On the extended Hecke group (H)over-bar (lambda(5))

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On the Extended Hecke Group $\overline{H}(\lambda_5)$

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Abstract. We consider the extended Hecke group $\overline{H}(\lambda_5)$ generated by $T(z) = -1/z$, $S(z) = -1/(z + \lambda_5)$ and $R(z) = 1/\overline{z}$ with $\lambda_5 = 2\cos(\pi/5) = (1 + \sqrt{5})/2$. In this paper, we study the abstract group structure of the extended Hecke group and its power subgroups $\overline{H}^{m}(\lambda_5)$. Also, we give relations between power subgroups and commutator subgroups of the extended Hecke group $\overline{H}(\lambda_5)$.

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1 Introduction

In [3], Erich Hecke introduced the groups $H(\lambda)$ generated by two linear fractional transformations

$$T(z) = -\frac{1}{z} \quad \text{and} \quad U(z) = z + \lambda,$$

where λ is a fixed positive real number. Let $S = TU$, i.e.,

$$S(z) = -\frac{1}{z + \lambda}\,.$$

E. Hecke showed that $H(\lambda)$ is Fuchsian if and only if $\lambda = \lambda_q = 2\cos\frac{\pi}{q}$ ($q \in \mathbb{N}$ with $q \geq 3$) or $\lambda \geq 2$ is real. In these two cases, $H(\lambda)$ is called a Hecke group. We consider the former case. Then the Hecke group $H(\lambda_q)$ is the discrete subgroup of $PSL(2, \mathbb{R})$ generated by T and S, and it is isomorphic to the free product of two finite cyclic groups of orders 2 and q. It has a presentation

$$H(\lambda_q) = \langle T, S \mid T^2 = S^q = I \rangle \cong C_2 * C_q. \tag{1}$$

Actually, the modular group $H(\lambda_3) = PSL(2, \mathbb{Z})$ has been worked intensively as an important Hecke group. In this $q = 3$ case, $\lambda_3 = 2\cos\frac{\pi}{3} = 1$, that is, all coefficients of the elements of $H(\lambda_3)$ are rational integers. Also, two most important Hecke groups are those for $q = 4$ and 6. In these cases, $\lambda_q = \sqrt{2}$ and $\sqrt{3}$, respectively, since they are the only Hecke groups whose elements are completely known.

The next most important q is 5 because it is the only other value of q for which $\mathbb{Q}(\lambda_q)$ is a quadratic field, and also because of Leutbecher's result [8], which essentially states that for $q = 5$ all elements of the field $\mathbb{Q}(\lambda_5)$ are cusp points. The elements of $H(\lambda_5)$ are worked out by D. Rosen in [10, 11]. Using continued λ-fractions, he gave necessary and sufficient conditions for a substitution to be an element of $H(\lambda_5)$.

For all other q, the degree is greater than 2. As a consequence, $q = 5$ is the next most workable and interesting q. Some of the classical results on the modular group can be generalized to $H(\lambda_5)$. The Hecke group $H(\lambda_5)$ and its normal subgroups have been studying extensively for many aspects in [2, 6, 7]. Also, the Hecke group $H(\lambda_3)$, the modular group $PSL(2,\mathbb{Z})$, has especially been of great interest in many fields of mathematics, for example, number theory, automorphic function theory and group theory.

The extended modular group $\overline{H}(\lambda_3)$ has been defined (see [4, 5, 15]) by adding the reflection $R(z) = 1/\overline{z}$ to the generators of the modular group $H(\lambda_3)$. Then the extended Hecke groups $\overline{H}(\lambda_q)$ have been defined similarly to the extended modular group case $\overline{H}(\lambda_3)$ in [12, 13]. They studied commutator subgroups $\overline{H}'(\lambda_q)$, $\overline{H}''(\lambda_q)$, even subgroups $\overline{H}_e(\lambda_q)$ and principal subgroups $\overline{H}_p(\lambda_q)$ of the extended Hecke groups $\overline{H}(\lambda_q)$. Also in [14], we investigated the power and free subgroups of the extended modular group $\overline{H}(\lambda_3)$ and the relations between power subgroups and commutator subgroups.

