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Optimal control of a linear time-invariant space–time fractional diffusion process

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Abstract

This paper presents a formulation and numerical solutions of an optimal control problem of a linear time-invariant space–time fractional diffusion equation. The main aim of this formulation is minimization of a performance index, which is a functional of both state and control functions of the diffusion system. The dynamics of the system are defined by the space–time fractional diffusion equation in the sense of Caputo and fractional Laplacian operators. The separation of variables technique and a spectral representation of a fractional Laplacian operator are applied to determine the eigenfunctions that represent the space parameters. Therefore, the state and control functions are defined by linear infinite combinations of eigenfunctions. Optimality conditions described by Euler–Lagrange equations are found by using a Lagrange multiplier technique. The Grünwald–Letnikov definition is used to approximate to the time fractional derivative. The applicapability and effectiveness of the numerical scheme are shown by comparison of analytical and numerical solutions for a numerical example. Finally, the variations of problem parameters are analyzed, with some figures obtained using MATLAB.

Keywords

Caputo derivative, fractional Laplacian operator, fractional optimal control problem, Grünwald–Letnikov approximation

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1. Introduction

Fractional calculus was born in 1965 when Leibniz and L'Hospital had correspondence where they tried to find the meaning of a derivative of order 1/2. It led to a paradox, from which one day useful consequences will be drawn. After this date, many famous mathematicians, including Laplace, Fourier, Abel, Liouville, Riemann, Grünwald, Letnikov, Levy, Marchaud, Erdelyi and Riesz, have been interested in this basic question and related considerations. However, especially in recent years, there has been growing interest in the applications of fractional calculus in many areas of science, engineering, finance and mathematics (Debnath, 2003; Machado et al., 2010, 2011). It has been recognized that many physical systems should be modeled more accurately by using fractional order operators than integer order ones, i.e. many researchers have pointed out that fractional derivatives and integrals are very suitable to define the memory and hereditary properties of materials and processes in the real physical world.

One of the application topics of fractional calculus is modeling of dynamical systems and their optimal control problems (OCPs), which are also considered in the present paper. A system whose dynamics are described by fractional differential equations is called a fractional dynamical system. In addition, the OCP of such systems is defined as a fractional optimal control problem (FOCP). As the demand for an accurate definition of dynamical systems increases, research about formulations and numerical solution schemes for FOCPs also increases. Therefore, it can be seen from the literature that studies related to FOCPs have grown rapidly.

A FOCP is an OCP in which the performance index and/or dynamic constraints of the system contain at least one fractional order derivative term. The first formulation and solution scheme for FOCP was studied by Agrawal (2004). In this work, Agrawal defined the general formulation in terms of the Riemann–Liouville (RL) fractional derivatives and used an approximation

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based on variational virtual work coupled with the Lagrange multiplier technique to find numerical solutions of FOCPs.

In addition, the formulations of FOCPs are developed out of fractional variational calculus, which is used to obtain Hamiltonian, Lagrangian and Euler–Lagrange equations, and other notions for the mechanics of systems (Baleanu et al., 2008; Abdeljawad et al., 2009; Jarad et al., 2010a,b).

A direct numerical technique using the Grünwald–Letnikov (GL) definition was applied to obtain approximation of fractional derivatives that describe the dynamics of the system by Agrawal and Baleanu (2007). Baleanu et al. (2009) also proposed a modified numerical scheme for a class of FOCPs that were formulated by Agrawal (2004). Biswas and Sen (2011a) presented a direct numerical technique based on the GL approximation for an OCP formulation and the solution of a fractional order system using a pseudo-state-space formulation. Frederico and Torres (2006, 2008a,b) used Agrawal's Euler–Lagrange equation and the Lagrange multiplier technique to obtain a Noether-like theorem for FOCPs in the terms of the Caputo fractional derivative and researched fractional conservation laws for FOCPs. Agrawal (2008a) analyzed a FOCP for a type of distributed system whose dynamics are defined in terms of the Caputo fractional derivative and also used eigenfunctions to define the problem in terms of a set of state and control variables. The main advantage of this consideration, which is also used for the present study, is that the main FOCP reduces to a multiFOCPs that are solved independently. Tangpong and Agrawal (2009) extended the numerical scheme, which was used for the scalar case in Agrawal (2008b), to the vector case, and the formulation was used to solve a continuum FOCP for different values of fractional order and different space–time discretization. Tricaud and Chen (2010a) introduced a formulation in which a rational approximation based on the Hankel data matrix of the impulse response was considered for fractional time OCPs, which are known as special classes of FOCPs. Tricaud and Chen (2010b) also developed a method to find the solution of FOCP by means of rational approximation, and proved that their methodology could be applied any type of FOCP such as linear/nonlinear, time-invariant/time-variant, Single Input Single Output (SISO)/Multiple Input Multiple Output (MIMO), state/input constrained, free terminal conditions, etc. Recently, Mophou and N'Guerekata (2011) considered OCPs of a fractional diffusion equation with the state constraint in a bounded domain. Dorville et al. (2011) studied an OCP for a nonhomogeneous Dirichlet boundary fractional diffusion equation. Yousefi et al. (2011) used Legendre multiwavelets, together with the collocation method to obtain the approximate solutions of FOCPs and also validate the applicability of this technique by using an illustrative example. Wang and Zhou (2011a) researched the existence of mild solutions that are associated with the probability density function and the semigroup property for semilinear fractional evolution equations and considered the Lagrange problem of such systems. Wang and Zhou (2011b) also studied approximate solutions of time optimal control for fractional evolution systems in Banach spaces using the method of reducing the main problem to a sequence of Meyer problems.

