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Coordinate Finite Type Rotational Surfaces in Euclidean Spaces

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Abstract. Submanifolds of coordinate finite-type were introduced in [10]. A submanifold of a Euclidean space is called a coordinate finite-type submanifold if its coordinate functions are eigenfunctions of Δ. In the present study we consider coordinate finite-type surfaces in $\mathbb{E}^4$. We give necessary and sufficient conditions for generalized rotation surfaces in $\mathbb{E}^4$ to become coordinate finite-type. We also give some special examples.

1. Introduction

Let M be a connected $n-$dimensional submanifold of a Euclidean space $\mathbb{E}^m$ equipped with the induced metric. Denote Δ by the Laplacian of M acting on smooth functions on M . This Laplacian can be extended in a natural way to $\mathbb{E}^m$ valued smooth functions on M. Whenever the position vector x of M in $\mathbb{E}^m$ can be decomposed as a finite sum of $\mathbb{E}^m$-valued non-constant functions of Δ, one can say that M is of *finite type*. More precisely the position vector x of M can be expressed in the form $x = x_0 + \sum_{i=1}^{k} x_i$, where x_0 is a constant map $x_1, x_2, ..., x_k$ non-constant maps such that $\Delta x = \lambda_i x_i$, $\lambda_i \in \mathbb{R}$, $1 \leq i \leq k$. If $\lambda_1, \lambda_2, ..., \lambda_k$ are different, then M is said to be of k-type. Similarly, a smooth map ϕ of an n-dimensional Riemannian manifold M of $\mathbb{E}^m$ is said to be of finite type if ϕ is a finite sum of $\mathbb{E}^m$-valued eigenfunctions of Δ ([2], [3]). For the position vector field $\overrightarrow{H}$ of M it is well known (see eg. [3]) that $\Delta x = -n\overrightarrow{H}$, which shows in particular that M is a minimal submanifold in $\mathbb{E}^m$ if and only if its coordinate functions are harmonic. In [13] Takahasi proved that an n-dimensional submanifold of $\mathbb{E}^m$ is of 1-type (i.e., $\Delta x = \lambda x$) if and only if it is either a minimal submanifold of $\mathbb{E}^m$ or a minimal submanifold of some hypersphere of $\mathbb{E}^m$. As a generalization of T. Takahashi's condition, O. Garay considered in [8], submanifolds of Euclidean space whose position vector field x satisfies the differential equation $\Delta x = Ax$, for some $m \times m$ diagonal matrix A with constant entries. Garay called such submanifolds *coordinate finite type submanifolds*. Actually coordinate finite type submanifolds are finite type submanifolds whose type number s are at most m. Each coordinate function of a coordinate finite type submanifold m is of 1-type, since it is an eigenfunction of the Laplacian [10].

In [7] by G. Ganchev and V. Milousheva considered the surface M generated by a W-curve γ in $\mathbb{E}^4$. They have shown that these generated surfaces are a special type of rotation surfaces which are introduced first by C. Moore in 1919 (see [12]). Vranceanu surfaces in $\mathbb{E}^4$ are the special type of these surfaces [14].

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This paper is organized as follows: Section 2 gives some basic concepts of the surfaces in $\mathbb{E}^4$. Section 3 tells about the generalised surfaces in $\mathbb{E}^4$. Further this section provides some basic properties of surfaces in $\mathbb{E}^4$ and the structure of their curvatures. In the final section we consider coordinate finite type surfaces in euclidean spaces. We give necessary and sufficient conditions for generalised rotation surfaces in $\mathbb{E}^4$ to become coordinate finite type.