In this work, we continue our study to extend to which properties of Hecke groups hold for the extended Hecke groups. Firstly, we give a proof of the fact that the extended Hecke group $\overline{H}(\lambda_5)$ is isomorphic to the free product of two finite dihedral groups of orders 4 and 10 with amalgamation $\mathbb{Z}_2$. Then we determine the group structure of the power subgroups $\overline{H}^m(\lambda_5)$, and finally, we give relations between the commutator subgroups and power subgroups of the extended Hecke groups.

2 Extended Hecke Group $\overline{H}(\lambda_5)$ and Its Decomposition

By (1), the Hecke group $H(\lambda_5)$ has a presentation

$$H(\lambda_5) = \langle T, S \mid T^2 = S^5 = I \rangle \cong C_2 * C_5.$$

We define the extended Hecke group $\overline{H}(\lambda_5)$ by adding the reflection $R(z) = 1/\overline{z}$ to the generators of $H(\lambda_5)$. Then we have

$$\overline{H}(\lambda_5) = \langle T, S, R \mid T^2 = S^5 = R^2 = I,\ RT = TR,\ RS = S^{-1}R \rangle$$

or

$$\overline{H}(\lambda_5) = \langle T, S, R \mid T^2 = S^5 = R^2 = (TR)^2 = (SR)^2 = I \rangle. \tag{2}$$

The Hecke group $H(\lambda_5)$ is a subgroup of index 2 in $\overline{H}(\lambda_5)$.

Now $H(\lambda_5)$ has trivial centre, and its outer automorphism class group $\operatorname{Out} H(\lambda_5) = \operatorname{Aut} H(\lambda_5)/\operatorname{Inn} H(\lambda_5)$ is generated by the automorphism fixing T and inverting S, so the action of $\overline{H}(\lambda_5)$ on $H(\lambda_5)$ by conjugation induces an isomorphism $\overline{H}(\lambda_5) \cong \operatorname{Aut} H(\lambda_5)$ with R corresponding to the required outer automorphism.

The function

$$\alpha : T \to RT, \quad S \to S, \quad R \to R$$

preserves the relations in (2), so it extends to an endomorphism of $\overline{H}(\lambda_5)$ since α^2 is the identity, α is an automorphism, which cannot be inner since $T \in H(\lambda_5) \trianglelefteq \overline{H}(\lambda_5)$ whereas $T\alpha = RT \notin H(\lambda_5)$. Therefore, the outer automorphism class group $\operatorname{Out} \overline{H}(\lambda_5) = \operatorname{Aut} \overline{H}(\lambda_5)/\operatorname{Inn} \overline{H}(\lambda_5)$ has order 2, being generated by α.

In terms of (2) we have

$$\alpha : T \to RT, \quad S \to S, \quad R \to R$$

so that

$$H(\lambda_5)\alpha = \langle RT, S \mid (RT)^2 = S^5 = I\rangle.$$

The Hecke group $H(\lambda_5)\alpha$ is a subgroup of index 2 in $\overline{H}(\lambda_5)$.

Now we give a theorem about the group structure of the extended Hecke group $\overline{H}(\lambda_5)$.

Theorem 2.1. *The extended Hecke group $\overline{H}(\lambda_5)$ is given directly as a free product of two groups G_1 and G_2 with amalgamated subgroup $\mathbb{Z}_2$, where G_1 is the dihedral group D_2 and G_2 is the dihedral group D_5, i.e., $\overline{H}(\lambda_5) \cong D_2 *_{\mathbb{Z}_2} D_5$.*

Proof. The result follows from a presentation of the extended Hecke group $\overline{H}(\lambda_5)$ given in (2).