The above-mentioned studies are very constitutive examples of one-dimensional (1D) FOCPs in Cartesian coordinates, but there is certainly other work about FOCPs for multidimensional cases and in different coordinate systems. Özdemir et al. (2009a) presented analytical and numerical solutions of FOCPs for a distributed system in two-dimensional Cartesian space and customized the main problem in polar coordinates by Özdemir et al. (2009b). Moreover, FOCPs were formulated in cylindrical coordinates by Özdemir et al. (2009c).

In this work, we research the exact and numerical solutions of such a FOCP that system dynamics are defined by a space–time fractional differential equation in terms of the Caputo and the fractional Laplacian operators. The organization of this work is as follows. In Section 2, we give some basic mathematical definitions and relations that are necessary for our formulation. In Section 3, we explain the general formulation of a FOCP in the literature. In Section 4, we formulate our main problem in terms of eigenfunctions obtained by a spectral representation for fractional Laplacian operators, the Lagrange multiplier technique and calculus of variations, and then find a set of time fractional differential equations. In Section 5, we calculate the analytical solution for the order of time derivative $\alpha = 1$. In Section 6, we apply the GL approximation to fractional differential equations obtained in Section 4. Moreover, we show the physical behavior of our problem and interpret the results for an initial condition function by the help of figures in this section. Finally, we conclude our work.

2. Mathematical tools

We briefly give some basic definitions and mathematical relations of fractional calculus that are necessary for our formulation. In fractional calculus, there are different definitions of the fractional derivative operators: RL, Caputo, GL, Weyl, Marchaud, Riesz, etc. (Oldham and Spanier, 1974; Miller and Ross, 1993; Samko et al., 1993; Podlubny, 1999; Kilbas et al., 2006). It is important that these operators are not separated from each other. Note that there is much more work related to the mathematical relations of these operators. For example, Ortigueira (2006, 2008) considered fractional centered differences, which led to centered derivatives similar to the GL derivatives, and proposed integral representations for these differences that revealed the generalizations of Riesz potential operators.

In this study, we formulate our problem in terms of the Caputo derivative and fractional Laplacian operators. From the physical and engineering point of view, the Caputo definition for the time fractional derivative is commonly preferred. This is because it is well known that the Caputo derivative of a constant is zero in every condition. It means that the Caputo definition can give a physical interpretation to any problem formulation. However, it is not always possible to say the same for the RL definition. This is because the RL derivative of a constant is not zero every time. This situation does not have a physical meaning in theory. Nevertheless, it is possible to find many well-organized works formulated in terms of the RL since this operator has a good mathematical construction.

In the literature, one can find much work related to space–time fractional diffusion equations. In these articles, the authors define the time fractional derivative in the sense of the RL and Caputo definitions. In addition, the space derivative is considered in terms of Riesz, Riesz–Feller and fractional Laplacian operators. A large number of papers are related to the analytical and numerical solutions of space–time fractional differential equations (Gorenflo et al., 1998; Ciesielski and Leszczynski, 2003, 2005, 2006; Huang and Liu, 2005; Ilic et al., 2005, 2006; Yang et al., 2009, 2010; Özdemir et al., 2011; Shen et al., 2011). For example, the Laplace and the Fourier transform methods are often used to find the exact solutions (see Povstenko (2011)). In addition, numerical techniques such as finite difference approximations, and matrix transform methods are numerous. For the space derivative term, it is important that some papers in the literature show the equivalence between the fractional Laplacian and Riesz operators in the infinite domain and the 1D case. However, in multidimensional spaces and different coordinate systems, one cannot have such a kind of relationship. For example, fractional Laplacian operators have invariant properties similar to standard Laplacian operators, and so it is possible to define this operator in different curvilinear coordinate systems. In addition, the eigenfunction expansion method can be applied to fractional differential equations in terms of fractional Laplacian operators. However, the Riesz and Riesz–Feller operators do not have invariant properties and so we cannot define these terms in different coordinate systems, such as cylindrical, spherical, polar, etc. The foundation of eigenfunctions of these operators is still an open problem in fractional calculus literature.

In this work, we define the space term in the sense of a fractional Laplacian operator in one dimension and in a finite domain. It allows us to use the eigenfunctions throughout the problem formulation. To obtain the numerical solutions, we first take into account the basic relationship between the RL and the Caputo derivatives, then we approximate the Caputo term with the GL definition.

The basic definitions and relations of our work are as follows.

Definition 1. *The operator ${}_0D_1^\alpha$ Caputo fractional derivative of order α $(n-1<\alpha\leq n)$ is defined as*

$$\left({}_0D_1^\alpha f\right)(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^1(t-\tau)^{n-\alpha-1}\frac{d^n f(\tau)}{d\tau^n}\,d\tau.$$

Ilic et al. (2005) produced a spectral representation for the foundation of eigenvalues and eigenfunctions that belong to the fractional Laplacian operator. Here, we give the outline of this work to show the starting point of our consideration. By using this spectral representation, we propose the optimal control of a space–time fractional diffusion problem in the present paper. It is a new construction for the FOCPs, as it can be seen in the literature that the system dynamics are only defined with time fractional differential equations. Here, we also consider the optimal control of a space–time fractional diffusion system.

