

## GENERALIZED SASAKIAN SPACE FORMS WITH SEMI-SYMMETRIC METRIC CONNECTIONS

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**Abstract.** The aim of the present paper is to introduce generalized Sasakian space forms endowed with semi-symmetric metric connections. We obtain the existence theorem of a generalized Sasakian space form with semi-symmetric metric connection and we give some examples by using warped products endowed with semi-symmetric metric connection.

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**Key words:** generalized Sasakian space form, warped product, semi-symmetric metric connection.

### 1. Introduction

The idea of a semi-symmetric linear connection in a differentiable manifold was introduced by FRIEDMANN and SCHOUTEN in [4]. Later, HAYDEN [5] introduced the idea of a metric connection with torsion in a Riemannian manifold. In [12], YANO studied semi-symmetric metric connection in a Riemannian manifold.

Furthermore, in [1], BLAIR, CARRIAZO and ALEGRE introduced the notion of a generalized Sasakian space form and proved some of its basic properties. Many examples of these manifolds were presented by using some different geometric techniques such as Riemannian submersions, warped products or conformal and related transformations. New results on generalized complex space forms were also obtained.

In [9], the present authors studied a warped product manifold endowed with a semi-symmetric metric connection and found relations between curvature tensors, Ricci tensors and scalar curvatures of the warped product manifold with this connection.

Motivated by the above studies, in the present study, we consider generalized Sasakian space forms admitting semi-symmetric metric connections. We obtain the existence theorem of a generalized Sasakian space form with a semi symmetric metric connection and give some new examples by the use of warped products.

The paper is organized as follows: In section 2, we give a brief introduction on semi-symmetric metric connections. In section 3, the definition of a generalized Sasakian space form is given and we introduce generalized Sasakian space forms endowed with a semi-symmetric metric connections. Some obstructions about a generalized Sasakian space form endowed with a semi-symmetric metric connection are given. In the last section, the existence theorem of a generalized Sasakian space form with a semi-symmetric metric connection is given by using warped product  $\mathbb{R} \times_f N$ , where  $N$  is a generalized complex space form. Moreover, in this section we obtain some examples of generalized Sasakian space forms with non-constant functions endowed with a semi-symmetric metric connection.

## 2. Semi-symmetric metric connection

Let  $M$  be an  $n$ -dimensional Riemannian manifold with Riemannian metric  $g$ . A linear connection  $\overset{\circ}{\nabla}$  on a Riemannian manifold  $M$  is called a *semi-symmetric connection* if the torsion tensor  $T$  of the connection  $\overset{\circ}{\nabla}$

$$(1) \quad T(X, Y) = \overset{\circ}{\nabla}_X Y - \overset{\circ}{\nabla}_Y X - [X, Y]$$

satisfies

$$(2) \quad T(X, Y) = \eta(Y)X - \eta(X)Y,$$

where  $\eta$  is a 1-form associated with the vector field  $\xi$  on  $M$  defined by

$$(3) \quad \eta(X) = g(X, \xi).$$

$\overset{\circ}{\nabla}$  is called a *semi-symmetric metric connection* if it satisfies  $\overset{\circ}{\nabla}g = 0$ . If  $\nabla$  is the Levi-Civita connection of a Riemannian manifold  $M$ , a semi-symmetric

metric connection  $\overset{\circ}{\nabla}$  is given by

$$(4) \quad \overset{\circ}{\nabla}_X Y = \nabla_X Y + \eta(Y)X - g(X, Y)\xi,$$

(see [12]).

Let  $R$  and  $\overset{\circ}{R}$  be curvature tensors of  $\nabla$  and  $\overset{\circ}{\nabla}$  of a Riemannian manifold  $M$ , respectively. Then  $R$  and  $\overset{\circ}{R}$  are related by

$$(5) \quad \begin{aligned} \overset{\circ}{R}(X, Y)Z &= R(X, Y)Z - \alpha(Y, Z)X \\ &\quad + \alpha(X, Z)Y - g(Y, Z)AX + g(X, Z)AY, \end{aligned}$$

for all vector fields  $X, Y, Z$  on  $M$ , where  $\alpha$  is the  $(0, 2)$ -tensor field defined by  $\alpha(X, Y) = (\nabla_X \eta)Y - \eta(X)\eta(Y) + \frac{1}{2}\eta(\xi)g(X, Y)$  and  $g(AX, Y) = \alpha(X, Y)$ , ([12]).