Let

$$G_1 = \langle T, R \mid T^2 = R^2 = (TR)^2 = I\rangle \cong D_2$$

and

$$G_2 = \langle S, R \mid S^5 = R^2 = (SR)^2 = I\rangle \cong D_5.$$

Then $\overline{H}(\lambda_5)$ is $G_1 * G_2$ with the identification $R = R$.

In G_1, the subgroup generated by R is $\mathbb{Z}_2$, this is also true in G_2. Therefore, the identification induces an isomorphism and $\overline{H}(\lambda_5)$ is a generalized free product with the subgroup $M \cong \mathbb{Z}_2$ amalgamated. □

3 Power Subgroups of $\overline{H}(\lambda_5)$

Let m be a positive integer. Let us define $\overline{H}^m(\lambda_5)$ to be the subgroup generated by the m-th powers of all elements of $\overline{H}(\lambda_5)$, called the *m-th power subgroup* of $\overline{H}(\lambda_q)$. As fully invariant subgroups, they are normal in $\overline{H}(\lambda_5)$. The power subgroups of $H(\lambda_5)$ were studied by İ.N. Cangül [1]. He proved that

$$\begin{aligned} &H^2(\lambda_5) = \langle S\rangle * \langle TST\rangle, \\ &H^5(\lambda_5) = \langle T\rangle * \langle STS^4\rangle * \langle S^2TS^3\rangle * \langle S^3TS^2\rangle * \langle S^4TS\rangle, \\ &H'(\lambda_5) = H^2(\lambda_5) \cap H^5(\lambda_5), \end{aligned}$$

and $H^{10k}(\lambda_5)$ are free groups.

Now we consider the presentation of the extended Hecke group $\overline{H}(\lambda_5)$ given in (2). Firstly we find a presentation for the quotient $\overline{H}(\lambda_5)/\overline{H}^m(\lambda_5)$ by adding the relation $X^m = I$ to the presentation of $\overline{H}(\lambda_5)$. The order of $\overline{H}(\lambda_5)/\overline{H}^m(\lambda_5)$ gives us the index. We have

$$\begin{aligned}\overline{H}(\lambda_5)/\overline{H}^m(\lambda_5) \cong \langle T, S, R \mid\ & T^2 = S^5 = R^2 = (TR)^2 = (SR)^2 = I,\\ & T^m = S^m = R^m = (TR)^m = (SR)^m = \cdots = I\rangle.\end{aligned} \tag{3}$$

Thus, we use the Reidemeister–Schreier process to find the presentation of the power subgroups $\overline{H}^m(\lambda_5)$. The idea is as follows: We first choose (not uniquely) a Schreier transversal Σ for $\overline{H}^m(\lambda_5)$. (This method, in general, applies to all normal subgroups.) Σ consists of some words in T, S and R. Then we take all possible products in the following order:

(element of Σ) $\times$ (generator) $\times$ (coset representative of the preceeding product)$^{-1}$.

We now see the situation for $m = 2$:

Theorem 3.1. *The normal subgroup $\overline{H}^2(\lambda_5)$ is isomorphic to the free product of two finite cyclic groups of order five. Also,*

$$\begin{aligned}&\overline{H}(\lambda_5)/\overline{H}^2(\lambda_5) \cong C_2 \times C_2,\\ &\overline{H}(\lambda_5) = \overline{H}^2(\lambda_5) \cup T\overline{H}^2(\lambda_5) \cup R\overline{H}^2(\lambda_5) \cup TR\overline{H}^2(\lambda_5),\\ &\overline{H}^2(\lambda_5) = \langle S\rangle * \langle TST\rangle.\end{aligned}$$

The elements of $\overline{H}^2(\lambda_5)$ are characterised by the requirement that the sum of the exponents of T is even.

