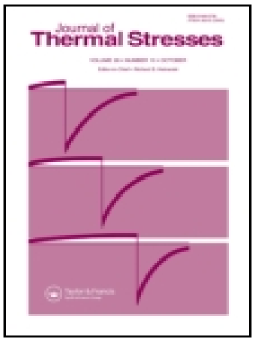

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OPTIMAL BOUNDARY CONTROL OF THERMAL STRESSES IN A PLATE BASED ON TIME-FRACTIONAL HEAT CONDUCTION EQUATION

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This article presents an optimal control problem for a fractional heat conduction equation that describes a temperature field. The main purpose of the research was to find the boundary temperature that takes the thermal stress under control. The fractional derivative is defined in terms of the Caputo operator. The Laplace and finite Fourier sine transforms were applied to obtain the exact solution. Linear approximation is used to get the numerical results. The dependence of the solution on the order of fractional derivative and on the nondimensional time is analyzed.

***Keywords*:** Fractional calculus; Non-Fourier heat conduction; Optimal control; Thermal stresses

INTRODUCTION

The classical thermoelasticity is based on the Fourier law, which gives the relation between the heat flux and the temperature gradient:

$$\mathbf{q} = -k \ \mathrm{grad}\, T \tag{1}$$

where k is the thermal conductivity of a solid. In combination with a law of conservation of energy, Eq. (1) leads to the parabolic heat conduction equation

$$\frac{\partial T}{\partial t} = a\Delta T \tag{2}$$

with a being the heat diffusivity coefficient.

From the mathematical point of view the Fourier law in the theory of heat conduction corresponds to Fick's law in the theory of diffusion:

$$\mathbf{J} = -k' \mathrm{grad}\, c \tag{3}$$

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where $\mathbf{J}$ is the matter flux, c is the concentration, k' is the diffusion conductivity. In combination with the balance equation for mass, Eq. (3) results in the classical diffusion equation:

$$\frac{\partial c}{\partial t} = a' \Delta c \tag{4}$$

with a' being the diffusivity coefficient.

The classical heat conduction and diffusion equations based on the Fourier and Fick laws, respectively, are quite acceptable for different physical situations. However, many theoretical and experimental studies testify that in media with complex internal structure (porous, random and granular materials, semiconductores, polymers, glasses, etc.) the standard parabolic equations are no longer sufficiently accurate. For an extensive bibliography on this subject and further discussion see Chandrasekharaiah [1], [2], Joseph and Preziosi [3], Ignaczak [4], Hetnarski and Ignaczak [5], Ignaczak and Ostoja-Starzewski [6] and references therein.

In nonclassical theories, the Fourier law and the parabolic heat conduction equation are replaced by more general equations. Gurtin and Pipkin [7] considered the general time-nonlocal dependence between the heat flux vector and the temperature gradient:

$$\mathbf{q}(t) = -k \int_0^t K(t-\tau) \operatorname{grad} T(\tau)\, d\tau \tag{5}$$

resulting in the heat conduction equation with memory [8], [9]:

$$\frac{\partial T}{\partial t} = a \int_0^t K(t-\tau) \Delta T(\tau)\, d\tau \tag{6}$$

Chandrasekharaiya [1], Nigmatullin [8], and Green and Naghdi [10] proposed the constitutive equation of heat conduction in the case of constant kernel (full memory with no memory decay):

$$\mathbf{q}(t) = -k \int_0^t \operatorname{grad} T(\tau) d\tau \tag{7}$$

The wave equation for temperature

$$\frac{\partial^2 T}{\partial t^2} = a \Delta T \tag{8}$$

obtained from Eq. (7) is a constituent part of thermoelasticity without energy dissipation [10].

Cattaneo [11], [12] and Vernotte [13] introduced the generalized constitutive equation for the heat flux, which can be rewritten in a nonlocal form with the "short-tail" exponential time-nonlocal kernel:

$$\mathbf{q}(t) = -\frac{k}{\zeta} \int_0^t \exp\left(-\frac{t-\tau}{\zeta}\right) \operatorname{grad} T(\tau)\, d\tau \tag{9}$$

where ζ is a nonnegative constant. This equation leads to the telegraph equation for temperature:

$$\frac{\partial T}{\partial t} + \zeta \frac{\partial^2 T}{\partial t^2} = a \Delta T \tag{10}$$