2. Basic Concepts

Let M be a smooth surface in $\mathbb{E}^n$ given with the patch $X(u,v) : (u,v) \in D \subset \mathbb{E}^2$. The tangent space to M at an arbitrary point $p = X(u,v)$ of M span $\{X_u, X_v\}$. In the chart (u,v) the coefficients of the first fundamental form of M are given by

$$E =< X_u, X_u >, F = \langle X_u, X_v \rangle, G = \langle X_v, X_v \rangle, \tag{1}$$

where $\langle , \rangle$ is the Euclidean inner product. We assume that $W^2 = EG - F^2 \neq 0$, i.e. the surface patch $X(u,v)$ is regular. For each $p \in M$, consider the decomposition $T_p\mathbb{E}^n = T_pM \oplus T_p^{\perp}M$ where $T_p^{\perp}M$ is the orthogonal component of T_pM in $\mathbb{E}^n$. Let $\tilde{\nabla}$ be the Riemannian connection of $\mathbb{E}^4$. Given orthonormal local vector fields X_1, X_2 tangent to M.

Let $\chi(M)$ and $\chi^{\perp}(M)$ be the space of the smooth vector fields tangent to M and the space of the smooth vector fields normal to M, respectively. Consider the second fundamental map: $h : \chi(M) \times \chi(M) \to \chi^{\perp}(M)$;

$$h(X_i, X_j) = \widetilde{\nabla}_{X_i} X_j - \nabla_{X_i} X_j \quad 1 \leq i, j \leq 2. \tag{2}$$

where $\widetilde{\nabla}$ is the induced. This map is well-defined, symmetric and bilinear.

For any arbitrary orthonormal normal frame field $\{N_1, N_2, ..., N_{n-2}\}$ of M, recall the shape operator $A : \chi^{\perp}(M) \times \chi(M) \to \chi(M)$;

$$A_{N_i} X_j = -(\widetilde{\nabla}_{X_j} N_k)^T, \quad X_j \in \chi(M),\ 1 \leq k \leq n-2 \tag{3}$$

This operator is bilinear, self-adjoint and satisfies the following equation:

$$\left\langle A_{N_k} X_j, X_i \right\rangle = \left\langle h(X_i, X_j), N_k \right\rangle = h_{ij}^k,\ 1 \leq i, j \leq 2. \tag{4}$$

The equation (2) is called Gaussian formula, and

$$h(X_i, X_j) = \sum_{k=1}^{n-2} h_{ij}^k N_k, \quad 1 \leq i, j \leq 2 \tag{5}$$

where c_{ij}^k are the coefficients of the second fundamental form.

Further, the Gaussian and mean curvature vector of a regular patch $X(u,v)$ are given by

$$K = \sum_{k=1}^{n-2} (h_{11}^k h_{22}^k - (h_{12}^k)^2), \tag{6}$$

and

$$H = \frac{1}{2} \sum_{k=1}^{n-2} (h_{11}^k + h_{22}^k) N_k, \tag{7}$$

respectively, where h is the second fundamental form of M. Recall that a surface M is said to be *minimal* if its mean curvature vector vanishes identically [2]. For any real function f on M the Laplacian of f is defined by

$$\Delta f = -\sum_i (\widetilde{\nabla}_{e_i} \widetilde{\nabla}_{e_i} f - \widetilde{\nabla}_{\nabla_{e_i} e_i} f). \tag{8}$$

3. Generalised Rotation Surfaces in $\mathbb{E}^4$

Let $\gamma = \gamma(s) : I \to \mathbb{E}^4$ be a W-curve in Euclidean 4-space $\mathbb{E}^4$ parametrized as follows:

$$\gamma(v) = (a\cos cv, a\sin cv, b\cos dv, b\sin dv),\ 0 \le v \le 2\pi,$$

where a, b, c, d are constants ($c > 0, d > 0$). In [7] G. Ganchev and V. Milousheva considered the surface M generated by the curve γ with the following surface patch:

$$X(u,v) = (f(u)\cos cv, f(u)\sin cv, g(u)\cos dv, g(u)\sin dv), \tag{9}$$

where $u \in J, 0 \le v \le 2\pi$, $f(u)$ and $g(u)$ are arbitrary smooth functions satisfying

$$c^2 f^{\,2} + d^2 g^2 > 0 \text{ and } (f\,')^2 + (g\,')^2 > 0.$$

These surfaces are first introduced by C. Moore in [12] , called *general rotation surfaces.* Note that X_u and X_v are always orthogonal and therefore we choose an orthonormal frame $\{e_1, e_2, e_3, e_4\}$ such that e_1, e_2 are tangent to M and e_3, e_4 normal to M in the following (see, [7]):

