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A DYNAMIC SYSTEM APPROACH FOR SOLVING NONLINEAR PROGRAMMING PROBLEMS WITH EXACT PENALTY FUNCTION

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Abstract: The Dynamic system has attracted increasing attention in recent years. In this paper, a dynamic system approach for solving Nonlinear Programming (NLP) problems with inequality constrained is presented. First, the system of differential equations based on exact penalty function is constructed. Furthermore, it is found that the equilibrium point of the dynamic system is converge to an optimal solution of the original optimization problem and is asymptotically stable in the sense of Lyapunov. Moreover, the Euler scheme is used for solving differential equations system. Finally, two practical examples are illustrated the effectiveness of the proposed dynamic system formulation.

Keywords: nonlinear programming, exact penalty function, dynamic system, lyapunov function.

1. Introduction

Nonlinear Programming (NLP) problems are commonly encountered in modern science and technology. Most of efficient methods have been developed for solving NLP problems. The method of penalty function has the attractive tool to solve the problems of NLP for many years. More extensive knowledge about them can be found in (Luenberger, 1973) and (Sun, 2006). Also differential equation methods are an alternative approach to solution of these problems. In this type of methods an optimization problem is formulated as a system of ordinary differential equations (ODEs) so that the equilibrium point of this system converges to the local minimum of the original problem.

The methods based on ODE for solving optimization problems have been proposed by Arrow and Hurwicz (Arrow, 1956), Fiacco and Mccormick (Fiacco, 1968) and Evtushenko (Evtushenko, 1994). However, Brown and Biggs (Brown, 1989) and Schropp (Schropp, 2000) were shown that ODE based methods for constrained optimization can be perform better than some conventional methods. Recently, Wang (Wang *et al.*, 2003) and Jin (Jin, 2007) have prepared a differential equation approach for solving NLP problems.

In this paper, we presented a differential equation approach for solving inequality-constrained optimization problems. This approach shows that the stable equilibrium point of the dynamic system is also an optimal solution of the corresponding optimization problem.

The rest of paper is organized as follow three sections. In Section 2, we will construct a dynamic system model based on exact penalty function (Meng *et al.*, 2004b). Moreover, we will prove that the system can converge to the optimal solution of the constrained optimization problem and discuss the stability of equilibrium point. A Lyapunov function is set up during the procedure. In Section 3, two illustrative examples will be given to emphasize the effectiveness of the proposed system. In Section 4, conclusions will be given.

2. A New Dynamic System Approach

Let us consider the following nonlinear programming problem

$$\begin{aligned} &\text{minimize} \quad f(x) \\ &\text{subject to} \quad g_i(x) \le 0 \quad i = 1,2,3,\ldots,m \end{aligned} \tag{1}$$

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where $x = (x_1, x_2, ..., x_n)^T \in R^n$, $f : R^n \to R$ and $g = (g_1, g_2, ..., g_m)^T : R^n \to R^m$ $(m \le n)$. The functions f and g are assumed to be convex and twice continuously differentiable. The objective penalty function of (1) is defined as

$$F(x,M) = (f(x) - M)^2 + \sum_{i=1}^{m} g_i^+(x)^2 \tag{2}$$

where $F : R^n \times R \to R$, $M \in R$ and $g_i^+(x) = \max\{0, g_i(x)\}$, $i = 1,2,...,m$. Therefore, the unconstrained optimization problem is define as follows

$$\underset{x \in R^n}{\text{minimize}}\ F(x,M). \tag{3}$$

The objective penalty function has been investigated by Meng in the reference (Meng *et al.*, 2004b). The authors proved that the objective penalty function $F(x,M)$ is exact if there is some M^* such that an optimal solution of (3) is also an optimal solution of (1) for any given $M \ge M^*$. Furthermore, they introduced a neural network for solving NLP problems based on exact penalty function, see Liu and Meng (Liu, 2003a).

By straightforward calculation, the gradient and Hessian matrix of exact penalty function are given as, respectively

$$\nabla_x F(x,M) = 2(f(x) - M)\nabla_x f(x) + 2\sum_{i=1}^{m} g_i^+(x)\nabla_x g_i(x)$$

$$\nabla_M F(x,M) = -2(f(x) - M)$$

$$\nabla_{xx}^2 F(x,M) = 2\nabla_x f(x)\nabla_x f(x)^T + 2(f(x) - M)\nabla_{xx}^2 f(x) + 2\sum_{i=1}^{m}\nabla_x g_i(x)\nabla_x g_i(x)^T + 2\sum_{i=1}^{m} g_i^+(x)\nabla_{xx}^2 g_i(x).$$

In order to solve the unconstrained optimization problem (3), the new dynamic system is defined by the following system of differential equations

$$\frac{dx}{dt} = -2(f(x) - M)\nabla_x f(x) - 2\sum_{i=1}^{m} g_i^+(x)\nabla_x g_i(x) \tag{4a}$$

$$\frac{dM}{dt} = 2(f(x) - M). \tag{4b}$$

Definition 1 A point (x^*, M^*) is called an equilibrium point of (4) if it satisfies the right hand side of the equations (4a) and (4b).

