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ORIGINAL RESEARCH

Contra g-α- and g-β-preirresolute functions on GTS's

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Abstract In this present paper, we define g-α-preirresolute, g-β-preirresolute, contra g-α-preirresolute and contra g-β-preirresolute functions on generalized topological spaces. We give some examples of this definitions. We investigate some properties and characterizations of this functions.

Keywords g-α-preirresolute · g-β-preirresolute · Contra g-α-preirresolute · Contra g-β-preirresolute

Mathematics Subject Classification 54A05 · 54C08

Introduction

Császár [2] introduced generalized open sets in 1997. Subsequently, he [3] defined generalized topology and generalized continuity in 2002. Also, (g_X, g_Y)-open functions [4] were introduced in 2003 and strong generalized topology [5] was presented in 2004. g-semi-open sets, g-preopen sets, g-α-open sets and g-β-open sets [6] were introduced by Császár in 2005. Also he [7] showed how the definition of the product of generalized topologies in 2009.

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In 2012, Jayanthi [8] introduced contra continuity on generalized topological space. Furthermore, Min [9] defined (α, g_Y)-continuous functions, (σ, g_Y)—continuous functions, (π, g_Y)-continuous functions and (β, g_Y)-continuous functions on generalized topological spaces in 2009. Additionally, Bai and Zuo [1] introduced g-α-irresolute functions in 2011. In 2009, Shen [10] studied the relationship between the product and some operations $(\sigma, \pi, \alpha$ and $\beta)$ of generalized topologies. Our aim in this paper, is to introduce g-α-preirresolute, g-β-preirresolute, contra g-α-preirresolute, contra g-β-preirresolute on generalized topological spaces. Also we obtain some properties and characterizations of this functions.

Preliminaries

Definition 2.1 [3] Let $X \neq \emptyset$ and $g \subseteq X$. Then g is called a generalized topology (briefly; GT) on X iff $\emptyset \in g$ and $G_i \in g$ for $i \in I \neq \emptyset$ implies $G = \bigcup_{i\in I} G_i \in g$. The pair (X, g) is called a generalized topological space (briefly; GTS) on X. The elements of g are called g-open sets and their complements are called g-closed sets.

Definition 2.2 [3] Let (X, g) be a generalized topological space and $A \subseteq X$.

(1) The closure of A is defined as follows:

$$c_g(A) = \bigcap\{F : F \textit{ is } g\textit{-closed},\ A \subseteq F\}.$$

(2) The interior of A is defined as follows:

$$i_g(A) = \bigcup\{G : G \textit{ is } g\textit{-open}, G \subseteq A\}.$$

Theorem 2.3 [3] *Let (X, g) be a generalized topological space. Then the following hold:*

(1) $c_g(A) = X - i_g(X - A)$.
(2) $i_g(A) = X - c_g(X - A)$.

Definition 2.4 [6] Let (X, g) be a generalized topological space and $A \subseteq X$. A is said to be

(1) g-semi-open if $A \subseteq c_g(i_g(A))$;
(2) g-preopen if $A \subseteq i_g(c_g(A))$;
(3) g-α-open if $A \subseteq i_g(c_g(i_g(A)))$;
(4) g-β-open if $A \subseteq c_g(i_g(c_g(A)))$.

The complement of g-semi-open (resp. g-preopen, g-α-open, g-β-open) is said to be g-semi-closed (resp. g -pre-closed, g-α-closed, g-β-closed). The set of all g-semi-open sets (resp. g-preopen sets, g-α-open sets, g-β-open sets) is denoted by $\sigma(g)$ (resp. $(\pi(g), \alpha(g), \beta(g))$.

The closure of g-semi-closed (resp. g-preclosed, g-α-closed, g-β-closed) sets is denoted by $c_\sigma(X)$ (resp. $c_\pi(X)$, $c_\alpha(X)$, $c_\beta(X)$). Also the interior of g-semi-open (resp. g-preopen, g-α-open, g-β-open) sets is denoted by $i_\sigma(X)$ (resp. $i_\pi(X)$, $i_\alpha(X)$, $i_\beta(X)$).

Definition 2.5 [4] Let (X, g_X) and (Y, g_Y) be GTS's. Then a function $f : X \to Y$ is said to be (g_X, g_Y)-open if $f(U) \in g_Y$ for each $U \in g_X$.

Definition 2.6 [3] Let (X, g_X) and (Y, g_Y) be GTS's. Then a function $f : X \to Y$ is said to be (g_X, g_Y)-continuous if $f^{-1}(V) \in g_X$ for each $V \in g_Y$.

Definition 2.7 [9] Let (X, g_X) and (Y, g_Y) be GTS's. Then a function $f : X \to Y$ is said to be

(1) (α, g_Y)-continuous if $f^{-1}(V)$ is g-α-open in X for each g-open set V in Y;
(2) (σ, g_Y)-continuous if $f^{-1}(V)$ is g-semi-open in X for each g-open set V in Y.
(3) (π, g_Y)-continuous if $f^{-1}(V)$ is g-preopen in X for each g-open set V in Y.
(4) (β, g_Y)-continuous if $f^{-1}(V)$ is g-β-open in X for each g-open set V in Y.

Definition 2.8 [8] Let (X, g_X) and (Y, g_Y) be GTS's. Then a function $f : X \to Y$ is said to be

(1) contra (g_X, g_Y)-continuous if $f^{-1}(V)$ is g-closed in X for each $V \in g_Y$.
(2) contra (α, g_Y)-continuous if $f^{-1}(V)$ is g-α-closed in X for each g-open set V in Y.
(3) contra (σ, g_Y)-continuous if $f^{-1}(V)$ is g-semi-closed in X for each g-open set V in Y.
(4) contra (π, g_Y)-continuous if $f^{-1}(V)$ is g-preclosed in X for each g-open set V in Y.
(5) contra (β, g_Y)-continuous if $f^{-1}(V)$ is g-β-closed in X for each g-open set V in Y.