$$\begin{aligned} e_1 &= \frac{X_u}{\|X_u\|},\ e_2 = \frac{X_v}{\|X_u\|} \\ e_3 &= \frac{1}{\sqrt{(f')^2 + (g')^2}}(g'\cos cv, g'\sin cv, -f'\cos dv, -f'\sin dv), \\ e_4 &= \frac{1}{\sqrt{c^2f^2 + d^2g^2}}(-dg\sin cv, dg\cos cv, cf\sin dv, -cf\cos dv). \end{aligned} \tag{10}$$

Hence the coefficients of the first fundamental form of the surface are

$$\begin{aligned} E &= \langle X_u, X_u\rangle = (f')^2 + (g')^2 \\ F &= \langle X_u, X_v\rangle = 0 \\ G &= \langle X_v, X_v\rangle = c^2f^2 + d^2g^2 \end{aligned} \tag{11}$$

where $\langle , \rangle$ is the standard scalar product in $\mathbb{E}^4$. Since

$$EG - F^2 = \left((f')^2 + (g')^2\right)\left(c^2f^2 + d^2g^2\right)$$

does not vanish, the surface patch $X(u,v)$ is regular. Then with respect to the frame field $\{e_1, e_2, e_3, e_4\}$, the Gaussian and Weingarten formulas (2)-(3) of M look like (see, [6]);

$$\begin{aligned} \tilde{\nabla}_{e_1}e_1 &= -A(u)e_2 + h^1_{11}e_3, \\ \tilde{\nabla}_{e_1}e_2 &= A(u)e_1 + h^2_{12}e_4, \\ \tilde{\nabla}_{e_2}e_2 &= h^1_{22}e_3, \\ \tilde{\nabla}_{e_2}e_1 &= h^2_{12}e_4, \end{aligned} \tag{12}$$

and

$$\begin{aligned} \tilde{\nabla}_{e_1}e_3 &= -h^1_{11}e_1 + B(u)e_4, \\ \tilde{\nabla}_{e_1}e_4 &= -h^2_{12}e_2 - B(u)e_3, \\ \tilde{\nabla}_{e_2}e_3 &= -h^1_{22}e_2, \\ \tilde{\nabla}_{e_2}e_4 &= -h^2_{12}e_1, \end{aligned} \tag{13}$$

where

$$\begin{aligned}
A(u) &= \frac{c^2 f f' + d^2 g g'}{\sqrt{(f')^2+(g')^2}(c^2f^2+d^2g^2)}, \\
B(u) &= \frac{cd(ff'+gg')}{\sqrt{(f')^2+(g')^2}(c^2f^2+d^2g^2)}, \\
h_{11}^1 &= \frac{d^2f'g - c^2fg'}{\sqrt{(f')^2+(g')^2}(c^2f^2+d^2g^2)}, \\
h_{22}^1 &= \frac{g'f''-f'g''}{((f')^2+(g')^2)^{\frac{3}{2}}}, \\
h_{12}^2 &= \frac{cd(f'g-fg')}{\sqrt{(f')^2+(g')^2}(c^2f^2+d^2g^2)}, \\
h_{11}^2 &= h_{22}^2 = h_{12}^1 = 0.
\end{aligned} \tag{14}$$

are the differentiable functions. Using (6)-(7) with (14) one can get the following results;

Proposition 3.1. *[1] Let M be a generalised rotation surface given by the parametrization (9), then the Gaussian curvature of M is*

$$K = \frac{(c^2f^2+d^2g^2)(g'f''\text{-}f'g'')(d^2gf'\text{-}c^2fg')\text{-}c^2d^2(gf'\text{-}fg')^2((f')^2+(g')^2)}{((f')^2+(g')^2)^2(c^2f^2+d^2g^2)^2}.$$

An easy consequence of Proposition 3.1 is the following.