### 3. Generalized Sasakian-space forms

Let  $M$  be an  $n$ -dimensional almost contact metric manifold [3] with an almost contact metric structure  $(\varphi, \xi, \eta, g)$  consisting of a  $(1, 1)$  tensor field  $\varphi$ , a vector field  $\xi$ , a 1-form  $\eta$  and a Riemannian metric  $g$  on  $M$  satisfying

$$(6) \quad \varphi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \varphi\xi = 0, \quad \eta \circ \varphi = 0,$$

$$(7) \quad g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \quad g(X, \xi) = \eta(X),$$

for all vector fields  $X, Y$  on  $M$ . Such a manifold is said to be a *contact metric manifold* if  $d\eta = \Phi$ , where  $\Phi(X, Y) = g(X, \varphi Y)$  is called the *fundamental 2-form* of  $M$  (see [3]).

On the other hand, the almost contact metric structure of  $M$  is said to be *normal* if

$$[\varphi, \varphi](X, Y) = -2d\eta(X, Y)\xi,$$

for any vector fields  $X, Y$  on  $M$ , where  $[\varphi, \varphi]$  denotes the Nijenhuis torsion of  $\varphi$ , given by

$$[\varphi, \varphi](X, Y) = \varphi^2[X, Y] + [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y].$$

A normal contact metric manifold is called a *Sasakian manifold* ([3]). It is well-known that an almost contact metric manifold is Sasakian if and only if

$$(8) \quad (\nabla_X \varphi)Y = g(X, Y)\xi - \eta(Y)X.$$

Moreover, the curvature tensor  $R$  of a Sasakian manifold satisfies

$$(9) \quad R(X, Y)\xi = \eta(Y)X - \eta(X)Y.$$

An almost contact metric manifold  $M$  is called a *trans-Sasakian manifold* ([8]) if there exist two functions  $\alpha$  and  $\beta$  on  $M$  such that

$$(10) \quad (\nabla_X \varphi)Y = \alpha[g(X, Y)\xi - \eta(Y)X] + \beta[g(\varphi X, Y)\xi - \eta(Y)\varphi X],$$

for any vector fields  $X, Y$  on  $M$ . From (10), it follows that

$$(11) \quad \nabla_X \xi = -\alpha\varphi X + \beta[X - \eta(X)\xi].$$

If  $\beta = 0$  (resp.  $\alpha = 0$ ), then  $M$  is said to be an  $\alpha$ -*Sasakian manifold* (resp.  $\beta$ -*Kenmotsu manifold*). Sasakian manifolds (resp. Kenmotsu manifolds [6]) appear as examples of  $\alpha$ -Sasakian manifolds ( $\beta$ -Kenmotsu manifolds), with  $\alpha = 1$  (resp.  $\beta = 1$ ).

Another kind of trans-Sasakian manifolds is that of *cosymplectic manifolds*, obtained for  $\alpha = \beta = 0$ . From (11), for a cosymplectic manifold it follows that  $\nabla_X \xi = 0$ , which implies that  $\xi$  is a Killing vector field for a cosymplectic manifold [2].

For an almost contact metric manifold  $M$ , a  $\varphi$ -*section* of  $M$  at  $p \in M$  is a section  $\pi \subseteq T_p M$  spanned by a unit vector  $X_p$  orthogonal to  $\xi_p$  and  $\varphi X_p$ . The  $\varphi$ -*sectional curvature* of  $\pi$  is defined by  $K(X \wedge \varphi X) = R(X, \varphi X, \varphi X, X)$ . A Sasakian manifold with constant  $\varphi$ -sectional curvature  $c$  is called a *Sasakian space form*. Similarly, a Kenmotsu manifold with constant  $\varphi$ -sectional curvature  $c$  is called a *Kenmotsu space form*. A cosymplectic manifold with constant  $\varphi$ -sectional curvature  $c$  is called a *cosymplectic space form*.