In here, we are going to prove that the equilibrium point of the corresponding dynamic system (4) is equivalent to the optimal solution of the problem (3) and vice versa.

Theorem 1 If (x^*, M^*) is the equilibrium point of the dynamic system (4), then x^* is an optimal solution to the unconstrained optimization problem (3).

Proof Since (x^*, M^*) is the equilibrium point of the dynamic system (4), we have

$$\frac{dx^*}{dt} = -2\left(f(x^*) - M^*\right)\nabla_x f(x^*) - 2\sum_{i=1}^{m} g_i^+(x^*)\nabla_x g_i(x^*) = 0 \tag{5}$$

$$\frac{dM^*}{dt} = 2\left(f(x^*) - M^*\right) = 0. \tag{6}$$

So clearly first order necessary condition is satisfied. To obtain the second order sufficient optimality condition we have to show that the Hessian matrix of the exact penalty function is positive definite. From the equations (5) and (6), we get

$$\sum_{i=1}^{m} g_i^+(x^*)\nabla_x g_i(x^*) = 0 \text{ and } f(x^*) = M^*. \tag{7}$$

Consequently, using the equations (7) the Hessian matrix is positive definite and so x^* is the optimal solution to the problem (3).

Theorem 2 If x^* is the optimal solution to the unconstrained optimization problem (3) for the penalty parameter M^*, then (x^*, M^*) is the equilibrium point of the dynamic system (4).
Proof According to the assumption, first order necessary condition is hold. That is

$$\nabla_x F(x^*, M^*) = 0 \quad \text{or} \quad \frac{dx^*}{dt} = 0. \tag{8}$$

Furthermore, in (Meng *et al.*, 2004b), it is shown that x^* is also a feasible solution for the problem (1). It means $g_i^+(x) = \max\{0, g_i(x)\} = 0$. From the equation (8) and the feasibility of x^*, we can say that $\frac{dM^*}{dt} = 0$. Thus (x^*, M^*) is the equilibrium point of the dynamic system (4).

2.1. Stability Analysis

Now, it will be discussed the stability of the dynamic system (4). Necessary definitions and theorems for the Lyapunov stability theorem can be found in the references (La Salle, 1961) and (Curtain, 1977).
Definition 2 Let Ω be an neighbourhood of (x^*, M^*). A continuously differentiable function V is said to be a Lyapunov function at the state (x^*, M^*) for dynamic system (4) if satisfy the following conditions:

a) $V(x, M)$ is equal to zero for the equilibrium point (x^*, M^*),
b) $V(x, M)$ is positive definite over Ω some neighbourhood of (x^*, M^*),
c) $\frac{dV}{dt}$ is semi-negative definite over Ω some neighbourhood of (x^*, M^*).

It is well known that the equilibrium point (x^*, M^*) is not only stable for a Lyapunov function which is satisfies above conditions, but also it is asymptotically stable if there exists a Lyapunov function V satisfies $\frac{dV}{dt} < 0$ in some neighbourhood Ω.

Now we define a suitable Lyapunov function for dynamic system (4) corresponding to the exact penalty function as follows

$$V(x, M) = F(x, M) = \left(f(x) - M\right)^2 + \sum_{i=1}^{m} g_i^+(x)^2 .$$

Theorem 3 If (x^*, M^*) is an equilibrium point of the dynamic system (4), then (x^*, M^*) is asymptotically stable for (4).
Proof It is clear that from assumption and optimality conditions, first two conditions are provided. Moreover, by the derivative of $V(x, M)$ with respect to t, we have

$$\begin{aligned}
\frac{dV}{dt} &= \frac{\partial V}{\partial x}\frac{dx}{dt} + \frac{\partial V}{\partial M}\frac{dM}{dt} \\
&= \begin{bmatrix} \frac{\partial V}{\partial x} & \frac{\partial V}{\partial M} \end{bmatrix} \begin{bmatrix} \frac{dx}{dt} & \frac{dM}{dt} \end{bmatrix}^T = \begin{bmatrix} \frac{\partial V}{\partial x} & \frac{\partial V}{\partial M} \end{bmatrix} \begin{bmatrix} -\frac{\partial V}{\partial x} & -\frac{\partial V}{\partial M} \end{bmatrix}^T \\
&= -\left(\left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial M}\right)^2 \right) < 0.
\end{aligned}$$

As a result, (x^*, M^*) is asymptotically stable for the dynamic system (4).