Corollary 3.2. *[1] The generalised rotation surface given by the parametrization (9) has vanishing Gaussian curvature if and only if the following equation*

$$(c^2f^2+d^2g^2)(g'f''\text{-}f'g'')(d^2gf'\text{-}c^2fg')\text{-}c^2d^2(gf'\text{-}fg')^2((f')^2+(g')^2) = 0,$$

holds.

The following results are well-known;

Proposition 3.3. *[1] Let M be a generalised rotation surface given by the parametrization (9), then the mean curvature vector of M is*

$$\begin{aligned}
\overrightarrow{H} &= \frac{1}{2}(h_{11}^1 + h_{22}^1)e_3 \\
&= \left(\frac{(c^2f^2+d^2g^2)(g'f''-f'g'') + (d^2gf'-c^2fg')((f')^2+(g')^2)}{2((f')^2+(g')^2)^{3/2}(c^2f^2+d^2g^2)}\right)e_3.
\end{aligned}$$

An easy consequence of Proposition 3.3 is the following.

Corollary 3.4. *[1] The generalised rotation surface given by the parametrization (9) is minimal surface in $\mathbb{E}^4$ if and only if the equation*

$$(c^2f^2+d^2g^2)(g'f''-f'g'') + (d^2gf'-c^2fg')((f')^2+(g')^2) = 0,$$

holds.

Definition 3.5. *The generalised rotation surface given by the parametrization*

$$f(u) = r(u)\cos u,\ g(u) = r(u)\sin u,\ c = 1, d = 1. \tag{15}$$

is called Vranceanu rotation surface in Euclidean 4-space $\mathbb{E}^4$ [14].

Remark 3.6. *Substituting (15) into the equation given in Corollary 3.2 we obtain the condition for Vranceanu rotation surface which has vanishing Gaussian curvature;*

$$r(u)r''(u) - (r'(u))^2 = 0. \tag{16}$$

Further, and easy calculation shows that $r(u) = \lambda e^{\mu u}$, $(\lambda, \mu \in R)$ is the solution is this second degree equation. So, we get the following result.

Corollary 3.7. *[15] Let M is a Vranceanu rotation surface in Euclidean 4-space. If M has vanishing Gaussian curvature, then $r(u) = \lambda e^{\mu u}$, where λ and μ are real constants. For the case, $\lambda = 1$, $\mu = 0, r(u) = 1$, the surface M is a Clifford torus, that is it is the product of two plane circles with same radius.*

Corollary 3.8. *[1] Let M is a Vranceanu rotation surface in Euclidean 4-space. If M is minimal then*

$$r(u)r''(u) - 3(r'(u))^2 - 2r(u)^2 = 0.$$

holds.

Corollary 3.9. *[1] Let M is a Vranceanu rotation surface in Euclidean 4-space. If M is minimal then*

$$r(u) = \frac{\pm 1}{\sqrt{a \sin 2u - b \cos 2u}}, \tag{17}$$

where, a and b are real constants.

Definition 3.10. *The surface patch $X(u, v)$ is called pseudo-umbilical if the shape operator with respect to H is proportional to the identity (see, [2]). An equivalent condition is the following:*

$$< h(X_i, X_j), H >= \lambda^2 < X_i, X_j >, \tag{18}$$

where, $\lambda = \|H\|$. It is easy to see that each minimal surface is pseudo-umbilical.

The following results are well-known;

Theorem 3.11. *[1] Let M be a generalised rotation surface given by the parametrization (9) is pseudo-umbilical then*

$$(c^2 f^2 + d^2 g^2)(g' f'' - f' g'') - (d^2 g f' - c^2 f g')((f')^2 + (g')^2) = 0. \tag{19}$$

The converse statement of Theorem 3.11 is also valid.