The time-nonlocal dependences between the heat flux vector and the temperature gradient with the "long-tail" power kernel [14], [15]:

$$\mathbf{q}(t) = -\frac{k}{\Gamma(\alpha)} \frac{\partial}{\partial t} \int_0^t (t-\tau)^{\alpha-1} \mathrm{grad}\, T(\tau)\, d\tau \quad 0 < \alpha < 1 \tag{11}$$

$$\mathbf{q}(t) = -\frac{k}{\Gamma(\alpha-1)} \int_0^t (t-\tau)^{\alpha-2} \mathrm{grad}\, T(\tau)\, d\tau \quad 1 < \alpha < 2 \tag{12}$$

result in the time-fractional heat conduction equation

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\, \Delta T \tag{13}$$

or in terms of diffusion

$$\frac{\partial^\alpha c}{\partial t^\alpha} = a'\, \Delta c \tag{14}$$

with the particular cases corresponding to subdiffusion (weak diffusion) ($0 < \alpha < 1$); normal diffusion ($\alpha = 1$); superdiffusion (strong diffusion) ($1 < \alpha < 2$), and ballistic diffusion ($\alpha = 2$).

In Eqs. (13) and (14) $\frac{\partial^\alpha}{\partial t^\alpha}$ is the Caputo fractional derivative (see Eq. (18)).

The fractional Cattaneo-type equation was considered in [16]:

$$\mathbf{q}(t) = -\frac{k}{\zeta} \int_0^t (t-\tau)^{\beta-2} E_{\beta-\alpha,\beta-1}\left[-\frac{(t-\tau)^{\beta-\alpha}}{\zeta}\right] \mathrm{grad}\, T(\tau)\, d\tau \tag{15}$$

where $E_{\alpha,\beta}(z)$ is the two-parameter Mittag-Leffler function [17]:

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)} \quad \alpha > 0 \quad \beta > 0 \tag{16}$$

being the generalization of the exponential function. The constitutive equation (15) leads to the fractional telegraph equation for temperature:

$$\frac{\partial^\alpha T}{\partial t^\alpha} + \zeta \frac{\partial^\beta T}{\partial t^\beta} = a \Delta T \tag{17}$$

Several particular cases of Eqs. (15) and (17) corresponding to different choices of α and β were analyzed in [18] (see also [19]).

Povstenko [14] first proposed the theory of thermoelasticity based on the time-fractional heat conduction equation and investigated the stresses corresponding to the fundamental solutions to the Cauchy problem for the one and two-dimensional fractional heat conduction equations. The central-symmetric thermal stresses in an infinite medium with a spherical [20] and cylindrical [21] cavity for different boundary conditions were analyzed. As a further generalization, a theory of thermal stresses for space-time fractional heat conduction equation was introduced [15]. In recent years, thermal stresses based on the fractional telegraph equation have also been researched [18], [22], [23]. Further discussion on different theories of generalized thermoelasticity can be found in references [1], [2], [4]–[6], [24], among others.

In this article, the fractional heat conduction equation in the case $0 < \alpha < 1$, called as "heat subconduction," is considered. An optimal control problem formulated on the basis of fractional heat conduction equation is studied.

In the classical scheme, we can cite many articles related to optimal control problems for thermoelastic structures. For example, the optimal heating mode with respect to stress over the thickness of a spherical shell in the absence of external force loading and with the zero initial condition was studied in [25]. The analytical solution of the optimal control problem with respect to the speed of response by means of heating and cooling of a body in the case of nonsteady one-dimensional temperature regime of a plate, a hollow cylinder, and a hollow sphere under the constraint on the control and on the mean temperature of the body was proposed in [26]. Similarly, a method was introduced to construct an optimal control for the heating of solids described by a two-dimensional nonsteady-state equation of thermal conductivity [27]. The proposed method in [27] was developed to apply to the solution of the problem of temperature-regime optimization with constraints on reduced stresses by using a nonlinear expression for energy-based strength criterion [28].

Another method was developed for stress optimization of the thermal conditions for heating of the glass plate materials with the constraints on the temperature of the heaters and the stress state of a plate [29]. A method for the inverse problem of thermomechanics and heat conduction were successfully applied to solve the optimal control problem of quasistatic thermoelastic stresses and displacements in the case of two-dimensional temperature field [30]. A method of the inverse thermoelasticity problem for investigation of optimal control of a two-dimensional nonaxisymmetric unsteady thermal regime in a long, hollow cylinder with the constraints on the thermoelastic stresses was developed, and a numerical algorithm was also presented for solving the optimization problem [31]. A stress-optimization problem of heating regimes of a piecewise-homogeneous cylindrical glass shell was studied and analytical/numerical solutions were obtained [32].

