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ON SUBMANIFOLDS SATISFYING CHEN'S EQUALITY IN A REAL SPACE FORM

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الخلاصــة

سوف ندرس - في هذا البحث - كلاً من: أينشتاين والتطابق المنبسط وشبه المتماثل، وجزئية ريتش – شبه المتماثلة التي تحقق مُساوية – تشن في صيغة القضاء الحقيقي. كما سنُبرهن أن التراكيب الجزئية التي لها احداثيات – n (حيث $n \geq 3$) في الفضاء الحقيقي تشكل $\widetilde{M}^{n+m}(c)$ وتحقق مُساوية – تشن تكون من الأنواع التالية: ١- أينشتاين إذا فقط إذا كانت جيوديسية كُلـيِّـة من خلال انحناء ثابت قدره (c). ٢- التطابق المنبسط فقط واذا فقط $(\inf K = c)$ حيث K ترمز جزئية الانحناء للتراكيب الجزئيه. كما سوف نصنف التراكيب الجزئية شبيه التماثل وريتشي – شبيه التماثل التي تحقق مساوية تشن في الفضاء الحقيقي.

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ABSTRACT

Einstein, conformally flat, semisymmetric, and Ricci-semisymmetric submanifolds satisfying Chen's equality in a real space form are studied. We prove that an n-dimensional ($n \geq 3$) submanifold of a real space form $\widetilde{M}^{n+m}(c)$ satisfying Chen's equality is (i) Einstein if and only if it is a totally geodesic submanifold of constant curvature c; and (ii) conformally flat if and only if $\inf K = c$, where K denotes the sectional curvatures of the submanifold. We also classify semisymmetric and Ricci-semisymmetric submanifolds satisfying Chen's equality in a real space form.

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Key words: Chen invariant, Chen's inequality, Einstein manifold, conformally flat manifold, semisymmetric manifold, Ricci-semisymmetric submanifold, totally geodesic submanifold, real space form, hyperbolic space form

ON SUBMANIFOLDS SATISFYING CHEN'S EQUALITY IN A REAL SPACE FORM

1. INTRODUCTION

In [1], B.-Y. Chen recalled that one of the basic interests of submanifold theory is to establish simple relationships between the main extrinsic invariants and the main intrinsic invariants of a submanifold. The main extrinsic invariant is the squared mean curvature and the main intrinsic invariants include the classical curvature invariants, namely the scalar curvature and the Ricci curvature, and the well known modern curvature invariant, namely Chen invariant [2].

In 1993, Chen obtained an interesting basic inequality for submanifolds in a real space form involving the squared mean curvature and the Chen invariant and found several of its applications (*cf.* Lemma 2.1). This inequality is now well known as Chen's inequality; and in the equality case it is known as Chen's equality. In [3], Dillen, Petrovic, and Verstraelen studied Einstein, conformally flat, and semisymmetric submanifolds satisfying Chen's equality in Euclidean spaces. Motivated by this study, in the present paper, we study Einstein, conformally flat, semisymmetric, and Ricci-semisymmetric submanifolds satisfying Chen's equality in real space forms.

The paper is organized as follows. In Section 2, we give the necessary details about Riemannian submanifolds and we state Chen's inequality. We also present some necessary formulas for sectional curvatures and Ricci tensor of a submanifold satisfying Chen's equality. In Section 3, we prove that an n-dimensional ($n \geq 3$) Einstein submanifold of a real space form $\widetilde{M}^{n+m}(c)$ satisfies Chen's equality if and only if it is a totally geodesic submanifold of constant curvature c. In Section 4, it is proved that an n-dimensional ($n > 3$) submanifold of a real space form $\widetilde{M}^{n+m}(c)$ satisfying Chen's equality is conformally flat if and only if $\inf K = c$. Section 5 contains a classification for semisymmetric submanifolds of a real space form satisfying Chen's equality, while in the last section we give a classification for Ricci-semisymmetric submanifolds of a real space form satisfying Chen's equality.

2. CHEN'S INEQUALITY

Let M^n be an n-dimensional submanifold of an $(n+m)$-dimensional Riemannian manifold $\widetilde{M}$ equipped with a Riemannian metric $\widetilde{g}$. We use the inner product notation $\langle , \rangle$ for both the metrics $\widetilde{g}$ of $\widetilde{M}$ and the induced metric g on the submanifold M.