Corollary 3.12. *[1] Let M be a Vranceanu rotation surface in Euclidean 4-space. If M pseudo-umbilical then $r(u) = \lambda e^{\mu u}$, where λ and μ are real constants.*

3.1. Coordinate Finite Type Surfaces in Euclidean Spaces

In the present section we consider coordinate finite type surfaces in Euclidean spaces $\mathbb{E}^{n+2}$. A surface *M* in Euclidean *m*-space is called coordinate finite type if the position vector field *X* satisfies the differential equation

$$\Delta X = AX, \tag{20}$$

for some $m \times m$ diagonal matrix *A* with constant entries. Using the Beltrami formula's $\Delta X = -2\overrightarrow{H}$, with (7) one can get

$$\Delta X = -\sum_{k=1}^{n} (h_{11}^k + h_{22}^k) N_k. \tag{21}$$

So, using (20) with (21) the coordinate finite type condition reduces to

$$AX = -\sum_{k=1}^{n}(h_{11}^{k} + h_{22}^{k})N_k \tag{22}$$

For a non-compact surface in $\mathbb{E}^4$ O.J.Garay obtained the following:

Theorem 3.13. *[9] The only coordinate finite type surfaces in Euclidean 4-space $\mathbb{E}^4$ with constant mean curvature are the open parts of the following surfaces:*
i) a minimal surface in $\mathbb{E}^4$,
ii) a minimal surface in some hypersphere $S^3(r)$,
iii) a helical cylinder,
iv) a flat torus $S^1(a) \times S^1(b)$ in some hypersphere $S^3(r)$.

3.2. Surface of Revolution of Coordinate Finite Type

A surface in $\mathbb{E}^3$ is called a surface of revolution if it is generated by a curve C on a plane Π when Π is rotated around a straight line L in Π. By choosing Π to be the xz-plane and line L to be the x axis the surface of revolution can be parameterized by

$$X(u, v) = (f(u), g(u)\cos v, g(u)\sin v), \tag{23}$$

where $f(u)$ and $g(u)$ are arbitrary smooth functions. We choose an orthonormal frame $\{e_1, e_2, e_3\}$ such that e_1, e_2 are tangent to M and e_3 normal to M in the following:

$$e_1 = \frac{X_u}{\|X_u\|},\ e_2 = \frac{X_v}{\|X_v\|},\ e_3 = \frac{1}{\sqrt{(f')^2 + (g')^2}}(g', -f'\cos v, -f'\sin v), \tag{24}$$

By covariant differentiation with respect to e_1, e_2 a straightforward calculation gives

$$\begin{aligned} \tilde{\nabla}_{e_1}e_1 &= h_{11}^{1}e_3, \\ \tilde{\nabla}_{e_2}e_2 &= -A(u)e_1 + h_{22}^{2}e_3, \\ \tilde{\nabla}_{e_2}e_1 &= A(u)e_2, \\ \tilde{\nabla}_{e_1}e_2 &= 0, \end{aligned} \tag{25}$$

where

$$\begin{aligned} A(u) &= \frac{g'}{g\sqrt{(f')^2 + (g')^2}}, \\ h_{11}^{1} &= \frac{g'f'' - f'g''}{((f')^2 + (g')^2)^{\frac{3}{2}}}, \\ h_{22}^{1} &= \frac{f'}{g\sqrt{(f')^2 + (g')^2}}, \\ h_{12}^{1} &= 0. \end{aligned} \tag{26}$$

are the differentiable functions. Using (6)-(7) with (26) one can get

$$\overrightarrow{H} = \frac{1}{2}\left(h_{11}^{1} + h_{22}^{1}\right)e_3 \tag{27}$$

where h_{11}^{1} and h_{22}^{1} are the coefficients of the second fundamental form given in (26).

A surface of revolution defined by (23) is said to be of polynomial kind if $f(u)$ and $g(u)$ are polynomial functions in u and it is said to be of rational kind if f is a rational function in g, i.e., f is the quotient of two polynomial functions in g [4].

