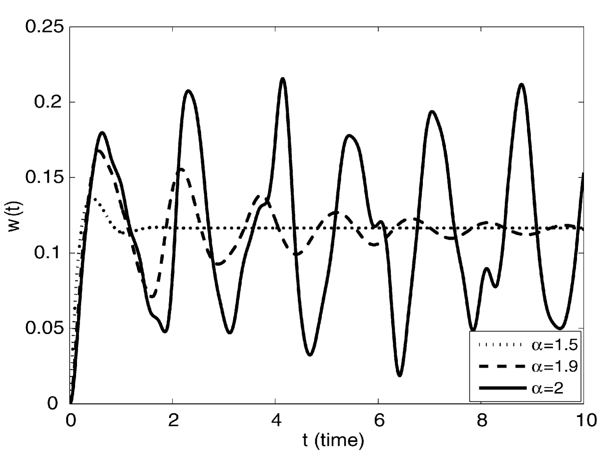

Fig. 6. Evolution of $w(r,z,t)$ for $r = 0.5$, $z = 0.3$, $h = 0.01$, $M = 5$ and $\alpha = 1.5, 1.9, 2$.

results, we take $R = L = c = u(r,z,t) = 1$ and change M, h and α variables. Here, M and h represent the number of the zeros of Bessel's function and step size, respectively. We first, compute $f_{ij}(t)$ using Eq. (12) for $i, j = 1, 2, \ldots, M$ and then solve Eqs. (10) and (11) using Grünwald–Letnikov approximation which is discussed in Section 3. Finally, we obtain analytical solutions of the system when $\alpha = 1$ and $\alpha = 2$ for comparison purpose. The series described the response of the system in Eq. (16) is truncated after M terms. Consequently, we explain the results of this work as follows:

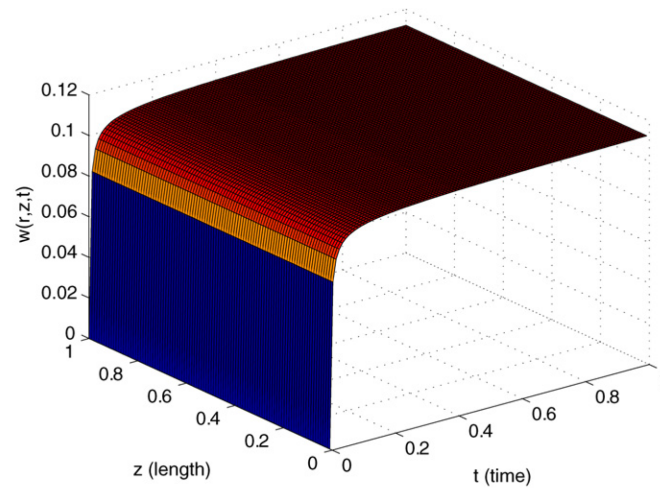

Fig. 7. Three-dimensional figure of $w(r,z,t)$ for $\alpha = 0.5$, $h = 0.01$, $r = 0.5$ and $M = 5$.

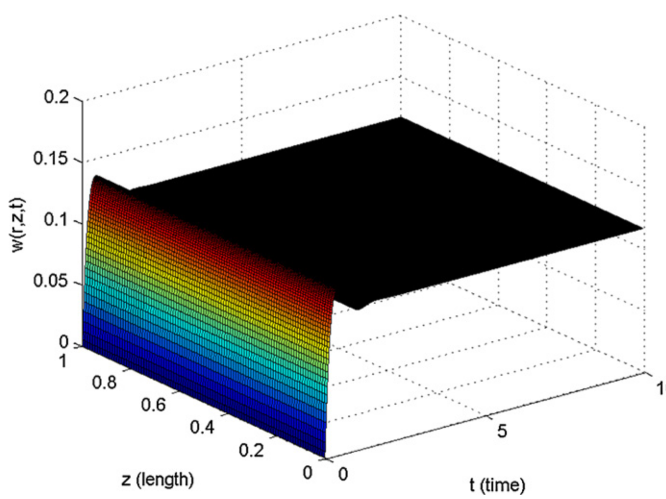

Fig. 8. Three-dimensional figure of $w(r,z,t)$ for $\alpha = 1.5$, $h = 0.01$, $r = 0.5$ and $M = 5$.

