BULLETIN *of the* *Bull . Malays . Math . Sci . Soc* $.(2)30(1)(2007), 37endash-four2$
MALAYSIAN MATHEMATICAL
SCIENCES SOCIETY
http : / / math . usm . my / bullet i n

On 2 - Quasi - Umbilical Pseudosymmetric Hypersurfaces in the Euclidean Space

$Cihan \ddot{O}_{zg\ddot{u}r}$

Balikesir University , Faculty of Art and Sciences ,
Department of Mathematics , 10 145 Balikesir , Turkey
cozgur @ balikesir . edu . tr

Abstract . In this paper , we investigate 2 - quasi - umbilical pseudosymmetric hy - persurfaces in the Euclidean space $\mathbb{E}^{n+1}$. We find the curvature characterization of pseudosymmetric hypersurfaces in the Euclidean space $\mathbb{E}n+1$.

2000 Mathematics Subject Classification : 53 C 40 , 53 C 42 , 53 C 25 , 53 B 50
Key words and phrases : Hypersurface , pseudosymmetry type manifolds .

1 . Introduction

Let (M,g) be an $n-$ dimensional , $n \geq 3$, connected Riemannian manifold of class C^{∞}. We denote by ∇, R, C, S and κ the Levi - Civita connection , the Riemann - Christoffel curvature tensor , the Weyl conformal curvature tensor , the Ricci tensor and the scalar curvature of (M,g) respectively . The Ricci operator $\mathcal{S}$ is defined by $g(\mathcal{S}X,Y)=S(X,Y)$, where $X,Y\in\chi(M), \chi(M)$ being Lie algebra of vector fields on M. We next define endomorphisms $X\wedge Y, \mathcal{R}(X,Y)$ and $\mathcal{C}(X,Y)$ of $\chi(M)$ by

$$(X\wedge Y)Z = g(Y,Z)X - g(X,Z)Y, \tag{1.1}$$

$$\mathcal{R}(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, \tag{1.2}$$

$$\mathcal{C}(X,Y)Z = \mathcal{R}(X,Y)Z - \frac{1}{n-2}(X\wedge\mathcal{S}Y + \mathcal{S}X\wedge Y - \frac{\kappa}{n-1}X\wedge Y)Z, \tag{1.3}$$

$$\text{respectively, where} X,Y,Z\in\chi(M).$$

The Riemannian Christoffel curvature tensor R and the Weyl conformal curvature tensor C of (M,g) are defined by

$$R(X,Y,Z,W) = g(\mathcal{R}(X,Y)Z,W), \tag{1.4}$$

$$C(X,Y,Z,W) = g(\mathcal{C}(X,Y)Z,W), \tag{1.5}$$

$$\text{respectively, where} W\in\chi(M).$$

Received : October 1 7 , 2005 ; *Accepted :* June 8 , 2006 .

For a $(0,k)-$ tensor field $T, k \geq 1$, on (M,g) we define the tensors $R.T$ and $Q(g,T)$ by

$$
\begin{aligned}
(R(X,Y)\cdot T)(X_1,...,X_k) &= -T(\mathcal{R}(X,Y)X_1,X_2,...,X_k) \\
&\quad - \cdots - T(X_1,...,X_{k-1},\mathcal{R}(X,Y)X_k), \quad (1.6)\\
Q(g,T)(X_1,...,X_k;X,Y) &= -T((X\wedge Y)X_1,X_2,...,X_k) \\
&\quad - \cdots - T(X_1,...,X_{k-1},(X\wedge Y)X_k), \quad (1.7)
\end{aligned}
$$

respectively .

If the tensors $R\cdot R$ and $Q(g,R)$ are linearly dependent then M is called *pseu - dosymmetric* . This is equivalent to

$$R\cdot R = L_R Q(g,R) \tag{1.8}$$

holding on the set $U_R = \{x \mid Q(g,R) \neq 0 \text{ at } x\}$, where L_R is some function on U_R. If $R\cdot R = 0$ then M is called *s emisymmetric* (see Deszcz [3]) .