Recently, a mathematical model based on the standard parabolic heat conduction equation describing the temperature field and assuring the stress under control with the linear boundary heating has been studied by Knopp [33]. The fractional generalization of this approach based on the heat conduction equation with the Caputo time-derivative was formulated in [34]. In the present article, we aim to develop the results of [34].

PRELIMINARIES

Here, we briefly give the basic definitions and relations necessary for problem formulation. It is well known in the fractional calculus literature that several definitions of a fractional derivative have been proposed: the Riemann–Liouville, Grünwald–Letnikov, Weyl, Caputo, Riesz derivative, etc. (see [17], [35]). In this article we use the Caputo derivative of the fractional order α ($n-1<\alpha\leq n$), which is defined as

$$\frac{d^\alpha f(t)}{dt^\alpha}=\frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\tau)^{n-\alpha-1}\frac{d^n f(\tau)}{d\tau^n}d\tau \tag{18}$$

and the Laplace transform rule for this operator has the form

$$\mathscr{L}\left\{\frac{d^\alpha f(t)}{dt^\alpha}\right\}=s^\alpha\mathscr{L}\{f(t)\}-\sum_{k=0}^{n-1}f^{(k)}(0^+)s^{\alpha-1-k} \tag{19}$$

where s is the transform variable.

This operator has wide applications because the initial conditions of fractional differential equations with Caputo derivatives should be expressed in terms of a given function and its derivatives of integer order. This allows us to get physically interpretable initial conditions for fractional differential equations.

The following formula for the inverse Laplace transform [17]:

$$\mathscr{L}^{-1}\left\{\frac{s^{\alpha-\beta}}{s^\alpha+b}\right\}=t^{\beta-1}E_{\alpha,\beta}(-bt^\alpha) \tag{20}$$

is applied in the problem analysis.

The finite Fourier sine transform,

$$\mathscr{F}\{f(x)\}=f_n^*=\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx \quad n=1,2,\ldots \tag{21}$$

with the inverse

$$\mathscr{F}^{-1}\{f_n^*\}=f(x)=\sum_{n=1}^{\infty}f_n^*\sin\left(\frac{n\pi x}{L}\right) \tag{22}$$

is used to eliminate the spatial coordinate x.

If $f(x,t)$ is a function of two variables, then

$$\mathscr{F}\{f(x,t)\}=f_n^*(t)=\frac{2}{L}\int_0^L f(x,t)\sin\left(\frac{n\pi x}{L}\right)dx \tag{23}$$

and

$$\mathscr{F}\left\{\frac{\partial^2 f(x,t)}{\partial x^2}\right\}=-\left(\frac{n\pi}{L}\right)^2\mathscr{F}\{f(x,t)\}+\frac{2n\pi}{L^2}[f(0,t)+(-1)^{n+1}f(L,t)] \tag{24}$$

PROBLEM FORMULATION

The considered uncoupled theory of thermal stresses [14] is governed by the equation of equilibrium in terms of displacements

$$\mu\Delta\mathbf{u}+(\lambda+\mu)\operatorname{grad}\operatorname{div}\mathbf{u}=\beta_T K_T\operatorname{grad}T \tag{25}$$

the stress-strain-temperature relation

$$\boldsymbol{\sigma} = 2\mu\mathbf{e} + (\lambda\,\mathrm{tr}\,\mathbf{e} - \beta_T K_T T)\mathbf{I} \tag{26}$$

and the fractional heat conduction equation with the Caputo time-derivative

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\Delta T \tag{27}$$

where **u** is the displacement vector, $\boldsymbol{\sigma}$ the stress tensor, **e** the linear strain tensor, a the diffusivity coefficient, λ and μ are Lamé constants, $K_T = \lambda + 2\mu/3$, β_T is the thermal coefficient of volumetric expansion, **I** denotes the unit tensor. In Eq. (27) we restrict ourselves to the case $0 < \alpha \leq 1$.

Now let us consider a finite plate $0 \leq x \leq L$ with temperature depending only on the spatial coordinate x and time t, i.e., $T(x, t)$. We also assume that the temperature is symmetric with respect to the middle point $x = L/2$. In this case, the thermoelastic stress $\sigma(x, t)$ is proportional to the distance from the average temperature [36]:

$$\sigma_{yy}(x, t) = -\frac{\alpha_T E}{1 - \nu}[T(x, t) - T_{average}(t)] \tag{28}$$

where

$$T_{average}(t) = \frac{1}{L}\int_0^L T(x, t)dx \tag{29}$$

Here, α_T is the linear thermal expansion coefficient, E is Young's modulus and ν denotes Poisson's ratio.