Suppose the Laplacian $(-\Delta)$ has a complete set of orthonormal eigenfunctions φ_n corresponding to eigenvalues λ_n^2 on a bounded region $\mathcal{D}$, i.e. $(-\Delta)\varphi_n=\lambda_n^2\varphi_n$ on $\mathcal{D}$; $\mathcal{B}(\varphi)=0$ on $\partial\mathcal{D}$, where $\mathcal{B}(\varphi)$ is one of the standard three homogeneous boundary conditions (Dirichlet, Neumann and Robin). Let

$$\mathcal{F}_\gamma=\left\{f=\sum_{n=1}^{\infty}c_n\varphi_n,\ c_n=\langle f,\varphi_n\rangle\ \middle|\ \sum_{n=1}^{\infty}|c_n|^2|\lambda_n|^\gamma<\infty,\ \gamma=\max(\alpha,0)\right\}.$$

Then, for any $f\in\mathcal{F}_\gamma$, $(-\Delta)^{\frac{\alpha}{2}}$ is defined by

$$(-\Delta)^{\frac{\alpha}{2}}f=\sum_{n=1}^{\infty}c_n\lambda_n^\alpha\varphi_n.$$

Let $\mathcal{H}$ be the real Hilbert space $L_2(0,\mathrm{L})$ with the usual inner product. Consider the operator $T:\mathcal{H}\to\mathrm{H}$ defined by $T\varphi=-\frac{d^2\varphi}{dx^2}=-\Delta\varphi$ on

$$\mathcal{H}=\left\{\varphi\in\mathrm{H};\varphi'\text{ is absolutely continuous},\varphi',\ \varphi''\in L_2(0,\mathrm{L}),\mathcal{B}(\varphi)=0\right\},$$

where $\mathcal{B}(\varphi)$ is one of the boundary conditions mentioned above. It is known that T is a closed, self-adjoint operator whose eigenfunctions $\{\varphi_n\}_{n=1}^{\infty}$ form an orthonormal basis for H. Thus, $T\varphi_n = \lambda_n \varphi_n$, $n = 1, 2, \ldots$ For any $\varphi \in$ H,

$$\varphi = \sum_{n=1}^{\infty} c_n \varphi_n,\, c_n = \langle \varphi, \varphi_n \rangle,$$

$$T\varphi = \sum_{n=1}^{\infty} \lambda_n c_n \varphi_n.$$

If ψ is a continuous function on $\mathbb{R}$, then

$$\psi(T)\varphi = \sum_{n=1}^{\infty} \psi(\lambda_n) c_n \varphi_n,$$

provided $\sum_{n=1}^{\infty} \left|\psi(\lambda_n)c_n\right|^2 < \infty$.

Hence, if the eigenvalue problem for T can be solved explicitly, then the following problem can be easily solved, where $\psi(t) = t^{\frac{\alpha}{2}}$.

Problem: solve the following boundary value problem in one dimension

$$\frac{\partial \varphi}{\partial t} = -\kappa \left(-\frac{\partial^2}{\partial x^2} \right)^{\frac{\alpha}{2}} \varphi, \quad 0 < x < L,$$

with the initial condition

$$\varphi(x, 0) = g(x),$$

together with one of the homogeneous Dirichlet, Neumann and Robin boundary conditions. The detailed analytical and numerical solutions of this problem can be found in Ilic et al. (2005).

In addition, there is another main relationship between the Riesz fractional derivative and the symmetric space fractional derivative $-(-\Delta)^{\frac{\beta}{2}}$ of order β $(1 < \beta \leq 2)$, which is defined by Gorenflo and Mainardi (1998), where Δ is the well-known Laplacian operator. Yang et al. (2010) derived that the Riesz fractional derivative is equivalent to the fractional power of the Laplacian operator, that is $-(-\Delta)^{\frac{\beta}{2}} f(x,t) = \frac{\partial^\beta f(x,t)}{\partial |x|^\beta}$, by assuming homogeneous Dirichlet boundary conditions. More detailed analysis of a time and space-symmetric fractional diffusion equation from the physical and mathematical point of view is given in Yang et al. (2009).

The mathematical relationship between the RL and Caputo definitions is given by Podlubny (1999) as

$${}^{RL}_{a}D^{\alpha}_{x} f(x) = {}^{C}_{a}D^{\alpha}_{x} f(x) + \sum_{k=0}^{n-1} \frac{d^k}{dx^k} f(x) \mid_{x=a} \frac{(x-a)^{k-\alpha}}{\Gamma(k-\alpha+1)},$$

$${}^{RL}_{x}D^{\alpha}_{b} f(x) = {}^{C}_{x}D^{\alpha}_{b} f(x) + \sum_{k=0}^{n-1} \frac{d^k}{dx^k} f(x) \mid_{x=b} \frac{(b-x)^{k-\alpha}}{\Gamma(k-\alpha+1)}.$$

GL numerical approximation of the left and the right RL fractional derivatives at node M can be defined as

$${}^{RL}_{0}D^{\alpha}_{t} f \approx \tfrac{1}{h^\alpha} \sum_{j=0}^{M} w_j^{(\alpha)} f(hM - jh),$$
$${}^{RL}_{t}D^{\alpha}_{1} f \approx \tfrac{1}{h^\alpha} \sum_{j=0}^{N-M} w_j^{(\alpha)} f(hM + jh),$$

where

$$w_0^{(\alpha)} = 1, \quad w_j^{(\alpha)} = \left(1 - \tfrac{\alpha+1}{j}\right) w_{j-1}^{(\alpha)}$$

and N is the number of subdomains that have $h = \frac{1}{N}$ lengths.

3. General formulation of a FOCP

General formulation of a FOCP is given as follows. The main aim of a FOCP is foundation of a control function $u(t)$ that minimizes the performance index

$$J(u) = \int_0^t F(x, u, t) dt,$$

subject to the dynamic constraints of the system

$${}_0D^{\alpha}_{t} x = G(x, u, t),$$

and the initial condition

$$x(0) = x_0,$$

where $x(t)$ defines the state of the system, F and G are arbitrary constants or vector functions and ${}_0D^{\alpha}_{t}$ represents the fractional derivative operator that is chosen with respect to the problem type. Note that when $\alpha = 1$, the FOCP reduces to a standard OCP.