Figs. 1 and 2 are obtained to compare the analytical and the numerical solutions for $\alpha = 1$ and $\alpha = 2$, respectively. In this work, we take $r = 0.5$, $z = 0.3$, $M = 5$. For $\alpha = 1$, we take $h = 0.001$ and for $\alpha = 2$, we take $h = 0.01$. For both cases, analytical and numerical results overlap. This shows that the numerical algorithm is stable. Note that, Fig. 1 shows that diffusion reaches a steady state position in a very short time. However, the system shows an undamped vibrational character in Fig. 2.

Fig. 3 shows the response of the system for $\alpha = 0.5$, $r = 0.5$, $z = 0.3$, $h = 0.01$ and different values of $M = 5, 10, 20$. While M values are more than 20, the obtained results converge to the exact solution. Therefore, we take $M = 20$. Fig. 4 gives the response of $w(r,z,t)$ for $\alpha = 1$, $r = 0.5$, $z = 0.3$, $M = 5$ and different $h = 0.1, 0.01, 0.001$ values. The solutions converge as the step size is reduced.

Fig. 5 shows $w(r,z,t)$ for $\alpha = 0.5, 0.7, 1$. It demonstrates that this process changes from sub-diffusion to diffusion. Fig. 6 shows also the response of the system for $\alpha = 1.5, 1.9, 2$, and process changes from diffusion-wave to wave. In both figures, not only α approaches to integer values but also the system approaches to the integer order system.

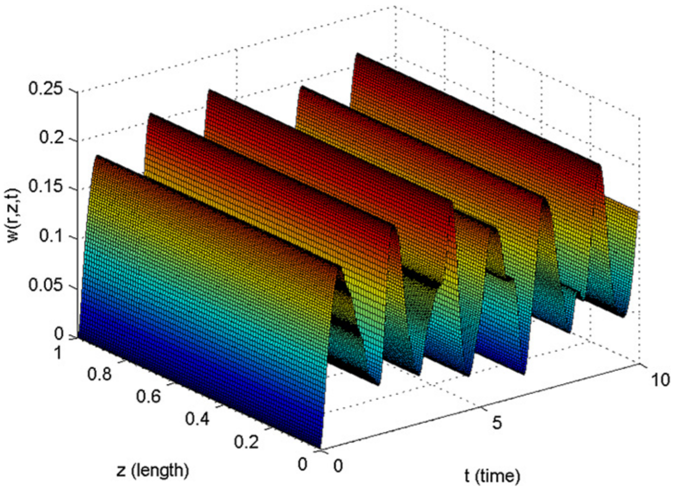

Fig. 9. Three-dimensional figure of $w(r,z,t)$ for $\alpha = 2$, $h = 0.01$, $r = 0.5$ and $M = 5$.

Figs. 7, 8 and 9 show the whole field response of the system for $\alpha = 0.5$, 1.5, and 2, respectively. In these figures, we plot $w(r,z,t)$ for z, t variables and fixed r. We use $M = 5$ and $h = 0.01$ values for these simulations. These figures show that the behavior of the system changes as α varies from 0.5 to 2.

6. Conclusions

The solution of an axis-symmetric fractional diffusion-wave problem defined in cylindrical coordinates was researched. Fractional derivative was defined in the sense of Riemann–Liouville. The method of separation of variables was used to find the closed form solution. Grünwald–Letnikov numerical approach was also used to obtain the numerical solutions of the problem. Simulation results were given for comparison of the numerical and the analytical solutions and it was showed that both solutions overlap for $\alpha = 1$ and 2. Simulation results were presented for different number of step size, zeros of the J_0 Bessel function and order of fractional derivative.

Acknowledgements

The author would like to thanks to the reviewers for carefully reading the manuscript and providing useful comments which led to more complete presentation.

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