The temperature field $T(x, t)$ satisfies the time-fractional heat conduction equation:

$$\frac{\partial^\alpha T(x, t)}{\partial t^\alpha} = a\frac{\partial^2 T(x, t)}{\partial x^2} \quad 0 < x < L \quad 0 < t < \infty \quad 0 < \alpha \leq 1 \tag{30}$$

We adopt the following initial:

$$T(x, 0) = 0 \tag{31}$$

and boundary conditions:

$$\begin{aligned} x = 0: &\quad T = g(t)T_0 \\ x = L: &\quad T = g(t)T_0 \end{aligned} \tag{32}$$

where $g(t)$ is the boundary control function, which we motivate to find the optimal regime to keep the thermal stress under constraint.

For convenience of calculations we introduce the nondimensional quantities:

$$\bar{x} = \frac{x}{L} \quad \tau = \frac{t}{t_0} \quad \overline{T} = \frac{T}{T_0} \quad \kappa^2 = \frac{a t_0^\alpha}{L^2} \tag{33}$$

where t_0 is the characteristic time.

Hence, the problem is reformulated as follows:

$$\frac{\partial^\alpha \overline{T}(\bar{x}, \tau)}{\partial \tau^\alpha} = \kappa^2 \frac{\partial^2 \overline{T}(\bar{x}, \tau)}{\partial \bar{x}^2} \quad 0 < \bar{x} < 1 \quad 0 < \tau < \infty \quad 0 < \alpha \leq 1 \tag{34}$$

$$\tau = 0 : \quad \overline{T} = 0 \tag{35}$$

$$\bar{x} = 0 : \quad \overline{T} = g(\tau) \tag{36}$$

$$\bar{x} = 1 : \quad \overline{T} = g(\tau) \tag{37}$$

To solve this problem, the Laplace transform with respect to time τ and the finite Fourier sine transform with respect to the spatial coordinate $\bar{x}$, respectively, are used. Applying the integral transforms, we obtain

$$\overline{T}^{**} = \frac{2\kappa^2 \xi_n}{s^\alpha + \kappa^2 \xi_n^2} g_n^*(s)[1 - (-1)^n] \tag{38}$$

where $\xi_n = n\pi$, and each of transforms is denoted by the asterisk. Taking the inverse Fourier and Laplace transforms leads to

$$\overline{T} = 2\kappa^2 \sum_{n=1}^{\infty} \xi_n [1 - (-1)^n] \sin(\bar{x}\xi_n) \int_0^\tau (\tau - u)^{\alpha-1} E_{\alpha,\alpha} \left[-\kappa^2 \xi_n^2 (\tau - u)^\alpha \right] g(u) du \tag{39}$$

Similarly, we calculate the average value $\overline{T}_{average}(\tau)$ using Eq. (39):

$$\overline{T}_{average}(\tau) = 2\kappa^2 \sum_{n=1}^{\infty} [1 - (-1)^n]^2 \int_0^\tau (\tau - u)^{\alpha-1} E_{\alpha,\alpha} \left[-\kappa^2 \xi_n^2 (\tau - u)^\alpha \right] g(u) du \tag{40}$$

Now, nondimensional stress can be introduced as

$$\bar{\sigma}_{yy}(\bar{x}, \tau) = \frac{1 - v}{\alpha_T E T_0} \sigma_{yy}(\bar{x}, \tau) \tag{41}$$

or

$$\bar{\sigma}_{yy}(\bar{x}, \tau) = -\left[\overline{T}(\bar{x}, \tau) - \overline{T}_{average}(\tau) \right] \tag{42}$$

Next, let us calculate the stress component at the boundary $\bar{\sigma}_{yy}(1, \tau)$ and assume that

$$|\bar{\sigma}_{yy}(1, \tau)| = \bar{\sigma}_{crit} \tag{43}$$

Taking into consideration that the maximal temperature and the maximal stress are reached at the boundary, $|\bar{\sigma}_{\max}(\tau)| = |\bar{\sigma}_{yy}(1, \tau)|$ we have

$$g(\tau) = \bar{\sigma}_{crit} + 2\kappa^2 \int_0^\tau \sum_{n=1}^{\infty} [1 - (-1)^n]^2 (\tau - u)^{\alpha-1} E_{\alpha,\alpha} \left[-\kappa^2 \xi_n^2 (\tau - u)^\alpha \right] g(u) du \tag{44}$$

Note that Eq. (44) is an integral equation for temperature control $g(\tau)$ for which we consider the numerical solution.