If the tensors $R\cdot S$ and $Q(g,S)$ are linearly dependent then M is called *Ricci - pseudosymmetric* . This is equivalent to

$$R\cdot S = L_S Q(g,S) \tag{1.9}$$

holding on the set $U_S = \{x \mid S \neq \frac{\kappa}{n} g \text{ at } x\}$, where L_S is some function on U_S. Every pseudosymmetric manifold is Ricci pseudosymmetric but the converse statement is not true . If $R\cdot S = 0$ then M is called *Ricci - s emisymmetric* (see Deszcz [3]) .

If the tensors $R\cdot C$ and $Q(g,C)$ are linearly dependent then M is called *Weyl - pseudosymmetric* . This is equivalent to

$$R\cdot C = L_C Q(g,C) \tag{1.10}$$

holding on the set $U_C = \{x \mid C \neq 0 \text{ at } x\}$. Every pseudosymmetric manifold is Weyl pseudosymmetric but the converse st atement is not true . If $R\cdot C = 0$ then M is called *Weyl - s emisymmetric* (see Deszcz [3]) .

The manifold M is a *manifold with pseudosymmetric Weyl t ensor* if and only if

$$C\cdot C = L_C Q(g,C) \tag{1.11}$$

holds on the set U_C, where L_C is some function on U_C(see Deszcz , Verstraelen , and Yaprak , [4]) . The tensor $C\cdot C$ is defined in the same way as the tensor $R\cdot R$.

2 . 2 - quasi umbilical hypersurfaces

Let M^n be an $n \geq 3$ dimensional connected hypersurface immersed isometrically in the Euclidean space $\mathbb{E}^{n+1}$. We denote by $\widetilde{\nabla}$ and ∇ the Levi - Civita connections corresponding to $\mathbb{E}^{n+1}$ and M^n, respectively . Let ξ be a lo cal unit normal vector field on M^n in $\mathbb{E}^{n+1}$. We can present the Gauss formula and the Weingarten formula of M^n in $\mathbb{E}^{n+1}$ of the form : $\widetilde{\nabla}_X Y = \nabla_X Y + h(X,Y)$ and $\widetilde{\nabla}_X \xi = -A_\xi(X) + D_X\xi$, respectively , where X, Y are vector fields tangent to M^n and D is the normal connection of M^n(see Chen [2]) .

For the plane section $e_i \wedge e_j$ of the tangent bundle TM^n spanned by the vectors e_i and $e_j (i \neq j)$ the scalar curvature of M^n is defined by $\kappa = \sum_{i,j=1}^{n} K(e_i \wedge e_j)$ where K denotes the sectional curvature of M^n. We denote by shortly $K_{rs} = K(e_r \wedge e_s)$.

Hypersurface M^n with three distinct principal curvatures , their multiplicities are 1 , 1 and $n-2$, is said to be *2 - quasi umbilical .* So the shape operator of a 2 - quasi - umbilical hypersurface is of the form

$$A_\xi = \begin{bmatrix} a & 0 & 0 & 0 & \cdots & 0 \\ 0 & b & 0 & 0 & \cdots & 0 \\ 0 & 0 & c & 0 & \cdots & 0 \\ 0 & 0 & 0 & c & \cdots & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & \cdots & c \end{bmatrix}. \tag{2.1}$$

2 - quasi - umbilical hypersurfaces are the extended class of quasi - umbilical hyper - surfaces . It is well - known that a hypersurface M^n which has a principal curvature with multiplicity $\geq n-1$ is said to be *quasi - umbilical* . The well - known result of E . Cartan gives us " A hypersurface $M^n, n \geq 4$, isometrically in the Euclidean space $\mathbb{E}^{n+1}$, is conformally flat if and only if it is quasi *umbilical* ". *In* $\ddot{O}_{\mathrm{zg}^{\ddot{u}}\mathrm{r}}$[8], the present author studied conformally flat submanifolds with flat normal connection .