Let us give the definitions of the necessary equations for optimality that are basic formulations of FOCPs from fractional variational calculus. To take the necessary optimality equations, the Lagrange multiplier technique and calculus of variations are used. Then, optimality conditions are obtained as

$${}_0D^{\alpha}_{t} x = G(x, u, t), \tag{1}$$

$${}_tD^{\alpha}_{1} \lambda = \frac{\partial F}{\partial x} + \lambda \frac{\partial G}{\partial x}, \tag{2}$$

$$\frac{\partial F}{\partial u} + \lambda \frac{\partial G}{\partial u} = 0, \tag{3}$$

where λ is the Lagrange multiplier, which is also known as the co-state variable and

$$x(0) = x_0,\ \lambda(1) = 0. \tag{4}$$

We note that more details on construction of a FOCP formulation can be found in Agrawal and Baleanu (2007).

4. OCP of the space–time fractional diffusion system

In this section, it is possible to reformulate the general form of FOCPs under our considerations and assumptions. Let us consider the following 1D system. The objective of this paper is to find an optimal control that minimizes the following quadratic performance index

$$J(u) = \frac{1}{2}\int_0^1 \int_0^L \left[Ax^2(y,t) + Bu^2(y,t)\right]dydt, \tag{5}$$

subjected to the dynamic constraints

$${}_0^C D_t^{\alpha} x(y,t) = -K_\beta(-\Delta)^{\frac{\beta}{2}} x(y,t) + u(y,t), \tag{6}$$

where

$$0 < \alpha \le 1 \text{ and } 1 < \beta \le 2, \tag{7}$$

with the initial condition

$$x(y,0) = x_0(y) \tag{8}$$

and the boundary conditions

$$x(0,t) = x(L,t) = 0, \tag{9}$$

where $x(y,\ t)$ and $u(y,\ t)$ are state and control functions defined on the $\{(y,\ t)\ y \in [0,\ L]\ \Lambda\ t \in [0,\ 1]\}$ domain, A and B are arbitrary constant coefficients that are determined by a real physical problem, K_β denotes the anomalous diffusion coefficient that changes with respect to the type of the diffusion process, ${}_0^C D_t^{\alpha}$ represents the well-known Caputo fractional derivative and $-(-\Delta)^{\frac{\beta}{2}}$ is the fractional Laplacian operator. Note that we consider a fixed final time and free final state FOCP with an integral cost function in this problem. This consideration can be changed with respect to the purpose of the problem formulation. For example, Biswas and Sen (2011b) took into account both cases of fixed and free final states for a fixed final time and obtained a general transversality condition due to the inclusion of a terminal cost function in the performance index.

First, we solve the following eigenvalue problem under the consideration of $x(y,\ t) = Y(y)T(t)$

$$(-\Delta)Y(y) = \mu Y(y), \tag{10}$$

with the boundary condition $Y(L) = 0$. Therefore, we assume that $x(y,\ t)$ and $u(y,\ t)$ functions can be represented by the following series expansions

$$x(y,t) = \sum_{n=1}^{m} x_n(t)\sin\left(\frac{n\pi y}{L}\right), \tag{11}$$

$$u(y,t) = \sum_{n=1}^{m} u_n(t)\sin\left(\frac{n\pi y}{L}\right), \tag{12}$$

where $\sin\left(\frac{n\pi y}{L}\right)$, $n = 1,\ 2,\ldots,\ m$, are the eigenfunctions that are found by solution of equation (10) . Moreover, the eigenvalues are obtained as $\mu_n = \frac{n^2\pi^2}{L^2}$, $n = 1,\ 2,\ldots,\ m$. It is important to emphasize that eigenfunctions are used to determine the space parameter of the problem and therefore, the distributed FOCP is reduced to a set of decoupled FOCPs that can be solved independently. In addition, equation (11) naturally satisfies the boundary condition given by equation (9) . By using the definition of the $(-\Delta)^{\frac{\beta}{2}}$ operator, we get

$$(-\Delta)^{\frac{\beta}{2}} x(y,t) = \sum_{n=1}^{m} x_n(t)\left(\frac{n^2\pi^2}{L^2}\right)^{\frac{\beta}{2}} \sin\left(\frac{n\pi y}{L}\right). \tag{13}$$

We must also emphasize that the upper limit of the series given by equations (11) and (12) is taken as a finite number m for computational reasons, whereas theoretically m should go to infinity. Let us determine the necessary optimality conditions with some manipulation. Firstly, we substitute equations (11) and (12) into equation (5) and therefore obtain

$$J(u) = \frac{L}{4}\int_0^1 \left\{\sum_{n=1}^{m}\left[Ax_n^2(t) + Bu_n^2(t)\right]\right\}dt. \tag{14}$$

By substituting equations (12) and (13) into equation (6) , we take

$${}_0^C D_t^{\alpha} x_n(t) = -\left(\frac{n^2\pi^2}{L^2}\right)^{\frac{\beta}{2}} x_n(t) + u_n(t) \quad (n = 1, 2, \ldots, m). \tag{15}$$

Substituting $x(y, t)$ in equation (11) into equation (8) and after some computation, the initial values for $x_n(t)$ are

$$x_n(0) = \frac{2}{L}\int_0^L x_0(\xi)\sin\left(\frac{n\pi\xi}{L}\right)d\xi. \tag{16}$$