Definition 2.9 [5] Let g be a GT on a set $X \neq \emptyset$. Then g is said to be strong if $X \in g$.

Definition 2.10 [7] Let $K \neq \emptyset$ be an index set, $X_k \neq \emptyset$ for $k \in K$ and $X = \prod_{k \in K} X_k$ the cartesian product of the sets X_k. Also $p_k : X \to X_k$ is the projection.

Let g_k be a given GT on X_k for $k \in K$. Then g is called the product of the GT's g_k.

Proposition 2.11 [10] *If every g_{X_k} is strong then each p_k is (g_X, g_{X_k})-continuous (resp. $(\alpha(g_X), \alpha(g_{X_k}))$-continuous, $(\sigma(g_X), \sigma(g_{X_k}))$-continuous, $(\pi(g_X), \pi(g_{X_k}))$-continuous, $(\beta(g_X), \beta(g_{X_k}))$-continuous) for $k \in K$.*

Theorem 2.12 [7] *Let $G = \prod_{k \in K} G_k$. Then*

(1) *If K is finite and every G_k is g-semi-open, then G is g-semi-open set.*
(2) *If K is finite and every G_k is g-preopen, then G is g-preopen set.*
(3) *If K is finite and every G_k is g-α-open, then G is g-α-open set.*
(4) *If K is finite and every G_k is g-β-open, then G is g-β-open set.*

Definition 2.13 [1] A function $f : X \to Y$ is said to be g-α-irresolute if $f^{-1}(V)$ is g-α-open in X for every g-α-open set V of Y.

g-α-Preirresolute and g-β-preirresolute functions

Definition 3.1 Let (X, g_X) and (Y, g_Y) be GTS's. Then a function $f : X \to Y$ is said to be g-α-preirresolute if $f^{-1}(V)$ is g-α-open in X for every g-preopen set V of Y.

Example 3.2 Let $X = \{x, y\}$, $Y = \{a, b\}$, $g_X = P(X)$ and $g_Y = \{\emptyset, \{a\}\}$. Then we obtain $\pi(g_Y) = \{\emptyset, \{a\}\}$.

$f : (X, g_X) \to (Y, g_Y)$ such that $f(x) = a, f(y) = b$.

Since $f^{-1}(\emptyset) = \emptyset$ and $f^{-1}(\{a\}) = \{x\}$ are g-α-open subsets of X, then f is g-α-preirresolute.

Definition 3.3 Let (X, g_X) and (Y, g_Y) be GTS's. Then a function $f : X \to Y$ is said to be g-β-preirresolute if $f^{-1}(V)$ is g-β-open in X for every g-preopen set V of Y.

Example 3.4 Let $X = \{x, y\}$, $Y = \{a, b, c\}$, $g_X = \{\emptyset, \{x\}\}$ and $g_Y = \{\emptyset, \{a\}, \{a, b\}\}$. Then we obtain $\pi(g_Y) = \{\emptyset, \{a\}, \{a, b\}\}$.

$f : (X, g_X) \to (Y, g_Y)$ such that $f(x) = f(y) = a$.

Since $f^{-1}(\emptyset) = \emptyset$, $f^{-1}(\{a\}) = X$ and $f^{-1}(\{a, b\}) = X$ are g-β-open subsets of X, then f is g-β-preirresolute.

Definition 3.5 Let (X, g_X) and (Y, g_Y) be GTS's. Then a function $f : X \to Y$ is said to be g-α-preirresolute at $x \in X$

if there exists a g-α-open set U of X containing x such that $f(U) \subseteq V$ for each g-preopen set V of Y containing $f(x)$.

Definition 3.6 Let (X, g_X) and (Y, g_Y) be GTS's. Then a function $f : X \to Y$ is said to be g-β-preirresolute at $x \in X$ if there exists a g-β-open set U of X containing x such that $f(U) \subseteq V$ for each g-preopen set V of Y containing $f(x)$.

Theorem 3.7 *Let (X, g_X), (Y, g_Y) be GTS's and $f : X \to Y$ be a function. The following conditions are equivalent:*

(1) *f is g-α-preirresolute;*
(2) *For each $x \in X$ and each g-preopen set V of Y containing $f(x)$, there exists a g-α-open set U of X containing x such that $f(U) \subseteq V$;*
(3) *$f^{-1}(V) \subseteq i_g(c_g(i_g(f^{-1}(V))))$ for every g-preopen set V of Y;*
(4) *$f^{-1}(V)$ is g-α-closed in X for every g-preclosed set V of Y;*
(5) *$c_g(i_g(c_g(f^{-1}(V)))) \subseteq f^{-1}(c_\pi(V))$ for every subset V of Y;*
(6) *$f(c_g(i_g(c_g(U)))) \subseteq c_\pi(f(U))$ for every subset U of X.*

Proof (1) $\Rightarrow$ (2). Let $x \in X$ and V be any g-preopen set of Y containing $f(x)$. By hypothesis, $f^{-1}(V)$ is g-α-open in X and contains x. Suppose $U = f^{-1}(V)$, then U is g-α-open set in X containing x and $f(U) \subseteq V$.

(2) $\Rightarrow$ (3). Let V be any g-preopen set of Y and $x \in f^{-1}(V)$. By hypothesis, there exists a g-α-open set U of X such that $f(U) \subseteq V$. Hence we obtain

$$x \in U \subseteq i_g(c_g(i_g(U))) \subseteq i_g(c_g(i_g(f^{-1}(V)))).$$

As a consequence, $f^{-1}(V) \subseteq i_g(c_g(i_g(f^{-1}(V))))$.

(3) $\Rightarrow$ (4). Let V be any g-preclosed of Y. Then $U = Y - V$ is g-preopen in Y. By (3), we have $f^{-1}(U) \subseteq i_g(c_g(i_g(f^{-1}(U))))$. Therefore

$$\begin{aligned} f^{-1}(U) = f^{-1}(Y - V) = X - f^{-1}(V) &\subseteq i_g(c_g(i_g(f^{-1}(U)))) \\ &= X - c_g(i_g(c_g(f^{-1}(V)))). \end{aligned}$$

As a consequence, we obtain $f^{-1}(V)$ is g-α-closed set in X.