By (2 . 1) for a 2 - quasi - umbilical hypersurface , one can get easily the following

corollaries :

Corollary 2 . 1 . *Let* M^n *be a 2 - quasi - umbilical hypersurface of* $\mathbb{E}^{n+1}, n \geq 4$, *then*

$$\begin{aligned} K_{12} &= ab, \\ K_{K^{1j}_{2j}} &=_= acbc^{;}_{,} \quad (^{(j}_{j} >^> 2^{2)}_{)} \\ K_{ij} &= c^2, \quad (i,j > 2). \end{aligned} \tag{2.2}$$

where $i,j>2$. *Furthermore* , $\mathcal{R}(e_i,e_j)e_k = 0$ *if* i,j *and* k *are mutually different .*

Theorem 2 . 1 . [5] . *Any 2 - quasi - umbilical hypersurface* $M^n, dim M^n \geq 4$, *immersed*

is ometrically in a s emi - Riemannian conformally flat manifold N is a manifold with pseudosymmetric Weyl t ensor .

On the other hand , it is known that in a hypersurface M^n of a Riemannian space of constant curvature $N^{n+1}(c), n \geq 4$, if M^n is a Ricci - pseudosymmetric manifold with pseudosymmetric Weyl tensor then it is a pseudosymmetric manifold (see Deszcz , Verstraelen , and Yaprak [4]) . Moreover from Arslan , Deszcz , and Yaprak [1] , we know that , in a hypersurface M^n of a Riemannian space of constant curvature

$N^{n+1}(c), n \geq 4$, the Weyl pseudosymmetry and the pseudosymmetry conditions are equivalent . So using the previous facts and Theorem 2 . 1 one can obtain the following corollary .

Corollary 2 . 2 . *In the class of 2 - quasi - umbilical hypersurfaces of th e Euclidean space*

$\mathbb{E}^{n+1}, n \geq 4,$ *the conditions of th e pseudosymmetry , Ricci pseudosymmetry and Weyl pseudosymmetry are equivalent .*

In Özgür and Arslan [9] , the present author and K . Arslan studied pseudosym - metry type hypersurfaces in the Euclidean space satisfying Chen ' s equality . It is known that , a hypersurface satisfying Chen ' s equality is a special 2 - quasi - umbilical hypersurface .

In this study , our aim is to generalize the study Özgür and Arslan [9] and to find the characterization of 2 - quasi - umbilical hypersurfaces satisfying pseudosymmetry curvature condition . Since pseudosymmetry , Ricci - pseudosymmetry and Weyl - pseudosymmetry curvature conditions for a 2 - quasi - umbilical hypersurface in the Euclidean space $\mathbb{E}^{n+1}$, are equivalent , it is sufficient to investigate only pseudosymmetry curvature condition .

Firstly we have :

Lemma 2 . 1 . *Let M^n be a 2 - quasi - umbilical hypersurface of $\mathbb{E}^{n+1}, n \geq 4$. Then*