Let us now rewrite the n components of $F(x, u, t)$ and $G(x, u, t)$ functions

$$F(x_n, u_n, t) = \frac{L}{4}\left[Ax_n^2(t) + Bu_n^2(t)\right], \tag{17}$$

$$G(x_n, u_n, t) = -\left(\frac{n^2\pi^2}{L^2}\right)^{\frac{\beta}{2}} x_n(t) + u_n(t), \tag{18}$$

and so the necessary optimality conditions of our problem are rearranged with respect to equations (1) to (3) by

$${}_0^C D_t^{\alpha} x_n(t) = -\left(\frac{n^2\pi^2}{L^2}\right)^{\frac{\beta}{2}} x_n(t) + u_n(t), \tag{19}$$

$${}_t^C D_1^{\alpha} \lambda_n(t) = A\frac{L}{2}x_n(t) - \left(\frac{n^2\pi^2}{L^2}\right)^{\frac{\beta}{2}} \lambda_n(t), \tag{20}$$

$$B\frac{L}{2}u_n(t) + \lambda_n(t) = 0 \tag{21}$$

and

$$x_n(0) = x_{n0}, \lambda_n(1) = 0 \quad (n = 1, 2, \ldots, m),$$

where $\lambda_n(t)$, $n = 1, 2, \ldots, m$, are the Lagrange multipliers. Using equations (20) and (21), we also obtain

$${}_t^C D_1^{\alpha} u_n(t) = -\frac{A}{B}x_n(t) - \left(\frac{n^2\pi^2}{L^2}\right)^{\frac{\beta}{2}} u_n(t). \tag{22}$$

After these calculations, we will take into account equations (15) and (22) for analytical and numerical solutions and then let us remember the equations

$$\begin{cases} {}_0^C D_t^{\alpha} x_n(t) = -\left(\frac{n^2\pi^2}{L^2}\right)^{\frac{\beta}{2}} x_n(t) + u_n(t), \\ {}_t^C D_1^{\alpha} u_n(t) = -\frac{A}{B}x_n(t) - \left(\frac{n^2\pi^2}{L^2}\right)^{\frac{\beta}{2}} u_n(t). \end{cases} \tag{23}$$

We will give analytical solution of equations (23) in the next section.

5. Analytical solution

To show the efficiency and applicability of the numerical method to such a type of problem, we first obtain the exact (i.e. analytical) solutions of $x_n(t)$ and $u_n(t)$ from equations (23) for $\alpha = 1$ as

$$\begin{aligned} \frac{d}{dt}x_n(t) &= -\left(\frac{n^2\pi^2}{L^2}\right)^{\frac{\beta}{2}} x_n(t) + u_n(t), \\ \frac{d}{dt}u_n(t) &= \frac{A}{B}x_n(t) + \left(\frac{n^2\pi^2}{L^2}\right)^{\frac{\beta}{2}} u_n(t) \quad (n = 1, 2, \ldots, m). \end{aligned} \tag{24}$$

For simplicity, we rename the coefficients of equations (24) and rewrite these equations

$$\begin{aligned} \frac{d}{dt}x_n(t) &= -E_n x_n(t) + u_n(t), \\ \frac{d}{dt}u_n(t) &= Fx_n(t) + E_n u_n(t) \quad (n = 1, 2, \ldots, m), \end{aligned} \tag{25}$$

where

$$F = \frac{A}{B}, E_n = \left(\frac{n^2\pi^2}{L^2}\right)^{\frac{\beta}{2}}, \tag{26}$$

and terminal conditions are

$$x_n(0) = x_{n0}, \quad u_n(1) = 0. \tag{27}$$

After some well-known manipulations in classical calculus, we obtain the analytical solutions of the eigencoordinates

$$x_n(t) = x_{n0}\frac{\left[(K_n - E_n)e^{-K_n(1-t)} - (K_n + E_n)e^{K_n(1-t)}\right]}{\left[((K_n - E_n)e^{-K_n} + (K_n + E_n)e^{K_n})\right]} \tag{28}$$

and

$$u_n(t) = Fx_{n0}\frac{\left[e^{-K_n(1-t)} - e^{K_n(1-t)}\right]}{\left[((K_n - E_n)e^{-K_n} + (K_n + E_n)e^{K_n})\right]}, \tag{29}$$

where

$$K_n = \sqrt{E_n^2 + F} \quad (n = 1, 2, \ldots, , m).$$

Consequently, by substituting equations (28) and (29) into equations (11) and (12) , respectively, we can obtain the analytical solutions of $x(y, t)$ and $u(y, t)$.

6. Illustrative example

In this section, we obtain approximate solutions of the problem by choosing an initial condition. To solve the problem numerically, we use the GL approximation for the Caputo derivative. It is well known that the GL approximation is based on discretization of the time interval into subintervals with fixed lengths. For this purpose, the entire domain is divided into N subdomains whose length are $h = \frac{1}{N}$, and each node is numbered 0, 1,..., N. Thus, let us analyze the numerical solutions of equations (23) under an initial condition function and variable order of α and β . By applying the GL definition to the relationship equations of the Caputo and the RL derivatives, the approximation of the Caputo derivative is obtained as

$$ {}_0^C D_{Mh}^{\alpha} x_n(Mh) \approx \frac{1}{h^{\alpha}} \sum_{j=0}^{M} w_j^{(\alpha)} x_n(Mh - jh) - x_n(0) \frac{[Mh]^{-\alpha}}{\Gamma(1-\alpha)}, \tag{30} $$

$$ {}_{Mh}^{C} D_1^{\alpha} u_n(Mh) \approx \frac{1}{h^{\alpha}} \sum_{j=0}^{N-M} w_j^{(\alpha)} u_n(Mh + jh) - \frac{u_n(Nh)([N-M]h)^{-\alpha}}{\Gamma(1-\alpha)}. \tag{31} $$

The initial condition function is chosen arbitrarily as

$$ x_0(y) = y^2(\pi - y), \tag{32} $$

which is applied to a numerical example in Yang et al. (2009). Note that it is possible to take different types of