(4) $\Rightarrow$ (5). Let V be any subset of Y. Since $c_\pi(V)$ is g-preclosed subset of Y, then $f^{-1}(c_\pi(V))$ is g-α-closed in X by (4). Hence

$$c_g(i_g(c_g(f^{-1}(c_\pi(V))))) \subseteq f^{-1}(c_\pi(V)).$$

Therefore we obtain $c_g(i_g(c_g(f^{-1}(V)))) \subseteq f^{-1}(c_\pi(V))$.

(5) $\Rightarrow$ (6). Let U be any subset of X. By hypothesis, we have

$$c_g(i_g(c_g(U))) \subseteq c_g(i_g(c_g(f^{-1}(f(U))))) \subseteq f^{-1}(c_\pi(f(U))).$$

As a consequence, $f(c_g(i_g(c_g(U)))) \subseteq c_\pi(f(U))$.

(6) $\Rightarrow$ (1). Let V be any g-preopen subset of Y. $f^{-1}(Y - V) = X - f^{-1}(V)$ is a subset of X and by hypothesis, we obtain

$$\begin{aligned} f(c_g(i_g(c_g(f^{-1}(Y - V))))) &\subseteq c_\pi(f(f^{-1}(Y - V))) \\ &\subseteq c_\pi(Y - V) = Y - i_\pi(V) \\ &= Y - V \end{aligned}$$

and so

$$\begin{aligned} &X - i_g(c_g(i_g(f^{-1}(V)))) = c_g(i_g(c_g(X - f^{-1}(V)))) = \\ &c_g(i_g(c_g(f^{-1}(Y - V)))) \subseteq f^{-1}(f(c_g(i_g(c_g(f^{-1}(Y - V)))))) \\ &\subseteq f^{-1}(Y - V) = X - f^{-1}(V). \end{aligned}$$

Thus $f^{-1}(V) \subseteq i_g(c_g(i_g(f^{-1}(V))))$ and $f^{-1}(V)$ is g-α-open set in X. As a consequence, f is g-α-preirresolute. □

Theorem 3.8 *Let (X, g_X), (Y, g_Y) be GTS's and $f : X \to Y$ be a function. The following conditions are equivalent:*

(1) *f is g-β-preirresolute;*
(2) *For each $x \in X$ and each g-preopen set V of Y containing $f(x)$, there exists a g-β-open set U of X containing x such that $f(U) \subseteq V$;*
(3) *$f^{-1}(V) \subseteq c_g(i_g(c_g(f^{-1}(V))))$ for every g-preopen set V of Y;*
(4) *$f^{-1}(V)$ is g-β-closed in X for every g-preclosed set V of Y;*
(5) *$i_g(c_g(i_g(f^{-1}(V)))) \subseteq f^{-1}(c_\pi(V))$ for every subset V of Y;*
(6) *$f(i_g(c_g(i_g(U)))) \subseteq c_\pi(f(U))$ for every subset U of X.*

Proof It is proved similar to the proof of Theorem 3.7. □

Theorem 3.9 *Let (X, g_X), (Y, g_Y) be GTS's and $f : X \to Y$ be a function. The following conditions are equivalent:*

(1) *f is g-α-preirresolute;*
(2) *$f^{-1}(F)$ is g-α-closed in X for every g-preclosed set F of Y;*
(3) *$f(c_\alpha(A)) \subseteq c_\pi(f(A))$ for every subset A of X;*
(4) *$c_\alpha(f^{-1}(B)) \subseteq f^{-1}(c_\pi(B))$ for every subset B of Y;*
(5) *$f^{-1}(i_\pi(B)) \subseteq i_\alpha(f^{-1}(B))$ for every subset B of Y;*
(6) *f is g-α-preirresolute at every $x \in X$.*

Proof (1) $\Rightarrow$ (2). It is obvious from Theorem 3.7.

(2) $\Rightarrow$ (3). Let $A \subseteq X$. Then $c_\pi(f(A))$ is a g-preclosed set of Y. By hypothesis, $f^{-1}(c_\pi(f(A)))$ is a g-α-closed set. Now $c_\alpha(A) \subseteq c_\alpha(f^{-1}(f(A))) \subseteq c_\alpha(f^{-1}(c_\pi(f(A)))) = f^{-1}(c_\pi(f(A)))$. Hence $f(c_\alpha(A)) \subseteq c_\pi(f(A))$.

(3) $\Rightarrow$ (4). Let $B \subseteq Y$. Then $f^{-1}(B) \subseteq X$. By hypothesis, $f(c_\alpha(f^{-1}(B))) \subseteq c_\pi(f(f^{-1}(B))) \subseteq c_\pi(B)$. Hence $c_\alpha(f^{-1}(B))$

$\subseteq f^{-1}(f(c_\alpha(f^{-1}(B)))) \subseteq f^{-1}(c_\pi(B))$. So we obtain $c_\alpha(f^{-1}(B)) \subseteq f^{-1}(c_\pi(B))$.

(4) $\Rightarrow$ (5). It is obvious from the complement of (4).

(5) $\Rightarrow$ (1). Let V be any g-preopen set of Y, then $V = i_\pi(V)$. By hypothesis, $f^{-1}(V) = f^{-1}(i_\pi(V)) \subseteq i_\alpha(f^{-1}(V)) \subseteq f^{-1}(V)$. Hence $f^{-1}(V) = i_\alpha(f^{-1}(V))$. Thus $f^{-1}(V)$ is a g-α-open set of X. As a consequence, f is g-α-preirresolute.