Given an almost contact metric manifold  $M$  with an almost contact metric structure  $(\varphi, \xi, \eta, g)$ ,  $M$  is called a *generalized Sasakian space form* if there exist three functions  $f_1, f_2$  and  $f_3$  on  $M$  such that

$$(12) \quad \begin{aligned} R(X, Y)Z &= f_1\{g(Y, Z)X - g(X, Z)Y\} \\ &+ f_2\{g(X, \varphi Z)\varphi Y - g(Y, \varphi Z)\varphi X + 2g(X, \varphi Y)\varphi Z\} \\ &+ f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\ &+ g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi\}, \end{aligned}$$

for any vector fields  $X, Y, Z$  on  $M$ , where  $R$  denotes the curvature tensor of  $M$ . If  $M$  is a Sasakian space form then  $f_1 = \frac{c+3}{4}$ ,  $f_2 = f_3 = \frac{c-1}{4}$ , if  $M$  is a

Kenmotsu space form then  $f_1 = \frac{c-3}{4}$ ,  $f_2 = f_3 = \frac{c+1}{4}$ , if  $M$  is a cosymplectic space form then  $f_1 = f_2 = f_3 = \frac{c}{4}$ .

Let  $\overset{\circ}{\nabla}$  be the semi-symmetric metric connection on an almost contact metric manifold  $M$  with closed 1-form  $\eta$ . We define that  $M$  is a *generalized Sasakian space form with semi-symmetric metric connection* if there exist three functions  $\tilde{f}_1, \tilde{f}_2$  and  $\tilde{f}_3$  on  $M$  such that

$$\begin{aligned} \overset{\circ}{R}(X, Y)Z &= \tilde{f}_1\{g(Y, Z)X - g(X, Z)Y\} \\ &+ \tilde{f}_2\{g(X, \varphi Z)\varphi Y - g(Y, \varphi Z)\varphi X + 2g(X, \varphi Y)\varphi Z\} \\ &+ \tilde{f}_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi\}, \end{aligned}$$

for any vector fields  $X, Y, Z$  on  $M$ , where  $\overset{\circ}{R}$  denotes the curvature tensor of  $M$  with respect to semi-symmetric metric connection  $\overset{\circ}{\nabla}$ .

Now we can state the following examples:

**Example 3.1.** A cosymplectic space form with the semi-symmetric metric connection, is a generalized Sasakian space form with the semi-symmetric metric connection such that  $\tilde{f}_1 = \tilde{f}_3 = \frac{c-4}{4}$  and  $\tilde{f}_2 = \frac{c}{4}$ .

**Example 3.2.** A Kenmotsu space form with the semi-symmetric metric connection is a generalized Sasakian space form with the semi-symmetric metric connection such that  $\tilde{f}_1 = \frac{c-15}{4}$ ,  $\tilde{f}_2 = \frac{c+1}{4}$  and  $\tilde{f}_3 = \frac{c-7}{4}$ .

**Remark 3.3.** A Sasakian space form with the semi-symmetric metric connection is not a generalized Sasakian space form with the semi-symmetric metric connection.

If  $(M, J, g)$  is a Kaehlerian manifold (i.e., a smooth manifold with a  $(1, 1)$ -tensor field  $J$  and a Riemannian metric  $g$  such that  $J^2 = -I$ ,  $g(JX, JY) = g(X, Y)$ ,  $\nabla J = 0$  for arbitrary vector fields  $X, Y$  on  $M$ , where  $I$  is identity tensor field and  $\nabla$  the Riemannian connection of  $g$ ) with constant holomorphic sectional curvature (i. e.  $K(X \wedge JX) = c$ ) then it is said to be a *complex space form* if its curvature tensor is given by

$$\begin{aligned} R(X, Y)Z &= \frac{c}{4}\{g(Y, Z)X - g(X, Z)Y \\ &+ g(X, JZ)JY - g(Y, JZ)JY + 2g(X, JY)JZ\}. \end{aligned}$$

Models for these spaces are  $\mathbb{C}^n$ ,  $\mathbb{C}P^n$  and  $\mathbb{C}H^n$ , depending on  $c = 0$ ,  $c > 0$  or  $c < 0$ .

More generally, if the curvature tensor of an almost Hermitian manifold  $M$  satisfies

$$R(X, Y)Z = F_1\{g(Y, Z)X - g(X, Z)Y\} \\ + F_2\{g(X, JZ)JY - g(Y, JZ)JY + 2g(X, JY)JZ\},$$

where  $F_1$  and  $F_2$  are differentiable functions on  $M$ , then  $M$  is said to be a *generalized complex space form* (see [10] and [11]).