For finite type surfaces of revolution B.Y. Chen and S. Ishikawa obtained in [5] the following results;

Theorem 3.14. *[5] Let M be a surface of revolution of polynomial kind. Then M is a surface of finite type if and only if either it is an open portion of a plane or it is an open portion of a circular cylinder.*

Theorem 3.15. *[5] Let M be a surface of revolution of rational kind. Then M is a surface of finite type if and only if M is an open portion of a plane.*

T. Hasanis and T. Vlachos proved the following.

Theorem 3.16. *[10] Let M be a surface of revolution. If M has constant mean curvature and is of finite type then M is an open portion of a plane, of a sphere or of a circular cylinder.*

We proved the following result;

Lemma 3.17. *Let M be a surface of revolution given with the parametrization (23). Then M is a surface of coordinate finite type if and only if diagonal matrix A is of the form*

$$A = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix} \tag{28}$$

where

$$\begin{aligned} a_{11} &= \frac{-g'(g\left(g'f'' - f'g''\right) + f'\left((f')^2 + (g')^2\right))}{fg\left((f')^2 + (g')^2\right)^2} \\ a_{22} &= a_{33} = \frac{f'\left(g(g'f'' - f'g'') + f'\left((f')^2 + (g')^2\right)\right)}{g^2\left((f')^2 + (g')^2\right)^2} \end{aligned} \tag{29}$$

are constant functions.

Proof. Assume that the surface of revolution M given with the parametrization (23). Then, from the equality (21)

$$\Delta X = -(h^1_{11} + h^1_{22})e_3. \tag{30}$$

Further, substituting (26) into (30) and using (24) we get the

$$\Delta X = \psi \begin{bmatrix} g' \\ -f'\cos v \\ -f'\sin v \end{bmatrix} \tag{31}$$

where

$$\psi = -\frac{g(g'f'' - f'g'') + f'\left((f')^2 + (g')^2\right)}{g\left((f')^2 + (g')^2\right)^2}$$

is differentiable function. Similarly, using (23) we get

$$AX = \begin{bmatrix} a_{11}f \\ a_{22}g\cos v \\ a_{33}g\sin v \end{bmatrix}. \tag{32}$$

Since, M is coordinate finite type then from the definition it satisfies the equality $AX = \Delta X$. Hence, using (31) and (32) we get the result. □

Remark 3.18. *If the diagonal matrix A is equivalent to a zero matrix then M becomes minimal. So the surface of revolution M is either an open portion of a plane or an open portion of a catenoid.*

Minimal rotational surfaces are of coordinate finite type.

For the non-minimal case we obtain the following result;

Theorem 3.19. *Let M be a non-minimal surface of revolution given with the parametrization (23). If M is coordinate finite type surface then*

$$ff' + \lambda g g\prime = 0 \tag{33}$$

holds, where λ is a nonzero constant.

Proof. Since the entries a_{11}, a_{22} and a_{33} of the diagonal matrix A are real constants then from the equality (29) one can get the following differential equations

$$\begin{aligned}
\frac{-g'\left(g(g'f'' - f'g'') + f'\left((f')^2 + (g')^2\right)\right)}{fg\left((f')^2 + (g')^2\right)^2} &= c_1 \\
\frac{f'\left(g(g'f'' - f'g'') + f'\left((f')^2 + (g')^2\right)\right)}{g^2\left((f')^2 + (g')^2\right)^2} &= c_2.
\end{aligned}$$

where c_1, c_2 are nonzero real constants. Further, substituting one into another we obtain the result. □

Example 3.20. *The round sphere given with the parametrization $f(u) = r\cos u$, $g(u) = r\sin u$ satisfies the equality (33). So it is a coordinate finite type surface.*

Example 3.21. *The cone $f(u) = g(u)$ satisfies the equality (33). So it is a coordinate finite type surface.*

3.3. Generalised Rotation Surfaces of Coordinate Finite Type

In the present section we consider generalised rotation surfaces of coordinate finite type surfaces in Euclidean 4-spaces $\mathbb{E}^4$.