(1) $\Rightarrow$ (6). Let f is g-α-preirresolute, $x \in X$ and any g-preopen set V of Y containing $f(x)$. Then $x \in f^{-1}(V)$ and $f^{-1}(V)$ is g-α-open set in X. Suppose $U = f^{-1}(V)$, then U is a g-α-open set of X and $f(U) \subseteq V$. Therefore f is g-α-preirresolute for each $x \in X$. □

Theorem 3.10 *Let (X, g_X), (Y, g_Y) be GTS's and $f : X \to Y$ be a function. The following conditions are equivalent:*

(1) *f is g-β-preirresolute;*
(2) *$f^{-1}(F)$ is g-β-closed in X for every g-preclosed set F of Y;*
(3) *$f(c_\beta(A)) \subseteq c_\pi(f(A))$ for every subset A of X;*
(4) *$c_\beta(f^{-1}(B)) \subseteq f^{-1}(c_\pi(B))$ for every subset B of Y;*
(5) *$f^{-1}(i_\pi(B)) \subseteq i_\beta(f^{-1}(B))$ for every subset B of Y;*
(6) *f is g-β-preirresolute at every $x \in X$.*

Proof It is proved by a similar way in Theorem 3.9. □

Theorem 3.11 *Let $f : X \to Y$ be a function from two GTS's. Then f is g-α-preirresolute if $f^{-1}(V) \subseteq i_g(c_g(i_g(f^{-1}(i_\pi(V)))))$ for every g-preopen subset V of Y.*

Proof Let V be g-preclosed set of Y. Then $Y - V$ is g-preopen set in Y. By hypothesis, $f^{-1}(Y - V) = X - f^{-1}(V) \subseteq i_g(c_g(i_g(f^{-1}(i_\pi(Y - V))))) = i_g(c_g(i_g(f^{-1}(Y - V)))) = X - c_g(i_g(c_g(f^{-1}(V))))$. Hence we obtain $c_g(i_g(c_g(f^{-1}(V)))) \subseteq f^{-1}(V)$. Therefore $f^{-1}(V)$ is g-α-closed set in X. As a consequence, f is g-α-preirresolute from Theorem 3.7(4). □

Theorem 3.12 *Let $f : X \to Y$ be a function from two GTS's. Then f is g-β-preirresolute if $f^{-1}(V) \subseteq c_g(i_g(c_g(f^{-1}(i_\pi(V)))))$ for every g-preopen subset V of Y.*

Proof It is similar to Theorem 3.11. □

Theorem 3.13 *Let $f : X \to Y$ be a function from two GTS's and g_X be a strong. f is g-α-preirresolute if the graph function $g : X \to X \times Y$ defined by $g(x) = (x, f(x))$ for each $x \in X$, is g-α-preirresolute.*

Proof Let $x \in X$ and V be any g-preopen set of Y containing $f(x)$. Then $X \times V$ is a g-preopen set of $X \times Y$ by Theorem 2.12 and contains $g(x)$. Since g is g-α-preirresolute, there exists a g-α-open U of X containing x such that $g(U) \subseteq X \times V$ and so $f(U) \subseteq V$. Thus f is g-α-preirresolute. □

Theorem 3.14 *Let $f : X \to Y$ be a function from two GTS's and g_X be a strong. f is g-β-preirresolute if the graph function $g : X \to X \times Y$ defined by $g(x) = (x, f(x))$ for each $x \in X$, is g-β-preirresolute.*

Proof The proof is similar to that of Theorem 3.13 □

Theorem 3.15 *Let g_{Y_k} be a given GT on Y_k for $k \in K$ and g_{Y_k} be a strong. If a function $f : X \to \prod Y_k$ is g-α-preirresolute, then $p_k \circ f : X \to Y_k$ is g-α-preirresolute for each $k \in K$, where p_k is the projection of $\prod Y_k$ onto Y_k.*

Proof Let V_k be any g-preopen set of Y_k. p_k is $(\pi(g_Y), \pi(g_{Y_k}))$-continuous from Proposition 2.11 since g_{Y_k} is strong and so $p_k^{-1}(V_k)$ is g-preopen set. Since f is g-α-preirresolute, $f^{-1}(p_k^{-1}(V_k)) = (p_k \circ f)^{-1}(V_k)$ is a g-α-open. As a consequence, we have $p_k \circ f$ is g-α-preirresolute for each $k \in K$. □

Theorem 3.16 *Let g_{Y_k} be a given GT on Y_k for $k \in K$ and g_{Y_k} be a strong. If a function $f : X \to \prod Y_k$ is g-β-preirresolute, then $p_k \circ f : X \to Y_k$ is g-β-preirresolute for each $k \in K$, where p_k is the projection of $\prod Y_k$ onto Y_k.*

Proof It is proved similar to that of Theorem 3.15. □

Theorem 3.17 *If the function $f : \prod X_k \to \prod Y_k$ defined by $f(\{x_k\}) = \{f_k(x_k)\}$ for each $\{x_k\} \in \prod X_k$, is g-α-preirresolute, then $f_k : X_k \to Y_k$ is g-α-preirresolute for each $k \in K$.*

Proof Let $k_0 \in K$ be an arbitrary fixed index and V_{k_0} be any g-preopen set of Y_{k_0}. Then $\prod Y_m \times V_{k_0}$ is g-preopen in $\prod Y_k$ by Theorem 2.12, where $k_0 \neq m \in K$. Since f is g-α-preirresolute, $f^{-1}(\prod Y_m \times V_{k_0}) = \prod X_m \times f_{k_0}^{-1}(V_{k_0})$ is g-α-open in $\prod X_k$ and $f_{k_0}^{-1}(V_{k_0})$ is g-α-open in X_{k_0}. As a consequence, f_{k_0} is g-α-preirresolute. □

Theorem 3.18 *If the function $f : \prod X_k \to \prod Y_k$ defined by $f(\{x_k\}) = \{f_k(x_k)\}$ for each $\{x_k\} \in \prod X_k$, is g-β-preirresolute, then $f_k : X_k \to Y_k$ is g-β-preirresolute for each $k \in K$.*