#### 4. Existence of a generalized Sasakian space form with semi-symmetric metric connection

Let  $(M_1, g_{M_1})$  and  $(M_2, g_{M_2})$  be two Riemannian manifolds and  $f$  is a positive differentiable function on  $M_1$ . Consider the product manifold  $M_1 \times M_2$  with its projections  $\pi : M_1 \times M_2 \rightarrow M_1$  and  $\sigma : M_1 \times M_2 \rightarrow M_2$ . The *warped product*  $M_1 \times_f M_2$  is the manifold  $M_1 \times M_2$  with the Riemannian structure such that

$$\|X\|^2 = \|\pi^*(X)\|^2 + f^2(\pi(p)) \|\sigma^*(X)\|^2,$$

for any vector field  $X$  on  $M$ . Thus we have

$$(13) \quad g = g_{M_1} + f^2 g_{M_2}$$

holds on  $M$ . The function  $f$  is called a *warping function* of the warped product [7]. If  $f = 1$  then we obtain the Riemannian product.

If  $N$  is a Kaehlerian manifold, it is well-known that  $M = N \times \mathbb{R}$  with its usual product almost contact metric structure is a cosymplectic manifold [3]. Given an almost Hermitian manifold  $(N, J, G)$ , the warped product  $M = \mathbb{R} \times_f N$ , where  $f > 0$  is a function on  $\mathbb{R}$ , can be endowed with an almost contact metric structure  $(\varphi, \xi, \eta, g)$ . In fact,

$$g = \pi^*(g_{\mathbb{R}}) + (f \circ \pi)^2 \sigma^*(G)$$

is the warped product metric, where  $\varphi(X) = (J\sigma_*X)^*$  for any vector field  $X$  on  $M$ .

We need the following lemmas from [7] and [9], respectively for later use:

**Lemma 4.1.** *Let us consider  $M = M_1 \times_f M_2$  and denote by  $\nabla$ ,  ${}^{M_1}\nabla$  and  ${}^{M_2}\nabla$  the Levi-Civita connections on  $M$ ,  $M_1$  and  $M_2$ , respectively. If  $X, Y$  are vector fields on  $M_1$  and  $V, W$  on  $M_2$ , then:*

- (i)  $\nabla_X Y$  is the lift of  ${}^{M_1}\nabla_X Y$ ;
- (ii)  $\nabla_X V = \nabla_V X = (Xf/f)V$ ;
- (iii) The component of  $\nabla_V W$  normal to the fibers is  $-(g(V, W)/f)\text{grad } f$ ;
- (iv) The component of  $\nabla_V W$  tangent to the fibers is the lift of  ${}^{M_2}\nabla_V W$ .

**Lemma 4.2.** *Let  $M = M_1 \times_f M_2$  be a warped product and  $R$  and  $\overset{\circ}{R}$  denote the Riemannian curvature tensors of  $M$  with respect to the Levi-Civita connection and the semi-symmetric metric connection, respectively. If  $X, Y, Z \in \chi(M_1)$ ,  $U, V, W \in \chi(M_2)$  and  $\xi \in \chi(M_1)$ , then:*

- (i)  $\overset{\circ}{R}(X, Y)Z \in \chi(M_1)$  is the lift of  ${}^{M_1}\overset{\circ}{R}(X, Y)Z$  on  $M_1$ ;
- (ii)  $\overset{\circ}{R}(V, X)Y = -[H^f(X, Y)/f + (\xi f/f)g(X, Y) + g(X, Y)\eta(\xi) + g(Y, \nabla_X \xi) - \eta(X)\eta(Y)]V$ ;
- (iii)  $\overset{\circ}{R}(X, Y)V = 0$ ;
- (iv)  $\overset{\circ}{R}(V, W)X = 0$ ;
- (v)  $\overset{\circ}{R}(X, V)W = g(V, W)[-(\nabla_X \text{grad } f)/f - (\xi f/f)X - \nabla_X \xi - \eta(\xi)X + \eta(X)\xi]$ ;
- (vi)  $\overset{\circ}{R}(U, V)W = {}^{M_2}R(U, V)W - \{\|\text{grad } f\|^2/f^2 + 2(\xi f/f) + \eta(\xi)\}[g(V, W)U - g(U, W)V]$ .