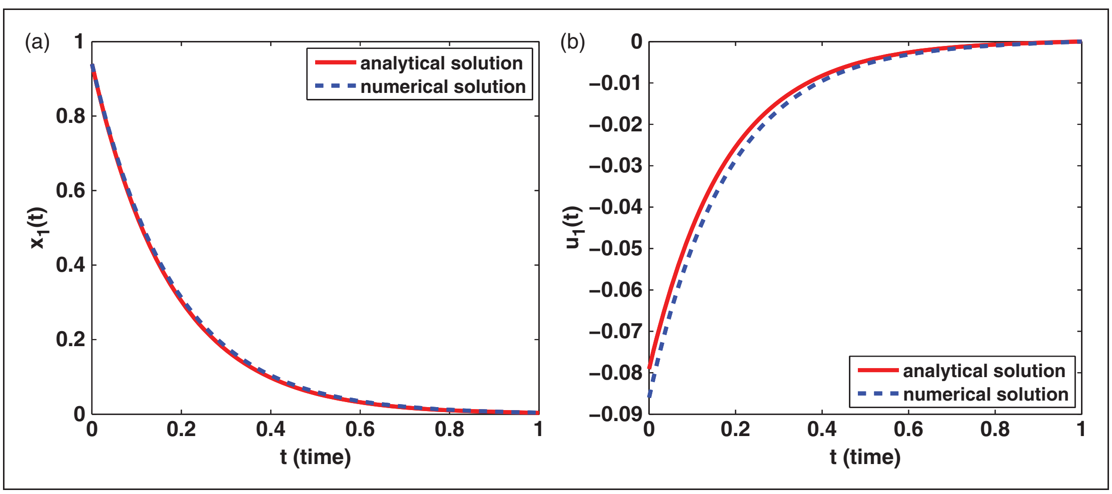

Figure 1. Comparison of the analytical and numerical solutions for $\alpha = 1$, $\beta = 1.5$ and $N = 100$.

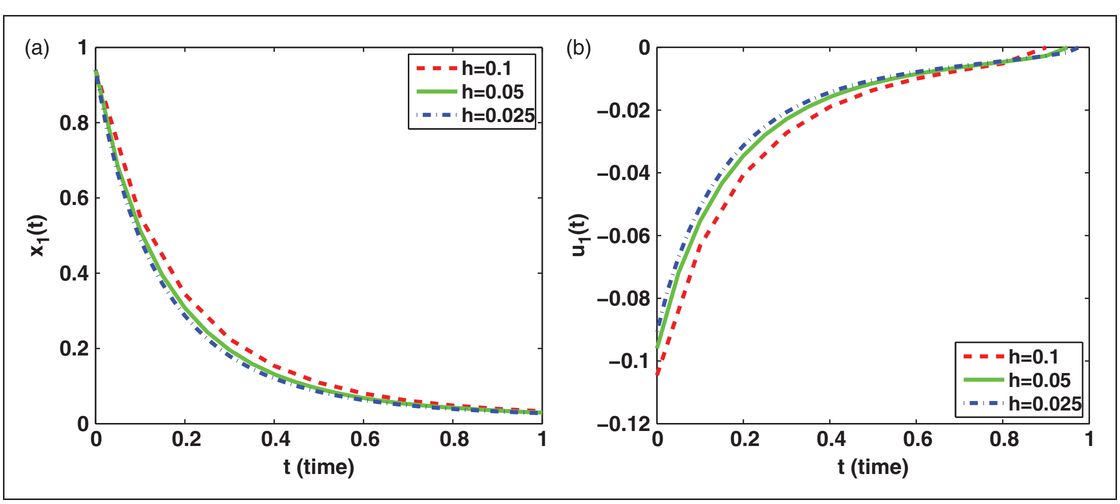

Figure 2. Contribution of the step sizes to the $x_1(t)$ and $u_1(t)$ for $\alpha = 0.9$, $\beta = 1.5$.

initial condition functions. We also assume the problem coefficients are $A = B = K_\beta = L = 1$ only for simplicity purposes. By substituting the initial condition function into the equation (16) , we obtain the initial condition $x_n(0)$ for equation (30) . Moreover, we take the final value of u_n as $u_n(1) = u_n(Nh) = 0$. Firstly, we compare the analytical and numerical solutions of the state $x_1(t)$ and control $u_1(t)$ components of the solution for $\alpha = 1$, $\beta = 1.5$ and term number $N = 100$ in Figure 1(a) and 1(b), respectively. The good agreement of analytical and numerical results shows that the applicability of the GL approximation is effective for such a problem. In Figure 2(a) and 2(b), we validate the effect of the variation of step sizes for $\alpha = 0.9$ and $\beta = 1.5$. For this purpose, we take the values of time step sizes as $h = 0.1$, 0.05, 0.025. While the time step sizes $h = \frac{1}{N}$ are decreasing, the smoothness is increasing for both state and control functions. We obtain the variation of the lower values of the α parameter for $\beta = 1.5$ and $N = 500$ in Figure 3(a) and 3(b). Similarly, we consider the higher values of the α parameter for $\beta = 1.5$ and $N = 100$ in Figure 4(a) and 4(b). In addition, the variation of β parameters when $\alpha = 1$ and $N = 100$ is shown in Figure 5(a) and 5(b). Up to now, we analyze the variations of the parameter only on $x_1(t)$ and $u_1(t)$ components of state and control functions. However, the whole solution of the main problem given by equations (11) and (12) is represented by the linear sum of $x_n(t)$ and $u_n(t)$. So, we change the values of $n = 1, \ldots, 5$ and take $\alpha = 0.9$, $\beta = 1.5$ and $N = 100$. Therefore, we show the contribution of term numbers to the solution in Figure 6(a) and 6(b). Finally, the surface of the state $x(y,t)$ for $\alpha = 0.5$, $\beta = 1.5$, $N = 100$ is given in Figure 7(a). Similarly, we obtain the whole solution