3. Illustrative Examples

Example 1 Let us consider the following nonlinear programming problem (Hock, 1981)

$$\text{minimize}\quad f(x) = 0.5x_1^2 + x_2^2 - x_1x_2 - 7x_1 - 7x_2$$
$$\text{subject to}\quad g(x) = 4x_1^2 + x_2^2 - 25 \le 0\,.$$

The optimal solution is $x^* = (2,3)^T$ and the optimal value of the objective function $f(x^*) = -30$. For solving above problem, we convert it to an unconstrained optimization problem with exact penalty function (2)

$$F(x,M) = \left(0.5x_1^2 + x_2^2 - x_1x_2 - 7x_1 - 7x_2 - M\right)^2 + \max\left\{0, 4x_1^2 + x_2^2 - 25\right\}^2,$$

where $M \in R$ is objective penalty parameter. The corresponding differential equations system is

$$\begin{aligned}
\frac{dx_1}{dt} &= -2(x_1 - x_2 - 7)\left(0.5x_1^2 + x_2^2 - x_1x_2 - 7x_1 - 7x_2 - M\right) - 16x_1 \max\left\{0, 4x_1^2 + x_2^2 - 25\right\}\\
\frac{dx_2}{dt} &= -2(2x_2 - x_1 - 7)\left(0.5x_1^2 + x_2^2 - x_1x_2 - 7x_1 - 7x_2 - M\right) - 4x_2 \max\left\{0, 4x_1^2 + x_2^2 - 25\right\}\\
\frac{dM}{dt} &= 2\left(0.5x_1^2 + x_2^2 - x_1x_2 - 7x_1 - 7x_2 - M\right).
\end{aligned} \tag{9}$$

The Euler method is used to solve the differential equations system (9) with initial points $(1,-1)$, $(-1,1)$, $(2,-2)$, $(-2,2)$ and $M_0 = -100$ with step size $\nabla t = 0.0001$, simulation result shows the trajectory of dynamic system converges to its optimal solution in the Fig. 1.

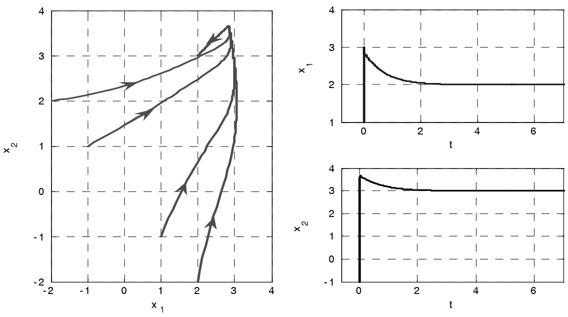

Fig. 1. Optimal solution for Example 1 with different initial points

Example 2 Consider the following nonlinear programming problem (Schittkowski, 1987)

$$\text{minimize}\quad f(x) = 100\left(x_2 - x_1^2\right)^2 + (1 - x_1)^2$$
$$\text{subject to}\quad g(x) = 0.25 - x_1^2 - x_2^2 \le 0.$$

The optimal solution of this problem is $x^* = (1,1)^T$ and the optimal value of the objective function $f(x^*) = 0$. Similar calculation as in Example 1, we have below objective penalty function (2) with objective penalty parameter $M \in R$,

$$F(x,M) = \left(100\left(x_2 - x_1^2\right)^2 + (1 - x_1)^2 - M\right)^2 + \max\left\{0, 0.25 - x_1^2 - x_2^2\right\}^2.$$

The corresponding differential equations system is

$$\begin{aligned}
\frac{dx_1}{dt} &= \left(800x_1\left(x_2 - x_1^2\right) + 4(1 - x_1)\right)\left(100\left(x_2 - x_1^2\right)^2 + (1 - x)^2 - M\right) + 4x_1 \max\left\{0, 0.25 - x_1^2 - x_2^2\right\}\\
\frac{dx_2}{dt} &= -400\left(x_2 - x_1^2\right)\left(100\left(x_2 - x_1^2\right)^2 + (1 - x)^2 - M\right) + 4x_2 \max\left\{0, 0.25 - x_1^2 - x_2^2\right\}\\
\frac{dM}{dt} &= 2\left(100\left(x_2 - x_1^2\right)^2 + (1 - x)^2 - M\right).
\end{aligned} \tag{10}$$

The differential equations system (10) is solved by using the Euler method. If we choose the initial values for $x_1(0)=0$, $x_2(0)=0$ and $M_0=-20$ with step size $\nabla t=0.00005$, simulation result gives the trajectory of dynamic system converges to its optimal solution in the Fig. 2.

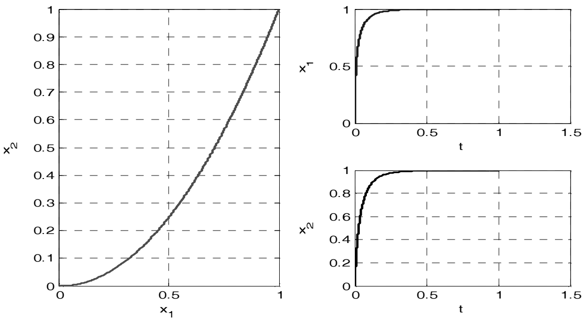

Fig. 2. Optimal solution for Example 2 with $M_0=-20$

4. Conclusions

We have considered a dynamic system approach for solving NLP problems (1) with inequality constrained in this work. Dynamic system model (4) has been constructed based on exact penalty function method (Meng et al., 2004). Through theoretical analysis and example computations, it has been shown that the equilibrium point of the dynamic system (4) can reach to the optimal solution of the NLP problem (1). In addition, the equilibrium point of the system was proved to be asymptotically stable in the sense of Lyapunov stability theorem. Hence, the simulation result illustrate that this approach is applicable.

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