Now, let's begin with the existence theorem of a generalized Sasakian space form with the semi-symmetric metric connection:

**Theorem 4.3.** *Let  $N(F_1, F_2)$  be a generalized complex space form. Then, the warped product  $M = \mathbb{R} \times_f N$  endowed with the almost contact metric structure  $(\varphi, \xi, \eta, g)$  with the semi-symmetric metric connection is a*

generalized Sasakian space form with the semi-symmetric metric connection such that

$$\begin{aligned}\tilde{f}_1 &= \frac{(F_1 \circ \pi) - (f' + f)^2}{f^2}, \quad \tilde{f}_2 = \frac{(F_2 \circ \pi)}{f^2}, \\ \tilde{f}_3 &= \frac{(F_1 \circ \pi) - (f' + f)^2}{f^2} + \frac{(f'' + f)}{f}.\end{aligned}$$

**Proof.** For any vector fields  $X, Y, Z$  on  $M$ , we can write  $X = \eta(X)\xi + U$ ,  $Y = \eta(Y)\xi + V$  and  $Z = \eta(Z)\xi + W$ , where  $U, V, W$  are vector fields on a generalized complex space form  $N$ . Since the structure vector field  $\xi$  is on  $\mathbb{R}$ , then by virtue of Lemma 4.2 we have

$$\begin{aligned}(14) \quad \overset{\circ}{R}(X, Y)Z &= \eta(X)\eta(Z)\left[\frac{H^f(\xi, \xi)}{f} + (\xi f/f)\right]V \\ &\quad - \eta(X)g(V, W)[(\nabla_\xi \text{grad} f)/f + (\xi f/f)\xi] \\ &\quad - \eta(Y)\eta(Z)\left[\frac{H^f(\xi, \xi)}{f} + (\xi f/f)\right]U \\ &\quad + \eta(Y)g(U, W)[(\nabla_\xi \text{grad} f)/f + (\xi f/f)\xi] + {}^N R(U, V)W \\ &\quad - \{\|\text{grad} f\|^2/f^2 + 2(\xi f/f) + \eta(\xi)\}[g(V, W)U - g(U, W)V].\end{aligned}$$

Since  $f = f(t)$ ,  $\text{grad} f = f'\xi$ . Therefore, we get  $\nabla_\xi \text{grad} f = f''\xi + f'\nabla_\xi \xi$ . By virtue of Lemma 4.1, since  $\nabla_\xi \xi = 0$ , the above equation reduces to

$$(15) \quad \nabla_\xi \text{grad} f = f''\xi.$$

Moreover, we have

$$(16) \quad H^f(\xi, \xi) = g(\nabla_\xi \text{grad} f, \xi) = f'',$$

$$(17) \quad \|\text{grad} f\|^2 = (f')^2, \quad \xi f = g(\text{grad} f, \xi) = f'.$$

In view of the equations (13), (15), (16) and (17) in (14) and by using the fact that  $N$  is a generalized complex space form, we have

$$\begin{aligned}\overset{\circ}{R}(X, Y)Z &= \left(\frac{f'' + f'}{f}\right) \{\eta(X)\eta(Z)V - \eta(Y)\eta(Z)U \\ &\quad + f^2 g_N(U, W)\eta(Y)\xi - f^2 g_N(V, W)\eta(X)\xi\} \\ &\quad + (F_1 \circ \pi)\{g_N(V, W)U - g_N(U, W)V\} \\ &\quad + (F_2 \circ \pi)\{g_N(U, JW)JV - g_N(V, JW)JU + 2g_N(U, JV)JW\} \\ &\quad + \left(\frac{f' + f}{f}\right)^2 \{f^2 g_N(U, W)V - f^2 g_N(V, W)U\}.\end{aligned}$$



Taking into account (13) and by the use of the relation between the vector fields  $X, Y, Z$  and  $U, V, W$ , the above equation turns into

$$\begin{aligned} \overset{\circ}{R}(X, Y)Z &= \left( \frac{(F_1 \circ \pi) - (f' + f)^2}{f^2} \right) \{g(Y, Z)X - g(X, Z)Y\} \\ &+ \left( \frac{F_2 \circ \pi}{f^2} \right) \{g(X, \varphi Z)\varphi Y - g(Y, \varphi Z)\varphi X + 2g(X, \varphi Y)\varphi Z\} \\ &+ \left( \frac{(F_1 \circ \pi) - (f' + f)^2}{f^2} + \frac{f'' + f'}{f} \right) \{ \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\ &- g(Y, Z)\eta(X)\xi + g(X, Z)\eta(Y)\xi \}. \end{aligned}$$

Hence, the proof of the theorem is completed.  $\square$

So we can state the following corollaries:

**Corollary 4.4.** *If  $N(a, b)$  is a generalized complex space form with constant functions, then we have a generalized Sasakian space form with the semi-symmetric metric connection as follows*

$$M \left( \frac{a - (f' + f)^2}{f^2}, \frac{b}{f^2}, \frac{a - (f' + f)^2}{f^2} + \frac{f'' + f'}{f} \right),$$

with non-constant functions.