The Gauss and Weingarten formulas are given respectively by

$$\widetilde{\nabla}_X Y = \nabla_X Y + \sigma(X,Y) \qquad \text{and} \qquad \widetilde{\nabla}_X N = -A_N X + \nabla^{\perp}_X N$$

for all $X, Y \in TM$ and $N \in T^{\perp}M$, where $\widetilde{\nabla}$, ∇, and $\nabla^{\perp}$ are respectively the Riemannian, induced Riemannian, and induced normal connections in $\widetilde{M}$, M, and the normal bundle $T^{\perp}M$ of M respectively, and σ is the second fundamental form related to the shape operator A by $\langle \sigma(X,Y), N \rangle = \langle A_N X, Y \rangle$. The equation of Gauss is given by

$$R(X,Y,Z,W) = \widetilde{R}(X,Y,Z,W) + \langle \sigma(X,W), \sigma(Y,Z) \rangle - \langle \sigma(X,Z), \sigma(Y,W) \rangle \tag{2.1}$$

for all $X, Y, Z, W \in TM$, where $\widetilde{R}$ and R are the curvature tensors of $\widetilde{M}$ and M respectively.

The mean curvature vector H is given by $H = \frac{1}{n}\,\mathrm{trace}(\sigma)$. The submanifold M is *totally geodesic* in $\widetilde{M}$ if $\sigma = 0$, and *minimal* if $H = 0$. If $\sigma(X,Y) = g(X,Y)\,H$ for all $X, Y \in TM$, then M is *totally umbilical* [4].

Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of the tangent space T_pM and ν_r $(r = 1, \ldots, m)$ belongs to an orthonormal basis $\{\nu_1, \ldots, \nu_m\}$ of the normal space $T_p^{\perp}M$. We put

$$\sigma_{ij}^r = \langle \sigma(e_i, e_j), \nu_r \rangle \quad \text{and} \quad \|\sigma\|^2 = \sum_{i,j=1}^{n} \langle \sigma(e_i, e_j), \sigma(e_i, e_j) \rangle$$

Let K_{ij} and $\widetilde{K}_{ij}$ denote the sectional curvatures of the plane section spanned by e_i and e_j at p in the submanifold M and in the ambient manifold $\widetilde{M}$ respectively. In view of (2.1), we have

$$K_{ij} = \widetilde{K}_{ij} + \sum_{r=1}^{m} \left(\sigma_{ii}^r \sigma_{jj}^r - (\sigma_{ij}^r)^2 \right)$$

from which we can get

$$2\tau(p) = n^2 \|H\|^2 - \|\sigma\|^2 + n(n-1)\,c \tag{2.2}$$

where $2\tau = \sum_{1 \le i,j \le n} K_{ij}$ is the scalar curvature of the submanifold M.

Recalling the Chen invariant [1]

$$\delta(p) = \tau(p) - \inf K(p)$$

where K denotes sectional curvature of a plane section of T_pM, we have a sharp inequality for submanifolds M^n in a real space form $\widetilde{M}^{n+m}(c)$ involving intrinsic invariant, namely Chen invariant of M; and the main extrinsic invariant, namely the squared mean curvature as follows.

Lemma 2.1. (Lemma 3.2, [1]). *Let M be an n-dimensional $(n \ge 3)$ submanifold of a real space form $\widetilde{M}^{n+m}(c)$. Then, for each point $p \in M$, we have*

$$\delta \equiv \tau - \inf K \le \frac{n^2(n-2)}{2(n-1)} \|H\|^2 + \frac{1}{2}\,(n+1)(n-2)c \tag{2.3}$$

The equality in (2.3) holds at $p \in M$ if and only if there exist an orthonormal basis $\{e_1, \ldots, e_n\}$ of T_pM and an orthonormal basis $\{\nu_1, \ldots, \nu_m\}$ of $T_p^{\perp}M$ such that ***(a)*** *$K = K_{12}$ and* ***(b)*** *the forms of shape operators $A_r \equiv A_{\nu_r}$, $r = 1, \ldots, m$, become*

$$A_1 = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & \mu I_{n-2} \end{pmatrix}, \qquad \mu = a + b \tag{2.4}$$

$$A_r = \begin{pmatrix} c_r & d_r & 0 \\ d_r & -c_r & 0 \\ 0 & 0 & 0_{n-2} \end{pmatrix}, \qquad r \in \{2, \ldots, m\} \tag{2.5}$$

The inequality (2.3) is well known as Chen's inequality. In case of equality, it is known as Chen's equality. For dimension $n = 2$, Chen's equality is always true.