We proved the following result;

Lemma 3.22. *Let M be a generalised rotation surface given with the parametrization (9). Then M is a surface of coordinate finite type if and only if diagonal matrix A is of the form*

$$A = \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 \\ 0 & 0 & a_{33} & 0 \\ 0 & 0 & 0 & a_{44} \end{bmatrix} \tag{34}$$

where

$$\begin{aligned}
a_{11} = a_{22} &= \frac{-g'\left(\left(d^2f'g - c^2fg'\right)\left((f')^2 + (g')^2\right) + \left(g'f'' - f'g''\right)\left(c^2f^2 + d^2g^2\right)\right)}{f\left((f')^2 + (g')^2\right)^2\left(c^2f^2 + d^2g^2\right)}, \\
a_{33} = a_{44} &= \frac{f'\left(\left(d^2f'g - c^2fg'\right)\left((f')^2 + (g')^2\right) + \left(g'f'' - f'g''\right)\left(c^2f^2 + d^2g^2\right)\right)}{g\left((f')^2 + (g')^2\right)^2\left(c^2f^2 + d^2g^2\right)},
\end{aligned} \tag{35}$$

are constant functions.

Proof. Assume that the generalised rotation surface given with the parametrization (9). Then, from the equality (21)

$$\Delta X = -(h^1_{11} + h^1_{22})e_3 - (h^2_{11} + h^2_{22})e_4. \tag{36}$$

Further, substituting (14) into (36) and using (10) we get the

$$\Delta X = \varphi \begin{bmatrix} g' \cos cv \\ g' \sin cv \\ -f' \cos dv \\ -f' \sin dv \end{bmatrix} \tag{37}$$

where

$$\varphi = -\frac{\left(d^2 f' g - c^2 f g'\right)\left((f')^2 + (g')^2\right) + (g' f'' - f' g'')\left(c^2 f^2 + d^2 g^2\right)}{\left((f')^2 + (g')^2\right)^2 \left(c^2 f^2 + d^2 g^2\right)}$$

is differentiable function. Also using (9) we get

$$AX = \begin{bmatrix} a_{11} f \cos cv \\ a_{22} f \sin cv \\ a_{33} g \cos dv \\ a_{44} g \sin dv \end{bmatrix}. \tag{38}$$

Since, M is coordinate finite type then from the definition it satisfies the equality $AX = \Delta X$. Hence, using (37) and (38) we get the result. □

If he matrix A is a zero matrix then M becomes minimal. So minimal rotational surfaces are of coordinate finite type.

For the non-minimal case we obtain the following result;

Theorem 3.23. *Let M be a generalised rotation surface given by the parametrization (9). If M is a coordinate finite type then*

$$f f' = \mu g g\prime$$

holds, where, μ is a real constant.

Proof. Since the entries a_{11}, a_{22}, a_{33} and a_{44} of the diagonal matrix A are real constants then from the equality (29) one can get the following differential equations

$$\frac{-g'\left(\left(d^2 f' g - c^2 f g'\right)\left((f')^2 + (g')^2\right) + \left(g' f'' - f' g''\right)\left(c^2 f^2 + d^2 g^2\right)\right)}{f\left((f')^2 + (g')^2\right)^2 \left(c^2 f^2 + d^2 g^2\right)} = d_1,$$
$$\frac{f'\left(\left(d^2 f' g - c^2 f g'\right)\left((f')^2 + (g')^2\right) + \left(g' f'' - f' g''\right)\left(c^2 f^2 + d^2 g^2\right)\right)}{g\left((f')^2 + (g')^2\right)^2 \left(c^2 f^2 + d^2 g^2\right)} = d_2,$$

where d_1, d_2 are nonzero real constants. Further, substituting one into another we obtain the result. □

An easy consequence of Theorem 3.23 is the following.

Corollary 3.24. *Let M be a Vranceanu rotation surface in Euclidean 4-space. If M is a coordinate finite type, then*

$$r r' \left(\cos^2 u - c \sin^2 u\right) = r^2 \cos u \sin u (1 + c)$$

holds, where, c is a real constant.

In [11] C. S. Houh investigated Vranceanu rotation surfaces of finite type and proved the following

Theorem 3.25. *[11] A flat Vranceanu rotation surface in $\mathbb{E}^4$ is of finite type if and only if it is the product of two circles with the same radius, i.e. it is a Clifford torus.*

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