**Corollary 4.5.** *If  $N(c)$  is a complex space form, we have*

$$M \left( \frac{c - 4(f' + f)^2}{4f^2}, \frac{c}{4f^2}, \frac{c - 4(f' + f)^2}{4f^2} + \frac{f'' + f'}{f} \right).$$

Thus, for example, the warped products  $\mathbb{R} \times_f \mathbb{C}^n$ ,  $\mathbb{R} \times_f \mathbb{C}P^n(4)$  and  $\mathbb{R} \times_f \mathbb{C}H^n(-4)$  are generalized Sasakian space forms with the semi-symmetric metric connections such that

$$\begin{aligned} \tilde{f}_1 &= -\frac{(f' + f)^2}{f^2}, \quad \tilde{f}_2 = 0, \quad \tilde{f}_3 = -\frac{(f' + f)^2}{f^2} + \frac{f'' + f'}{f}, \\ \tilde{f}_1 &= \frac{1 - (f' + f)^2}{f^2}, \quad \tilde{f}_2 = \frac{1}{f^2}, \quad \tilde{f}_3 = \frac{1 - (f' + f)^2}{f^2} + \frac{f'' + f'}{f}, \\ \tilde{f}_1 &= \frac{-1 - (f' + f)^2}{f^2}, \quad \tilde{f}_2 = -\frac{1}{f^2}, \quad \tilde{f}_3 = \frac{-1 - (f' + f)^2}{f^2} + \frac{f'' + f'}{f}, \end{aligned}$$

respectively.

Hence, this method gives us some examples of generalized Sasakian space forms with semi-symmetric metric connections with arbitrary dimensions and non-constant functions.

On the other hand, the following theorem gives us some information about the structure of these warped product manifolds endowed with semi-symmetric metric connections.

**Theorem 4.6.** *Let  $N$  be an almost Hermitian manifold. Then,  $\mathbb{R} \times_f N$  is a  $(0, \beta)$  trans-Sasakian manifold endowed with semi-symmetric metric connection such that  $\beta = \frac{f'+f}{f}$  if and only if  $N$  is a Kaehlerian manifold.*

**Proof.** Similar to the proof of Theorem 4.3, for any vector fields  $X, Y$  on  $M$ , we can write  $X = \eta(X)\xi + U$  and  $Y = \eta(Y)\xi + V$ , where  $U, V$  are vector fields on an almost Hermitian manifold  $N$ .

By direct covariant differentiation with semi-symmetric metric connection, we have  $(\overset{\circ}{\nabla}_X \varphi)Y = \overset{\circ}{\nabla}_X \varphi Y - \varphi \overset{\circ}{\nabla}_X Y$ . In view of (4), the above equation gives us

$$(18) \quad (\overset{\circ}{\nabla}_X \varphi)Y = (\nabla_X \varphi)Y + g(\varphi X, Y)\xi - \eta(Y)\varphi X.$$

On the other hand, by the use of Lemma 4.1, we have

$$(19) \quad (\nabla_X \varphi)Y = \frac{f'}{f}[g(\varphi X, Y)\xi - \eta(Y)\varphi X] + (\nabla_U^N J)V.$$

Then using (19) in (18), we obtain

$$(\overset{\circ}{\nabla}_X \varphi)Y = \left( \frac{f' + f}{f} \right) \{g(\varphi X, Y)\xi - \eta(Y)\varphi X\} + (\nabla_U^N J)V,$$

which implies that  $\mathbb{R} \times_f N$  is a  $(0, \beta)$  trans-Sasakian manifold endowed with semi-symmetric metric connection such that  $\beta = \frac{f'+f}{f}$  if and only if  $N$  is a Kaehlerian manifold. Thus, the proof is completed.  $\square$

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