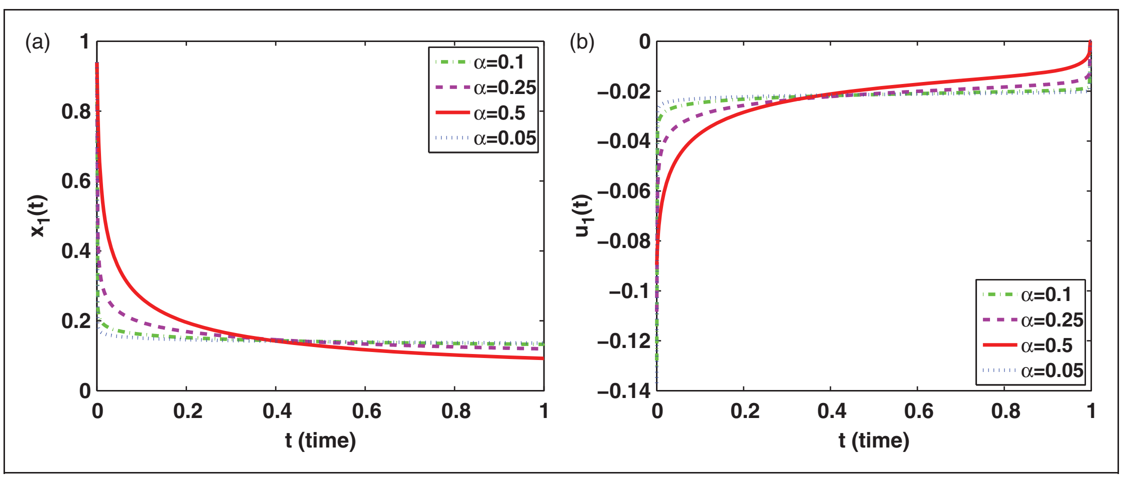

Figure 3. Dependence of solution on lower values of α parameter for $\beta = 1.5$ and $N = 500$.

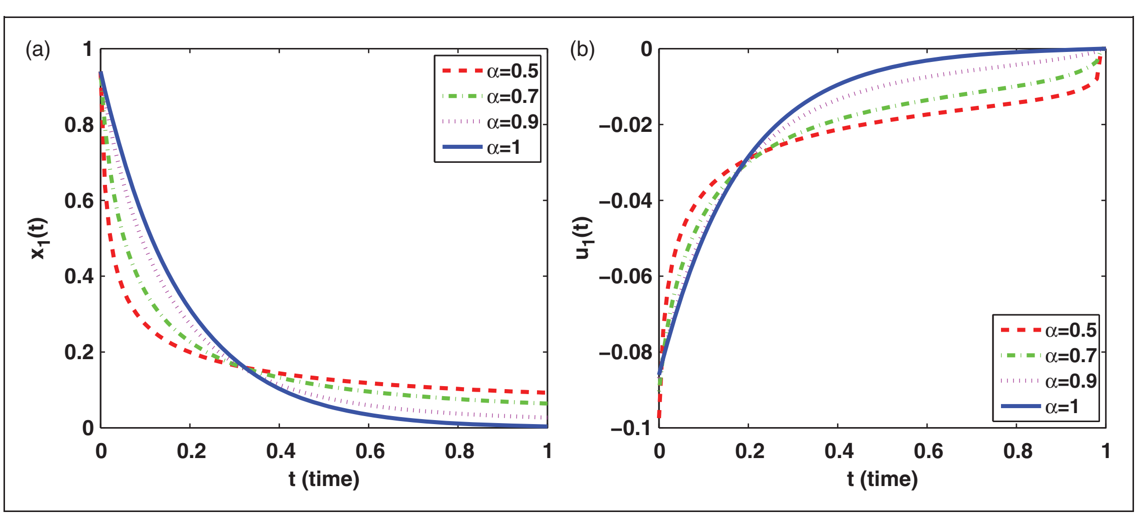

Figure 4. Dependence of solution on higher values of α parameter for $\beta = 1.5$ and $N = 100$.

of $u(y, t)$ for the values of $\alpha = 0.99$, $\beta = 1.9$ and $N = 100$ in Figure 7(b). Thus, we characterize the behaviors of $x(y, t)$ and $u(y, t)$ in three-dimensional space.

7. Conclusions

In this work, optimal control of a space–time fractional diffusion equation in a 1D domain has been proposed. The minimization of a quadratic performance index was the aim. Dynamic constraints of the system on which the problem is formulated have been defined in terms of the Caputo time and fractional Laplacian space operators. In general, one can find some papers related to FOCPs whose dynamics are defined only with time fractional derivatives in the sense of the Caputo or the RL operators. However, anomalous diffusion processes are described by the space–time fractional differential equations. For this reason, consideration of OCPs for an anomalous diffusion process is a new viewpoint that is analyzed in the present paper. The eigenfunctions of the fractional Laplacian operator has been used to eliminate the state and control functions. To obtain numerical results, a GL approximation has been applied to an illustrative example. The validity of this numerical scheme has been analyzed by comparison of analytical and numerical solutions. MATLAB has been used for plots of the variation of the problem parameters. In addition, the