$$(\mathcal{R}(e_1,e_3)\cdot\mathcal{R})(e_2,e_3)e_1 = abc[c-a]e_2, \tag{2.3}$$

$$(\mathcal{R}(e_2,e_3)\cdot\mathcal{R})(e_1,e_3)e_2 = abc[c-b]e_1. \tag{2.4}$$

Proof . Using (1 . 6) we have

$$\begin{aligned} &(\mathcal{R}(e_1,e_3)\cdot\mathcal{R})(e_2,e_3)e_1 \\ &= \mathcal{R}(e_1,e_3)(\mathcal{R}(e_2,e_3)e_1) - \mathcal{R}(\mathcal{R}(e_1,e_3)e_2,e_3)e_1 \\ &-\mathcal{R}(e_2,\mathcal{R}(e_1,e_3)e_3)e_1 - \mathcal{R}(e_2,e_3)(\mathcal{R}(e_1,e_3)e_1) \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} &(\mathcal{R}(e_2,e_3)\cdot\mathcal{R})(e_1,e_3)e_2 \\ &= \mathcal{R}(e_2,e_3)(\mathcal{R}(e_1,e_3)e_2) - \mathcal{R}(\mathcal{R}(e_2,e_3)e_1,e_3)e_2 \\ &-\mathcal{R}(e_1,\mathcal{R}(e_2,e_3)e_3)e_2 - \mathcal{R}(e_1,e_3)(\mathcal{R}(e_2,e_3)e_2). \end{aligned} \tag{2.6}$$

Since $\mathcal{R}(e_i,e_j)e_k = (A_\xi e_i \wedge A_\xi e_j)e_k$, using (2 . 2) , one can get

$$\begin{aligned} \mathcal{R}(e_1,e_3)e_1 &= -K_{13}e_3, \quad \mathcal{R}(e_1,e_3)e_3 = K_{13}e_1 \\ \mathcal{R}(e_2,e_1)e_1 &= K_{12}e_2, \quad \mathcal{R}(e_2,e_1)e_2 = -K_{12}e_1 \\ \mathcal{R}(e_2,e_3)e_2 &= -K_{23}e_3, \quad \mathcal{R}(e_2,e_3)e_3 = K_{23}e_2. \end{aligned} \tag{2.7}$$

So substituting (2 . 7) and (2 . 2) into (2 . 5) and (2 . 6) , respectively we obtain (2 . 3) and (2 . 4) .

Lemma 2 . 2 . *Let M be a 2 - quasi - umbilical hypersurface of $\mathbb{E}^{n+1}, n \geq 4$. Then*

$$Q(g,\mathcal{R})(e_2,e_3,e_1;e_1,e_3) = b[c-a]e_2, \tag{2.8}$$

$$Q(g,\mathcal{R})(e_1,e_3,e_2;e_2,e_3) = a[c-b]e_1. \tag{2.9}$$

Proof . Using the relation (1 . 7) we obtain

$$\begin{aligned} &Q(g,\mathcal{R})(e_2,e_3,e_1;e_1,e_3) \\ &= (e_1\wedge e_3)\mathcal{R}(e_2,e_3)e_1 - \mathcal{R}((e_1\wedge e_3)e_2,e_3)e_1 \\ &-\mathcal{R}(e_2,(e_1\wedge e_3)e_3)e_1 - \mathcal{R}(e_2,e_3)((e_1\wedge e_3)e_1) \end{aligned} \tag{2.10}$$

and

$$
\begin{aligned}
&Q(g,\mathcal{R})(e_2,e_3,e_2;e_2,e_3)\\
&= (e_2\wedge e_3)\mathcal{R}(e_2,e_3)e_2 - \mathcal{R}((e_2\wedge e_3)e_2,e_3)e_2\\
&-\mathcal{R}(e_2,(e_2\wedge e_3)e_3)e_2 - \mathcal{R}(e_2,e_3)((e_2\wedge e_3)e_2).
\end{aligned} \tag{2.11}
$$

So substituting (2 . 7) and (2 . 2) into (2 . 1 0) and (2 . 1 1 , respectively we obtain (2 . 8)
and (2 . 9) .

Using Lemma 2 . 1 and Lemma 2 . 2 we have the following theorem :

Theorem 2 . 2 . *Let* M^n *be a 2 - quasi - umbilical hypersurface of* $\mathbb{E}^{n+1}, n \geq 4$. *Then*
M^n *is proper pseudosymmetric if and only if* $a = b$ *and* $L_R = ac$ *holds on* M^n.