Let M be an n-dimensional $(n \geq 3)$ submanifold of a real space form $\widetilde{M}^{n+m}(c)$ satisfying Chen's equality, then from Lemma 2.1 we immediately have the following:

$$K_{12} = c + ab - \sum_{r=1}^{m} (c_r^2 + d_r^2) \tag{2.6}$$

$$K_{1j} = c + a\mu \tag{2.7}$$

$$K_{2j} = c + b\mu \tag{2.8}$$

$$K_{ij} = c + \mu^2 \tag{2.9}$$

$$S(e_1, e_1) = K_{12} + (n-2)(c + a\mu) \tag{2.10}$$

$$S(e_2, e_2) = K_{12} + (n-2)(c + b\mu) \tag{2.11}$$

$$S(e_i, e_i) = (n-2)\mu^2 + (n-1)\, c \tag{2.12}$$

where $i, j > 2$. Consequently,

$$\frac{2\tau}{(n-1)(n-2)} = \frac{2}{(n-1)(n-2)} K_{12} + \mu^2 + \left(\frac{n+1}{n-1}\right) c \tag{2.13}$$

Furthermore, $R(e_i, e_j)e_k = 0$ if i, j and k are mutually different.

3. EINSTEIN SUBMANIFOLDS SATISFYING CHEN'S EQUALITY

In this section, we consider Einstein submanifolds satisfying Chen's equality in real space forms. We have the following:

Theorem 3.1. *Let M be an n-dimensional $(n \geq 3)$ submanifold of a real space form $\widetilde{M}^{n+m}(c)$ satisfying Chen's equality. Then, M is Einstein if and only if it is a totally geodesic submanifold $M^n(c)$ of constant curvature c.*

Proof. Let M be Einstein. Then from (2.10) and (2.11) we get

$$(a - b)\mu = 0$$

If $\mu = 0$ then from (2.10) and (2.12) we get

$$-\, a^2 = \sum_{r=1}^{m} (c_r^2 + d_r^2)$$

which implies that $a = b = c_r = d_r = 0$. Hence M is totally geodesic.

If $a = b$ then $\mu = 2a$ and from (2.10) and (2.12) we get

$$(5 - 2n)\, a^2 = \sum_{r=1}^{m} (c_r^2 + d_r^2)$$

which again implies that $a = b = c_r = d_r = 0$. Hence again M is totally geodesic.

Now, if M is totally geodesic then in view of (2.6)–(2.9) we see that M is of constant curvature c.

The converse is easy to follow. □

As a corollary, we have the following:

Corollary 3.2. (Theorem 1, [3]). *Let M^n, $n \geq 3$, submanifold of $\mathbb{E}^{n+m}$ satisfying Chen's equality. Then M^n is Einstein if and only if it is a totally geodesic n-plane in $\mathbb{E}^{n+m}$.*

4. CONFORMALLY FLAT SUBMANIFOLDS SATISFYING CHEN'S EQUALITY

The Weyl conformal curvature tensor C of an n-dimensional Riemannian manifold is defined by [5]

$$\begin{aligned} C(X,Y,Z,W) &= R(X,Y,Z,W) \\ &- \frac{1}{n-2}\{S(Y,Z)\,g(X,W) - S(X,Z)\,g(Y,W) \\ &+ S(X,W)\,g(Y,Z) - S(Y,W)\,g(X,Z)\} \\ &+ \frac{2\tau}{(n-1)(n-2)}\{g(X,W)\,g(Y,Z) - g(Y,W)\,g(X,Z)\} \end{aligned} \tag{4.1}$$

for all $X, Y, Z, W \in TM$, where 2τ is the scalar curvature of M.

Now, we prove the following:

Theorem 4.1. *Let M be an n-dimensional $(n > 3)$ submanifold of a real space form $\widetilde{M}^{n+m}(c)$ satisfying Chen's equality. Then M is conformally flat if and only if $\inf K = c$.*

Proof. Let M be conformally flat. Then using (2.10)–(2.13) in (4.1) we get

$$C_{1221} = \frac{n-3}{n-1}(K_{12} - c)$$

$$C_{1331} = \frac{3-n}{(n-1)(n-2)}(K_{12} - c) = C_{2332}$$

$$C_{ijji} = \frac{2}{(n-1)(n-2)}(K_{12} - c), \qquad i, j > 2$$

The last three equations give us $\inf K = c$. The converse is easily verified. □

Using the above theorem we have the following corollary:

Corollary 4.2. *Let M be an n-dimensional $(n > 3)$ conformally flat submanifold of a real space form $\widetilde{M}^{n+m}(c)$ satisfying Chen's equality. Then M is minimal if and only if it is totally geodesic.*