Numerical Algorithm

Here, we rearrange Eq. (44) by a successive change of variables. In the first step, we take $y = \tau - u$ and so the integral in (44) reduces to

$$I = \int_0^\tau \sum_{n=1}^{\infty} c_n E_{\alpha,\alpha}\left[-\kappa^2 \xi_n^2 y^\alpha\right] y^{\alpha-1} g(\tau - y)dy \tag{45}$$

where $c_n = [1-(-1)^n]^2$. The second change of variable is $z = y^\alpha$, which leads to

$$I = \frac{1}{\alpha}\int_0^{\tau^\alpha} \sum_{n=1}^{\infty} c_n E_{\alpha,\alpha}\left[-\kappa^2 \xi_n^2 z\right] g\left(\tau - z^{\frac{1}{\alpha}}\right) dz \tag{46}$$

and so the integral Eq. (44) for $g(\tau)$ becomes

$$g(\tau) = \bar{\sigma}_{crit} + \frac{2\kappa^2}{\alpha}\int_0^{\tau^\alpha} \sum_{n=1}^{\infty} c_n E_{\alpha,\alpha}\left[-\kappa^2 \xi_n^2 z\right] g\left(\tau - z^{\frac{1}{\alpha}}\right) dz \tag{47}$$

Let us explain the numerical iterations applied to Eq. (47). The iterative form is the following:

$$g_{m+1}(\tau) = \bar{\sigma}_{crit} + \frac{2\kappa^2}{\alpha}\int_0^{\tau^\alpha} \sum_{n=1}^{\infty} c_n E_{\alpha,\alpha}\left[-\kappa^2 \xi_n^2 z\right] g_m\left(\tau - z^{\frac{1}{\alpha}}\right) dz \quad m = 0, 1, 2, \ldots \tag{48}$$

where we assume the initial values $g_0(\tau) = \bar{\sigma}_{crit} = 1$. Next, we calculate the iterative values $g_m(\tau)$ $(m = 1, 2, \ldots)$. Note that we have to know the values of $g_m(\tau)$ at

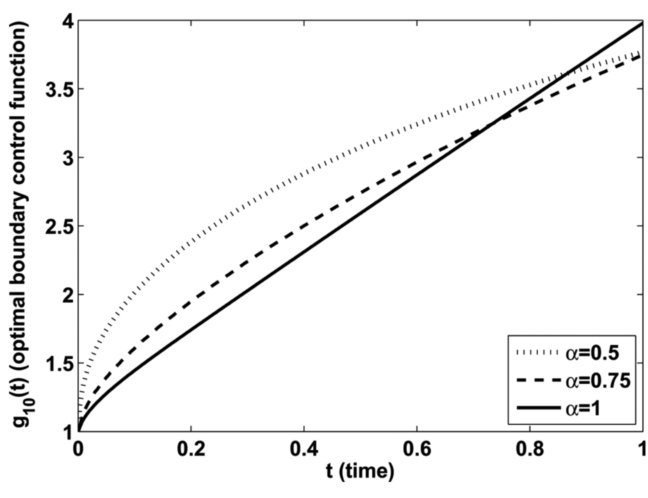

Figure 1 Change of the optimal boundary control with respect to the variation of α for $N = 300$ and $\kappa = 0.5$.

the time nodes $\tau - z^{\frac{1}{\alpha}}$. They may not be calculated in the first iteration because of time discretization. If we take the time interval $[0, T]$ and divide it into N equal subintervals, we only know the values of $g_m(\tau)$ at $\tau = Nh$. To calculate the other values of $g_m(\tau)$ for the values $lh < \tau - z^{\frac{1}{\alpha}} < (l+1)h$, $(l = 1, 2, \ldots, N)$, we use a linear approximation. After all, we plot some figures under the variation of problem parameters. In all the figures, we take the upper limit of the sum in Eq. (48) equal to 20. First, we show the effect of the variation of fractional order α on $g_{10}(\tau)$ for the step number $N = 300$ and $\kappa = 0.5$ in Figure 1. Note that, we calculate the $10th$ iteration value of $g_m(\tau)$ because of the convergence reason demonsrated by Figure 2