Proof It is proved by a similar way in Theorem 3.17. □

Theorem 3.19 *If $f : X \to Y$ is g-α-preirresolute and A is a g-α-open in X, then the restriction $f|A : A \to Y$ is g-α-preirresolute.*

Proof Let V be any g-preopen set in Y. Then we have $f^{-1}(V)$ is a g-α-open set in Y. Since the set A is a g-α-open set, we have $(f|A)^{-1}(V) = A \cap f^{-1}(V)$ is g-α-open. Therefore $f|A$ is g-α-preirresolute. □

Theorem 3.20 *If $f : X \to Y$ is g-β-preirresolute and A is g-open in X , then the restriction $f|A : A \to Y$ is g-β-preirresolute.*

Proof It is proved by a similar way of that of Theorem 3.19. □

Definition 3.21 Let (X, g_X) and (Y, g_Y) be GTS's. Then a function $f : X \to Y$ is said to be g-preirresolute if $f^{-1}(V)$ is g-preopen in X for every g-preopen set V of Y.

Theorem 3.22 *Let (X, g_X), (Y, g_Y) and (Z, g_Z) be GTS's. If $f : X \to Y$ is g-α-preirresolute and $g : Y \to Z$ is g-preirresolute, then the composition $g \circ f : X \to Z$ is g-α-preirresolute.*

Proof Let V be any g-preopen subset of Z. Since g is g-preirresolute, $g^{-1}(V)$ is g-preopen in Y. Since f is g-α-preirresolute, then $f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V)$ is g-α-open in X. As a consequence, $g \circ f$ is g-α-preirresolute. □

Theorem 3.23 *Let (X, g_X), (Y, g_Y) and (Z, g_Z) be GTS's. If $f : X \to Y$ is g-β-preirresolute and $g : Y \to Z$ is g-preirresolute, then the composition $g \circ f : X \to Z$ is g-β-preirresolute.*

Proof It is similar to that of Theorem 3.22 □

Definition 3.24 Let (X, g_X) and (Y, g_Y) be GTS's. Then a function $f : X \to Y$ is said to be g-α-pre-continuous if $f^{-1}(V)$ is g-preopen in X for every g-α-open set V of Y.

Definition 3.25 Let (X, g_X) and (Y, g_Y) be GTS's. Then a function $f : X \to Y$ is said to be almost g-α-irresolute if $f^{-1}(V)$ is g-β-open in X for every g-α-open set V of Y.

From the definitions stated above, we obtain the following diagram:

$$\begin{tikzcd}
g\text{-}\alpha\text{-preirresolute} \arrow[r] \arrow[d] & g\text{-preirresolute} \arrow[r] \arrow[d] & g\text{-}\beta\text{-preirresolute} \arrow[d] \\
g\text{-}\alpha\text{-irresolute} \arrow[r] & g\text{-}\alpha\text{-pre-continuity} \arrow[r] & \text{almost } g\text{-}\alpha\text{-irresolute}
\end{tikzcd}$$

Remark 3.26 The following examples enables us to realize that none of these implications is reversible.

Example 3.27 Let $X = Y = \{a, b, c, d\}$, $g_X = \{\emptyset, \{a\}, \{d\}, \{b, c\}, \{a, b, c\}, \{b, c, d\}, \{a, d\}\}$ and $g_Y = \{\emptyset, Y, \{b\}\}$. The identity function $f : X \to Y$ is g-α-pre-continuous function, but it is not g-α-irresolute. Also, f is g-preirresolute, but it is not g-α-preirresolute.

Example 3.28 Let $X = Y = \{a, b, c, d\}$, $g_X = \{\emptyset, X, \{a\}, \{c\}, \{a, c\}\}$ and $g_Y = \{\emptyset, Y, \{c, d\}\}$. The identity function $f : X \to Y$ is almost g-α-irresolute function, but it is neither g-α-pre-continuous nor g-β-preirresolute.

Example 3.29 Let $X = Y = \{a, b\}$ and $g_X = g_Y = \{\emptyset, \{a\}\}$. We define the function $f : X \to Y$ such that $f(a) = f(b) = a$. Then f is g-β-preirresolute function, but it is not g-preirresolute.

Example 3.30 Let $X = Y = \{a, b, c\}$ and $g_X = g_Y = \{\emptyset, X, \{a\}, \{b, c\}\}$. The identity function $f : X \to Y$ is g-α-irresolute function, but it is not g-α-preirresolute.

Contra g-α-preirresolute and contra g-β-preirresolute functions

Definition 4.1 Let (X, g_X) and (Y, g_Y) be GTS's. Then a function $f : X \to Y$ is said to be contra g-α-preirresolute if $f^{-1}(V)$ is g-α-closed in X for every g-preopen V of Y.

Example 4.2 Let $X = \{x, y\}$, $Y = \{a, b, c\}$, $g_X = \{\emptyset, \{y\}, X\}$ and $g_Y = \{\emptyset, \{a\}, \{c\}, \{a, c\}\}$. Then we obtain $\pi(g_Y) = \{\emptyset, \{a\}, \{c\}, \{a, c\}\}$.
$f : (X, g_X) \to (Y, g_Y)$ such that

$f(x) = a, f(y) = b.$

Since $f^{-1}(\emptyset) = \emptyset$, $f^{-1}(\{a\}) = \{x\}$, $f^{-1}(\{c\}) = \emptyset$ and $f^{-1}(\{a, c\}) = \{x\}$ are g-α-closed subsets of X, then f is contra g-α-preirresolute.

Definition 4.3 Let (X, g_X) and (Y, g_Y) be GTS's. Then a function $f : X \to Y$ is said to be contra g-β-preirresolute if $f^{-1}(V)$ is g-β-closed in X for every g-preopen V of Y.