Proof. If M is minimal then $0 = \mu = a + b$; so $a = -b$. From the Proof of Theorem 4.1, M is conformally flat if and only if $K_{12} = c$. Hence from (2.6), it follows that

$$c = c - b^2 - \sum_{r=1}^{m}(c_r^2 + d_r^2)$$

Then

$$b^2 = -\sum_{r=1}^{m}(c_r^2 + d_r^2)$$

The above equation gives $b = c_r = d_r = 0$; and hence M becomes totally geodesic. The converse statement is trivial. □

5. SEMISYMMETRIC SUBMANIFOLDS SATISFYING CHEN'S EQUALITY

As a generalization of locally symmetric spaces, many geometers have considered semisymmetric spaces and in turn their generalizations. A Riemannian manifold M is known to be semisymmetric if its curvature tensor R satisfies

$$R(X,Y)\cdot R = 0, \qquad X,Y \in TM$$

where $R(X,Y)$ acts on R as a derivation.

Now, let M^n, $n > 3$, be a submanifold of a real space form $\widetilde{M}^{n+m}(c)$ satisfying Chen's equality. Then the only non-zero possible terms of type $(R(e_i,e_j)\cdot R)(e_l,e_k)e_u$ are $(R(e_i,e_j)\cdot R)(e_l,e_j)e_i$, where i, j, and l are mutually different. So we have

$$\begin{aligned}(R(e_i,e_j)\cdot R)(e_l,e_j)e_i &= R(e_i,e_j)R(e_l,e_j)e_i - R(R(e_i,e_j)e_l,e_j)e_i \\ &\quad - R(e_l,R(e_i,e_j)e_j)e_i - R(e_l,e_j)R(e_i,e_j)e_i \\ &= -\,R(e_l,K_{ij}e_i)e_i - R(e_l,e_j)\,(-K_{ij}e_j)\end{aligned}$$

which implies that

$$(R(e_i,e_j)\cdot R)(e_l,e_j)e_i = (K_{jl} - K_{il})\,K_{ij}e_l \tag{5.1}$$

Now, we give the following classification theorem:

Theorem 5.1. *Let M be an n-dimensional $(n > 3)$ submanifold of a real space form $\widetilde{M}^{n+m}(c)$ satisfying Chen's equality. Then M is semisymmetric if and only if one of the following statements are true:*

- **(a)** *M is $(n-2)$-ruled submanifold of the Euclidean space $\mathbb{E}^{n+m}$.*
- **(b)** *M is a totally geodesic submanifold of constant curvature $c \neq 0$.*
- **(c)** *M is a round hypercone in some totally geodesic subspace $\mathbb{E}^{n+1}$ of the Euclidean space $\mathbb{E}^{n+m}$.*
- **(d)** *M is a hypersurface of a hyperbolic space form $\widetilde{M}^{n+m}\left(-2a^2\right)$ with the shape operator of the form*

$$A = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 2aI_{n-2} \end{pmatrix} \tag{5.2}$$

Proof. Suppose that M is semisymmetric. Then from (5.1) we have

$$(K_{jl} - K_{il})\, K_{ij} = 0 \tag{5.3}$$

Since for $i, j, l > 3$, the above equation becomes an identity, therefore we need to consider only the following three equations:

$$K_{12}\,(a-b)\,\mu = 0, \qquad i=1, j=2, l>2 \tag{5.4}$$

$$(c + b\mu - K_{12})\,(c + a\mu) = 0, \qquad i=1, j>2, l=2 \tag{5.5}$$

$$(c + a\mu - K_{12})\,(c + b\mu) = 0, \qquad i=2, j>2, l=1 \tag{5.6}$$

We see that Equations (5.5) and (5.6) imply (5.4). We have the following cases.

Case I: $\mu = 0$. In this case M is minimal. Moreover, from (5.5) or (5.6), in view of (2.6) we find that

$$\left(a^2 + \sum_{r=1}^{m}(c_r^2 + d_r^2)\right) c = 0 \tag{5.7}$$

Now, if $c = 0$ then M is an $(n-2)$-ruled submanifold of the Euclidean space $\mathbb{E}^{n+m}$ [3]. If $c \neq 0$ then M is totally geodesic.

Case II: $\mu \neq 0$. Then from (5.4) we get

$$K_{12}\,(a-b) = 0 \tag{5.8}$$

Now, we have two subcases:

Case II(*a*): $a \neq b$ and $\mu \neq 0$. Then we get $\inf K = 0$. So from (5.5)

$$(c + b\mu)\,(c + a\mu) = 0$$

Hence either $c = -b(a+b)$ or $c = -a(a+b)$. So from (2.6), we have either

$$b = c = c_r = d_r = 0$$

or

$$a = c = c_r = d_r = 0$$

respectively. In both cases M is a round hypercone in some totally geodesic subspace $\mathbb{E}^{n+1}$ of the Euclidean space $\mathbb{E}^{n+m}$ [3].