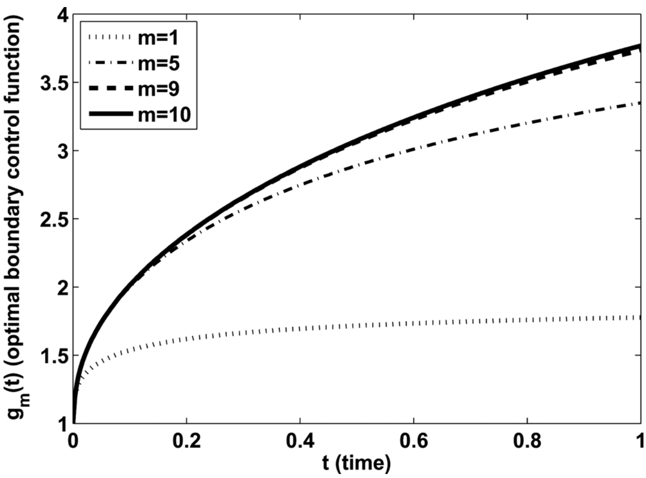

Figure 2 Convergence of the optimal boundary control with respect to the iteration number for $\alpha = 0.5$, $N = 300$, and $\kappa = 0.5$.

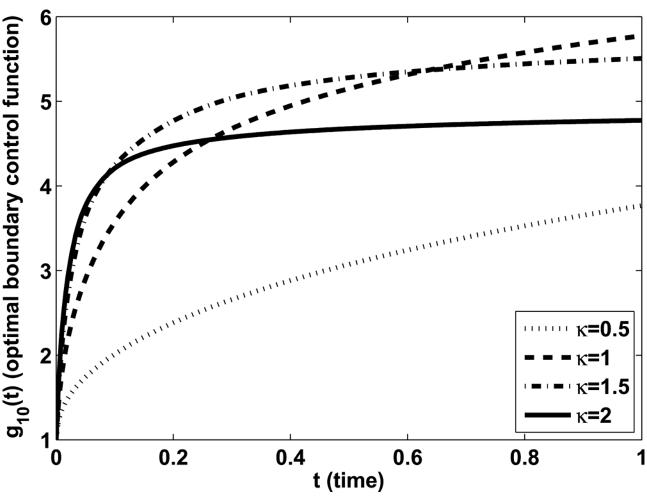

Figure 3 Dependence of the optimal boundary control on the variation of κ for $\alpha = 0.5$ and $N = 300$.

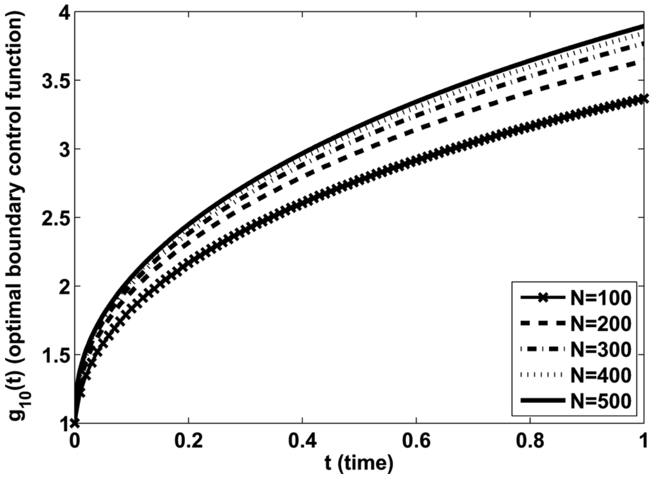

Figure 4 Dependence of the optimal boundary control on the variation of time step number for $\alpha = \kappa = 0.5$.

that points out the solutions overlap for $m \geq 10$. We also take $N = 300$ and $\kappa = 0.5$ for Figure 2. In Figure 3, we analyze the dependence of the optimal boundary control on the nondimensional parameter κ. Finally, we evaluate the change of time step number N using the discretization of integral given by Eq. (46) for the values $\alpha = \kappa = 0.5$ in Figure 4.

CONCLUSION

In this work, an optimal control problem for a temperature field defined by a time-fractional heat subconduction equation has been formulated. In the description of the problem, the Caputo fractional derivative has been used. The purpose was to take the thermal stress under control with an optimal boundary temperature function. Therefore, the problem constructed in [33] has been generalized by the usage of fractional tools. Successive iterations and linear approximation have been applied to calculate the solution numerically. MATLAB 7.1 was used to show the influence of nondimensional parameters on the solution.

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