Example 4.4 Let $X = \{x, y, z\}$, $Y = \{a, b\}$, $g_X = \{\emptyset, \{x\}\}$ and $g_Y = \{\emptyset, \{a\}\}$. Then we obtain $\pi(g_Y) = \{\emptyset, \{a\}\}$.
$f : (X, g_X) \to (Y, g_Y)$ such that

$f(x) = a, f(y) = f(z) = b.$

Since $f^{-1}(\emptyset) = \emptyset$ and $f^{-1}(\{a\}) = \{x\}$ are g-β-closed subsets of X, then f is contra g-β-preirresolute.

Definition 4.5 Let (X, g_X) and (Y, g_Y) be GTS's. Then a function $f : X \to Y$ is said to be contra g-α-preirresolute at $x \in X$ if there exists a g-α-open set U containing x such that $f(U) \subseteq V$ for each g-preclosed V of Y containing $f(x)$.

Definition 4.6 Let (X, g_X) and (Y, g_Y) be GTS's. Then a function $f : X \to Y$ is said to be contra g-β-preirresolute at $x \in X$ if there exists a g-β-open set U containing x such that $f(U) \subseteq V$ for each g-preclosed V of Y containing $f(x)$.

Theorem 4.7 *Let $f: X \to Y$ be a function from two GTS's. Then the following are equivalent*:

(1) *f is contra g-α-preirresolute*;
(2) *$f^{-1}(F)$ is g-α-open set in X for each g-preclosed set F of Y*;
(3) For each *$x \in X$ and each g-preopen set V of Y with $f(x) \notin V$, there exists g-α-closed set U in X such that $x \notin U$ and $f^{-1}(V) \subseteq U$*;
(4) *f is contra g-α-preirresolute at any $x \in X$*;
(5) *$f^{-1}(V) \subseteq i_\alpha(f^{-1}(V))$ for any g-preclosed set V of Y*;
(6) *$c_\alpha(f^{-1}(U)) \subseteq f^{-1}(U)$ for any g-preopen set U of Y*;
(7) *$c_\alpha(f^{-1}(i_\pi(A))) \subseteq f^{-1}(i_\pi(A))$ for any $A \subseteq Y$*;
(8) *$f^{-1}(c_\pi(A)) \subseteq i_\alpha(f^{-1}(c_\pi(A)))$ for any $A \subseteq Y$.*

Proof $(1) \Rightarrow (2)$. Let F be a g-preclosed set in Y. Then $Y - F$ is a g-preopen set in Y. By (1), $f^{-1}(Y - F) = X - f^{-1}(F)$ is a g-α-closed set in X. Hence $f^{-1}(F)$ is a g-α-open set in X.

$(1) \Rightarrow (3)$. Let $x \in X$ and V be a g-preopen set of Y with $f(x) \notin V$. Then $x \notin f^{-1}(V)$. By (1), $f^{-1}(V)$ is a g-α-closed set in X. Suppose $U = f^{-1}(V)$. Then $f^{-1}(V) \subseteq U$ and $x \notin U$.

$(3) \Rightarrow (1)$. Let V be a g-preopen set of Y. For each $x \in f^{-1}(Y - V), f(x) \notin V$. By (3), there exists a g-α-closed set U_x in X such that $x \notin U_x$ and $f^{-1}(V) \subseteq U_x$. Then $X - U_x \subseteq X - f^{-1}(V) = f^{-1}(Y - V)$. Hence we have

$$\bigcup_{x \in f^{-1}(Y-V)} \{x\} \subseteq \bigcup_{x \in f^{-1}(Y-V)} (X - U_x) \subseteq f^{-1}(Y - V).$$

Thus $f^{-1}(Y - V) = \bigcup_{x \in f^{-1}(Y-V)}(X - U_x)$ is a g-α-open set in X. As a consequence, $f^{-1}(V)$ is a g-α-closed set in X and so f is g-α-preirresolute.

$(2) \Rightarrow (4)$. Let $x \in X$ and V be a g-preclosed set of Y containing $f(x)$. By (2), $f^{-1}(V)$ is a g-α-open set in X containing x. Put $U = f^{-1}(V)$. Thus we obtain U is a g-α-open set in X containing x and $f(U) \subseteq V$.

$(4) \Rightarrow (5)$. Let V be a g-preclosed set of Y. For each $x \in f^{-1}(V), f(x) \in V$. By (4), there exists a g-α-open set U in X containing x such that $f(U) \subseteq V$. Since $x \in U \subseteq f^{-1}(V)$, we obtain $x \in i_\alpha(f^{-1}(V))$. Thus $f^{-1}(V) \subseteq i_\alpha(f^{-1}(V))$.

$(5) \Rightarrow (6)$. Let U be a g-preopen set of Y. Then $Y - U$ is a g-preclosed set of Y. By (5), $X - f^{-1}(U) = f^{-1}(Y - U) \subseteq i_\alpha(f^{-1}(Y - U)) = i_\alpha(X - f^{-1}(U)) = X - c_\alpha(f^{-1}(U))$. Thus $c_\alpha(f^{-1}(U))) \subseteq f^{-1}(U)$.

$(6) \Rightarrow (7)$. Let $A \subseteq Y$. Since $i_\pi(A)$ is a g-preopen set of Y, by (6), we obtain $c_\alpha(f^{-1}(i_\pi(A))) \subseteq f^{-1}(i_\pi(A))$.

$(7) \Rightarrow (8)$. Let $A \subseteq Y$. Then $Y - A \subseteq Y$. By (7), $c_\alpha(f^{-1}(i_\pi(Y - A))) = c_\alpha(f^{-1}(Y - c_\pi(A))) = c_\alpha(X - f^{-1}(c_\pi(A))) = X - i_\alpha(f^{-1}(c_\pi(A))) \subseteq f^{-1}(i_\pi(Y - A)) = f^{-1}(Y - c_\pi(A)) = X - f^{-1}(c_\pi(A))$. Thus $f^{-1}(c_\pi(A)) \subseteq i_\alpha(f^{-1}(c_\pi(A)))$.