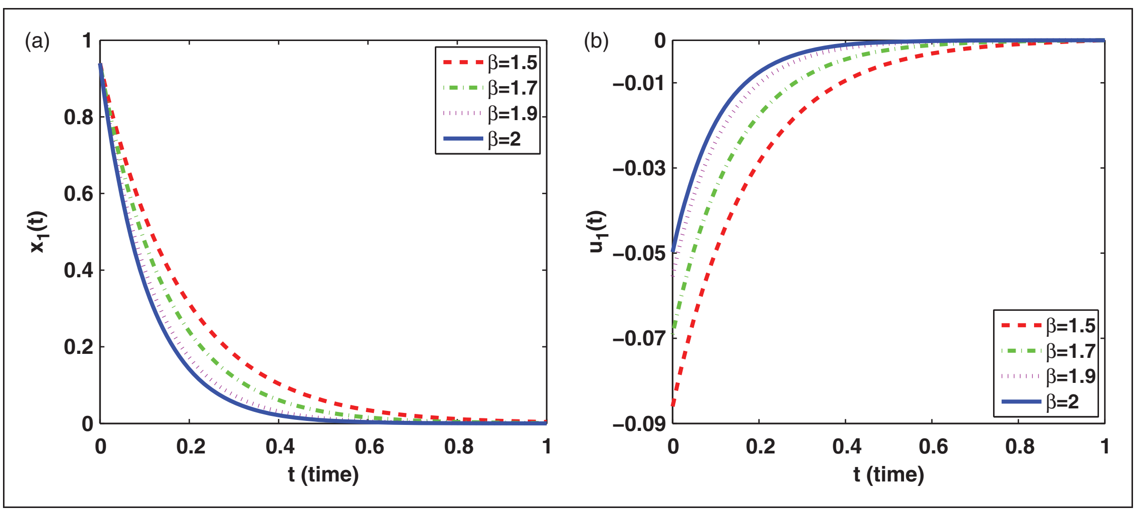

Figure 5. Dependence of solution on higher values of β parameter for $\alpha = 1$ and $N = 100$.

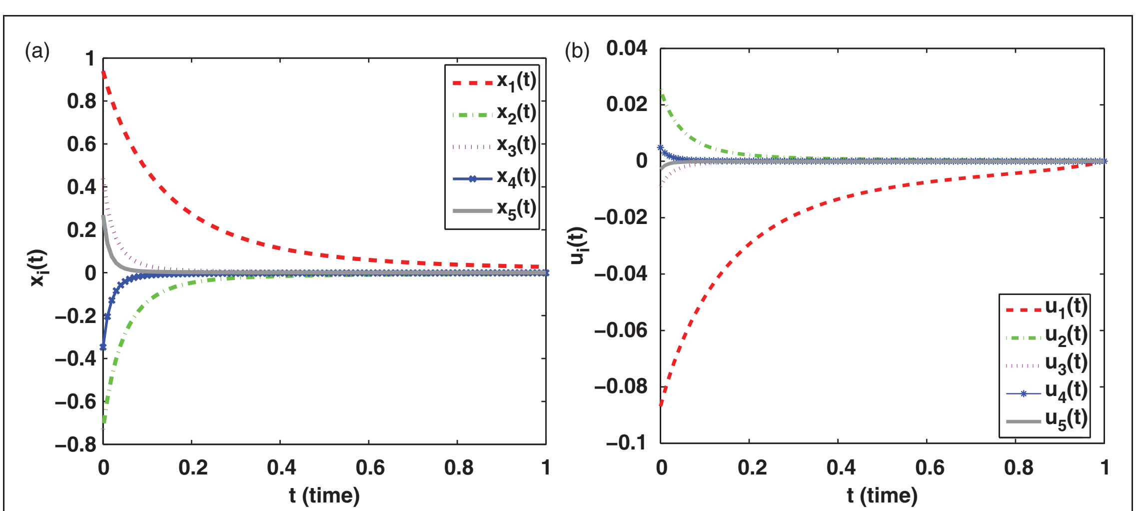

Figure 6. Contribution of term number to the $x(y, t)$ and $u(y, t)$ for $\alpha = 0.9$, $\beta = 1.5$ and $N = 100$.

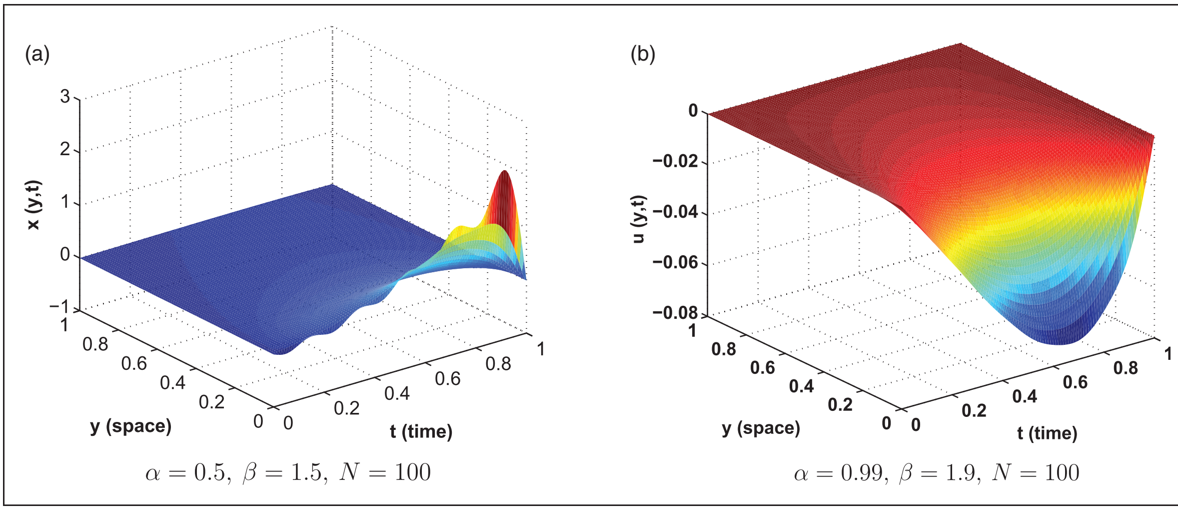

Figure 7. Surfaces of the state and control functions.

effectiveness of the numerical approximation for such a type of problem has been shown with the figures.

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