Case II(*b*): $a = b$ and $\mu \neq 0$. Then $\mu = 2a$ and $a = b \neq 0$. Then from (5.5) and (2.6) we get

$$\left(a^2 + \sum_{r=1}^{m}(c_r^2 + d_r^2)\right)\left(c + 2a^2\right) = 0 \tag{5.9}$$

Since $a \neq 0$, from the above equation we get $c = -2a^2$. In this case the ambient manifold is a hyperbolic space form $\widetilde{M}^{n+m}\left(-2a^2\right)$ and M becomes a hypersurface of $\widetilde{M}^{n+m}\left(-2a^2\right)$ with the shape operator of the form (5.2).

The converse is easily verified. □

6. RICCI-SEMISYMMETRIC SUBMANIFOLDS SATISFYING CHEN'S EQUALITY

Theorem 6.1. *Let M be an n-dimensional $(n>3)$ submanifold of a real space form $\widetilde{M}^{n+m}(c)$ satisfying Chen's equality. Then M is Ricci-semisymmetric if and only if one of the following statements are true:*

- **(*a*)** *M is $(n-2)$-ruled submanifold of the Euclidean space $\mathbb{E}^{n+m}$.*
- **(*b*)** *M is a totally geodesic submanifold of constant curvature $c \neq 0$.*
- **(*c*)** *M is a round hypercone in some totally geodesic subspace $\mathbb{E}^{n+1}$ of the Euclidean space $\mathbb{E}^{n+m}$.*
- **(*d*)** *The ambient manifold is a hyperbolic space form $\widetilde{M}^{n+m}\left(-2a^2\right)$.*

Proof. Suppose that M is Ricci-semisymmetric. Then

$$(S_{ii} - S_{jj})\, K_{ij} = 0, \qquad i, j = 1, \ldots, n \tag{6.1}$$

Therefore we have the following three equations:

$$K_{12}\,(a-b)\,\mu = 0 \tag{6.2}$$

$$(c + a\mu)\,\{K_{12} - c - (n-2)\, b\mu\} = 0 \tag{6.3}$$

$$(c + b\mu)\,\{K_{12} - c - (n-2)\, a\mu\} = 0 \tag{6.4}$$

We see that Equations (6.3) and (6.4) imply (6.2). We have the following cases:

Case I: $\mu = 0$. In this case M is minimal. Moreover, from (6.3) or (6.4), in view of (2.6) we find that

$$\left(a^2 + \sum_{r=1}^{m} (c_r^2 + d_r^2)\right) c = 0 \tag{6.5}$$

Now, if $c = 0$ then M is an $(n-2)$-ruled submanifold of the Euclidean space $\mathbb{E}^{n+m}$ [3]. If $c \neq 0$ then M is totally geodesic.

Case II: $\mu \neq 0$. In this case, from (6.2) we get

$$K_{12}\,(a-b) = 0 \tag{6.6}$$

Now, we have two subcases:

Case II(*a*): $a \neq b$ and $\mu \neq 0$. Then we get $K_{12} = 0$. So from (6.3) and (6.4) we get

$$\left(a^2 - b^2\right) c = 0$$

Since $a \neq b$ and $\mu \neq 0$ therefore $c = 0$. In this case M is a round hypercone in some totally geodesic subspace $\mathbb{E}^{n+1}$ of the Euclidean space $\mathbb{E}^{n+m}$ [3].

Case II(*b*): $a = b$ and $\mu \neq 0$. Then $\mu = 2a$ and $a = b \neq 0$. Then from (6.3) we get

$$\left(c + 2a^2\right) \left\{ a^2\,(1 - 2\,(n-2)) - \sum_{r=1}^{m} (c_r^2 + d_r^2) \right\} = 0 \tag{6.7}$$

Since $a \neq 0$, from the above equation we get $c = -2a^2$. In this case the ambient manifold is a hyperbolic space form $\widetilde{M}^{n+m}\left(-2a^2\right)$.

The converse is easily verified. □

Remark 6.2. In view of Theorem 5.1 and Theorem 6.1, we observe that if the ambient real space form is not hyperbolic, then for a submanifold satisfying Chen's equality, the conditions of semisymmetry and Ricci-semisymmetry are equivalent.

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