$(8) \Rightarrow (1)$. Let V be a g-preopen set of Y. Then $Y - V$ is g-preclosed set of Y. By (8), $f^{-1}(c_\pi(Y - V)) = f^{-1}(Y - V) = X - f^{-1}(V) \subseteq i_\alpha(f^{-1}(c_\pi(Y - V))) = i_\alpha(f^{-1}(Y - V)) = i_\alpha(X - f^{-1}(V)) = X - c_\alpha(f^{-1}(V))$. Thus we obtain $c_\alpha(f^{-1}(V)) \subseteq f^{-1}(V)$. As a consequence, $f^{-1}(V)$ is a g-α-closed set in X and f is contra g-α-preirresolute. □

Theorem 4.8 *Let $f: X \to Y$ be a function from two GTS's. Then the following are equivalent:*

(1) *f is contra g-β-preirresolute*;
(2) *$f^{-1}(F)$ is g-β-open set in X for each g-preclosed set F of Y*;
(3) For each *$x \in X$ and each g-preopen set V of Y with $f(x) \notin V$, there exists g-β-closed set U in X such that $x \notin U$ and $f^{-1}(V) \subseteq U$*;
(4) *f is contra g-β-preirresolute at any $x \in X$*;
(5) *$f^{-1}(V) \subseteq i_\beta(f^{-1}(V))$ for any g-preclosed set V of Y*;
(6) *$c_\beta(f^{-1}(U)) \subseteq f^{-1}(U)$ for any g-preopen set U of Y*;
(7) *$c_\beta(f^{-1}(i_\pi(A))) \subseteq f^{-1}(i_\pi(A))$ for any $A \subseteq Y$*;
(8) *$f^{-1}(c_\pi(A)) \subseteq i_\beta(f^{-1}(c_\pi(A)))$ for any $A \subseteq Y$.*

Proof It is similar to that of Theorem 4.7 □

Theorem 4.9 *Let $f: X \to Y$ be a function from two GTS's. Then the following are equivalent:*

(1) *f is contra g-α-preirresolute*;
(2) For each *g-preclosed set F of Y, $f^{-1}(F)$ is g-α-open in X*;
(3) *$f^{-1}(B) \subseteq i_g(c_g(i_g(f^{-1}(c_\pi(B)))))$ for every subset B of Y.*

Proof $(1) \Leftrightarrow (2)$: It is obvious from Definition 4.1 and Theorem 4.7.

$(2) \Rightarrow (3)$: Let $B \subseteq Y$. Since the set $c_\pi(B)$ is g-preclosed in $Y, f^{-1}(c_\pi(B))$ is g-α-open and so

$$f^{-1}(c_\pi(B)) \subseteq i_g(c_g(i_g(f^{-1}(c_\pi(B))))).$$

As a consequence, we obtain

$$f^{-1}(B) \subseteq i_g(c_g(i_g(f^{-1}(c_\pi(B))))).$$

$(3) \Rightarrow (1)$: Let V be a g-preopen in Y. Then $Y - V$ is a subset of Y. By (3),

$$f^{-1}(Y - V) \subseteq i_g(c_g(i_g(f^{-1}(c_\pi(Y - V))))).$$

Hence we obtain

$$c_g(i_g(c_g(f^{-1}(V)))) = c_g(i_g(c_g(f^{-1}(i_\pi(V))))) \subseteq f^{-1}(V).$$

As a consequence, $f^{-1}(V)$ is g-α-closed. □

Theorem 4.10 *Let $f : X \to Y$ be a function from two GTS's. Then the following are equivalent:*

(1) *f is contra g-β-preirresolute;*
(2) *For each g-preclosed set F of Y, $f^{-1}(F)$ is g-β-open in X;*
(3) *$f^{-1}(B) \subseteq c_g(i_g(c_g(f^{-1}(c_\pi(B)))))$ for every subset B of Y.*

Proof It is proved by a similar way of that of Theorem 4.9. □

Theorem 4.11 *Let $f : X \to Y$ be a function from two GTS's. Suppose one of the following conditions hold:*

(1) *$f(c_\alpha(A)) \subseteq i_\pi(f(A))$ for each subset A in X.*
(2) *$c_\alpha(f^{-1}(B)) \subseteq f^{-1}(i_\pi(B))$ for each subset B in Y.*
(3) *$f^{-1}(c_\pi(B)) \subseteq i_\alpha(f^{-1}(B))$ for each subset B in Y.*

Then f is contra g-α-preirresolute.

Proof $(1) \Rightarrow (2)$: Let $B \subseteq Y$. Then $f^{-1}(B) \subseteq X$. By hypothesis, $f(c_\alpha(f^{-1}(B))) \subseteq i_\pi(f(f^{-1}(B))) \subseteq i_\pi(B)$. Then $f^{-1}(f(c_\alpha(f^{-1}(B)))) \subseteq f^{-1}(i_\pi(B))$. Hence $c_\alpha(f^{-1}(B)) \subseteq f^{-1}(f(c_\alpha(f^{-1}(B)))) \subseteq f^{-1}(i_\pi(B))$. As a consequence, (2) is obtained.

$(2) \Rightarrow (3)$: It is obvious from the complement of (2).

Suppose (3) holds: Let $B \subseteq Y$ be g-preclosed. Then by (3), $f^{-1}(c_\pi(B)) \subseteq i_\alpha(f^{-1}(B))$. Thus $f^{-1}(B) = f^{-1}(c_\pi(B)) \subseteq i_\alpha(f^{-1}(B))$. Hence $f^{-1}(B)$ is a g-α-open in X. As a consequence, we obtain f is contra g-α-preirresolute. □

Theorem 4.12 *Let $f : X \to Y$ be a function from two GTS's. Suppose one of the following conditions hold:*

(1) *$f(c_\beta(A)) \subseteq i_\pi(f(A))$ for each subset A in X.*
(2) *$c_\beta(f^{-1}(B)) \subseteq f^{-1}(i_\pi(B))$ for each subset B in Y.*
(3) *$f^{-1}(c_\pi(B)) \subseteq i_\beta(f^{-1}(B))$ for each subset B in Y.*

Then f is contra g-β-preirresolute.