Proof . Let M^n be a pseudosymmetric hypersurface in $\mathbb{E}^{n+1}$. Then by definition one can write

$$(\mathcal{R}(e_1, e_3) \cdot \mathcal{R})(e_2, e_3)e_1 = L_R Q(g, \mathcal{R})(e_2, e_3, e_1; e_1, e_3) \tag{2.12}$$

and

$$(\mathcal{R}(e_2, e_3) \cdot \mathcal{R})(e_1, e_3)e_2 = L_R Q(g, \mathcal{R})(e_2, e_3, e_2; e_2, e_3). \tag{2.13}$$

Since M^n is 2 - quasi - umbilical then by Lemma 2 . 1 and Lemma 2 . 2 the equations (2 . 1 2) and (2 . 1 3) turn into respectively

$$b(c - a)(L_R - ac) = 0 \tag{2.14}$$

and

$$a(c - b)(L_R - bc) = 0, \tag{2.15}$$

respectively . Extracting the equations (2 . 1 4) and (2 . 1 5) we get

$$cL_R(b - a) = 0. \tag{2.16}$$

Since M^n is proper pseudosymmetric , it is not semisymmetric . Then the equation (2 . 1 6) gives us $b =$ a. So the principal curvatures of M^n must be of the form $(a, a, c, ..., c)$, which gives us

$$(\mathcal{R}(e_1, e_3) \cdot \mathcal{R})(e_2, e_3)e_1 = a^2 c[c - a]e_2, \tag{2.17}$$
$$Q(g, \mathcal{R})(e_2, e_3, e_1; e_1, e_3) = a[c - a]e_2, \tag{2.18}$$
$$(\mathcal{R}(e_2, e_3) \cdot \mathcal{R})(e_1, e_3)e_2 = a^2 c[c - a]e_1, \tag{2.19}$$

and

$$Q(g, \mathcal{R})(e_1, e_3, e_2; e_2, e_3) = a[c - a]e_1. \tag{2.20}$$

So from (2 . 1 7) – (2 . 1 8) and (2 . 1 9) – (2 . 20) we obtain $L_R = ac$. This completes the proof of the theorem .

References

[1] K . Arslan , R . Deszcz and Ş. Yaprak , On Weyl pseudosymmetric hypersurfaces , *Colloq . Math .* **72** (2) (1 997) , 353 – 36 1 . [2] B . Chen , *Geometry of Submanifolds and its Applications* , Sci . Univ . Tokyo , Tokyo , 1 98 1 . [3] R . Deszcz , On pseudosymmetric spaces , *Bull . Soc . Math . Belg . Sér . A* **44** (1) (1 992) , 1 – 34 . [4] R . Deszcz , L . Verstraelen and Ş. Yaprak , On hypersurfaces with pseudosymmetric Weyl tensor , in *Geometry and topology of submanifolds , VIII (Brussels , 1 995 / Nordfjordeid , 1 995)* , 1 1 1 – 1 20 , World Sci . Publ . , River Edge , NJ . [5] R . Deszcz , L . Verstraelen and Ş. Yaprak , On 2 - quasi - umbilical hypersurfaces in conformally flat spaces , *Acta Math . Hungar .* **78** (1 – 2) (1 998) , 45 – 57 .

[6] R . Deszcz , L . Verstraelen and Ş . Yaprak , Hypersurfaces with pseudosymmetric Weyl ten - sor in conformally flat manifolds , in *Geometry and topology of submanifolds , IX (Valenci - ennes / Lyon / Leuven , 1 997)* , 108 – 1 1 7 , World Sci . Publ . , River Edge , NJ . [7] F . Dillen , M . Petrovic and L . Verstraelen , Einstein , conformally flat and semi - symmetric sub - manifolds satisfying Chen ' s equality , *Israel J . Math .* **1 0** (1 997) , 1 63 – 1 69 . [8] C .Özgür, Submanifolds satisfying some curvature conditions imposed on the Weyl tensor , *Bull . Austral . Math . Soc .* **67** (1) (2003) , 95 – 10 1 .
[9] C .Özgür and K . Arslan , On some class of hypersurfaces in $\mathbb{E}^{n+1}$ satisfying Chen ' s equality , Turkish J . Math . **26** (3) (2002) , 283 – 293 .