Proof It is similar to proof of Theorem 4.11 □

Theorem 4.13 *Let $f : X \to Y$ be a function from two GTS's and g_X is a strong. f is contra g-α-preirresolute if the graph function $g : X \to X \times Y$ defined by $g(x) = (x, f(x))$ for each $x \in X$, is contra g-α-preirresolute.*

Proof Let $x \in X$ and V be g-preopen containing $f(x)$ in Y. Then $X \times V$ is a g-preopen set of $X \times Y$ by Theorem 2.12 and contains $g(x)$. Then $g^{-1}(X \times V)$ is a g-α-closed set in X. Since $g^{-1}(X \times V) = f^{-1}(V)$, $f^{-1}(V)$ is a g-α-closed set in X. As a consequence, f is contra g-α-preirresolute. □

Theorem 4.14 *Let $f : X \to Y$ be a function from two GTS's and g_X is a strong. f is contra g-β-preirresolute if the graph function $g : X \to X \times Y$ defined by $g(x) = (x, f(x))$ for each $x \in X$, is contra g-β-preirresolute.*

Proof It is proved similar to that of Theorem 4.13. □

Theorem 4.15 *Let g_{Y_k} be a given GT on Y_k for $k \in K$ and g_{Y_k} be a strong. If a function $f : X \to \prod Y_k$ is contra g-α-preirresolute, then $p_k \circ f : X \to Y_k$ is contra g-α-preirresolute for each $k \in K$, where p_k is the projection of $\prod Y_k$ onto Y_k.*

Proof Let V_k be any g-preopen set of Y_k. p_k is $(\pi(g_Y), \pi(g_{Y_k}))$-continuous from Proposition 2.11 since g_{Y_k} is strong and so $p_k^{-1}(V_k)$ is g-preopen set. Since f is contra g-α-preirresolute, $f^{-1}(p_k^{-1}(V_k)) = (p_k \circ f)^{-1}(V_k)$ is a g-α-closed. As a consequence, we have $p_k \circ f$ is contra g-α-preirresolute for each $k \in K$. □

Theorem 4.16 *Let g_{Y_k} be a given GT on Y_k for $k \in K$ and g_{Y_k} be a strong. If a function $f : X \to \prod Y_k$ is contra g-β-preirresolute, then $p_k \circ f : X \to Y_k$ is contra g-β-preirresolute for each $k \in K$, where p_k is the projection of $\prod Y_k$ onto Y_k.*

Proof It is similar to that of Theorem 4.15 □

Theorem 4.17 *If the function $f : \prod X_k \to \prod Y_k$ defined by $f(\{x_k\}) = \{f_k(x_k)\}$ for each $\{x_k\} \in \prod X_k$, is contra g-α-preirresolute, then $f_k : X_k \to Y_k$ is contra g-α-preirresolute for each $k \in K$.*

Proof Let $k_0 \in K$ be an arbitrary fixed index and V_{k_0} be any g-preopen set of Y_{k_0}. Then $\prod Y_m \times V_{k_0}$ is g-preopen in $\prod Y_k$ by Theorem 2.12, where $k_0 \neq m \in K$. Since f is contra g-α-preirresolute, $f^{-1}(\prod Y_m \times V_{k_0}) = \prod X_m \times f_{k_0}^{-1}(V_{k_0})$ is g-α-closed in $\prod X_k$ and $f_{k_0}^{-1}(V_{k_0})$ is g-α-closed in X_{k_0}. As a consequence, f_{k_0} is contra g-α-preirresolute. □

Theorem 4.18 *If the function $f : \prod X_k \to \prod Y_k$ defined by $f(\{x_k\}) = \{f_k(x_k)\}$ for each $\{x_k\} \in \prod X_k$, is contra g-β-preirresolute, then $f_k : X_k \to Y_k$ is contra g-β-preirresolute for each $k \in K$.*

Proof It is proved similar to Theorem 4.17. □

Theorem 4.19 *If $f : X \to Y$ is contra g-α-preirresolute and A is a g-αclosed in-X, then the restriction $f|A : A \to Y$ is contra g-α-preirresolute.*

Proof Let V be any g-preopen set in Y. Then we have $f^{-1}(V)$ is a g-α-closed set in Y. Since the set A is g-α-closed set, we have $(f|A)^{-1}(V) = A \cap f^{-1}(V)$ is g-α-closed. Therefore $f|A$ is contra g-α-preirresolute. □

Theorem 4.20 *If $f : X \to Y$ is contra g-β-preirresolute and A is a g-β-closed in X, then the restriction $f|A : A \to Y$ is contra g-β-preirresolute.*

Proof It is proved by a similar way as that of Theorem 4.19. □

Theorem 4.21 *Let (X, g_X), (Y, g_Y) and (Z, g_Z) be GTS's. If $f : X \to Y$ is contra g-α-preirresolute and $g : Y \to Z$ is g-preirresolute, then the composition $g \circ f : X \to Z$ is contra g-α-preirresolute.*

Proof Let V be any g-preopen subset of Z. Since g function is g-preirresolute, $g^{-1}(V)$ is g-preopen in Y. Since f is contra g-α-preirresolute, then $f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V)$ is g-α-closed in X. As a consequence, $g \circ f$ is contra g-α-preirresolute.

□

Theorem 4.22 *Let (X, g_X), (Y, g_Y) and (Z, g_Z) be GTS's. If $f : X \to Y$ is contra g-β-preirresolute and $g : Y \to Z$ is g-preirresolute, then the composition $g \circ f : X \to Z$ is contra g-β-preirresolute.*

Proof It is proved similar to that of Theorem 4.21. □

Conclusion

The concepts of g-α-preirresolute, g-β-preirresolute, contra g-α-preirresolute, contra g-β-preirresolute have been introduced on generalized topological spaces and some properties of this continuity have been investigated. These concepts may be used in other topological spaces and can be defined in different forms.

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