Proof. By (3), we obtain $T^2 = R^2 = I$ and $S = I$ from the relations $S^5 = S^2 = I$. Then we get

$$\overline{H}(\lambda_5)/\overline{H}^2(\lambda_5) \cong \langle T, R \mid T^2 = R^2 = (TR)^2 = I\rangle \cong C_2 \times C_2,$$

and therefore,

$$\left|\overline{H}(\lambda_5) : \overline{H}^2(\lambda_5)\right| = 4.$$

Now we choose $\{I, T, R, TR\}$ as a Schreier transversal for $\overline{H}^2(\lambda_5)$. According to the Reidemeister–Schreier method, we can form all possible products:

$$\begin{array}{lll} I.T.(T)^{-1} = I, & I.S.(I)^{-1} = S, & I.R.(R)^{-1} = I,\\ T.T.(I)^{-1} = I, & T.S.(T)^{-1} = TST, & T.R.(TR)^{-1} = I,\\ R.T.(TR)^{-1} = RTRT, & R.S.(R)^{-1} = RSR, & R.R.(I)^{-1} = I,\\ TR.T.(R)^{-1} = TRTR, & TR.S.(TR)^{-1} = TRSRT, & TR.R.(T)^{-1} = I.\end{array}$$

We see that S and TST are the generators since $RTRT = I$, $TRTR = I$, $RSR = S^{-1}$ and $TRSRT = TS^{-1}T = (TST)^{-1}$. Thus, we have

$$\overline{H}^2(\lambda_5) = \langle S, TST \mid S^5 = (TST)^5 = I\rangle \cong C_5 * C_5,$$

and

$$\overline{H}(\lambda_5) = \overline{H}^2(\lambda_5) \cup T\overline{H}^2(\lambda_5) \cup R\overline{H}^2(\lambda_5) \cup TR\overline{H}^2(\lambda_5). \qquad \square$$

Theorem 3.2. $\overline{H}^5(\lambda_5) = \overline{H}(\lambda_5)$.

Proof. By (3), we find $S = T = R = I$ from the relations

$$R^2 = R^5 = I, \quad S^5 = S^5 = (SR)^2 = (SR)^5 = I, \quad T^2 = T^5 = I.$$

Thus, we have

$$\left|\overline{H}(\lambda_5) : \overline{H}^5(\lambda_5)\right| = 1,$$

i.e., $\overline{H}^5(\lambda_5) = \overline{H}(\lambda_5)$. $\square$

We can now obtain a classification of these subgroups:

Theorem 3.3. *Let m be a positive integer.*
(i) $\overline{H}^m(\lambda_5) = \overline{H}(\lambda_5)$ *if* $2 \nmid m$.
(ii) $\overline{H}^m(\lambda_5) = \langle S\rangle * \langle TST\rangle$ *if* $2 \mid m$ *but* $10 \nmid m$.

Proof. (i) If $2 \nmid m$, then by (3), we find $S = T = R = I$ from the relations

$$R^2 = R^m = I, \quad S^5 = S^m = (SR)^2 = I = I, \quad T^2 = T^m = I.$$

Thus, $\overline{H}(\lambda_5)/\overline{H}^m(\lambda_5)$ is trivial, and hence, $\overline{H}^m(\lambda_5) = \overline{H}(\lambda_5)$.

(ii) If $2 \mid m$ but $10 \nmid m$, then $(m, 5) = 1$. By (3), we obtain $S = T^2 = R^2 = I$ from the relations

$$R^2 = R^m = I, \quad S^5 = S^m = I, \quad T^2 = T^m = I$$

as $2 \mid m$ but $10 \nmid m$. These show that

$$\overline{H}(\lambda_5)/\overline{H}^m(\lambda_5) \cong \langle T, R \mid T^2 = R^2 = (TR)^2 = I\rangle \cong D_2$$

and

$$\left|\overline{H}(\lambda_5) : \overline{H}^m(\lambda_5)\right| = 4.$$

Now we choose $\{I, T, R, TR\}$ as a Schreier transversal for $\overline{H}^2(\lambda_5)$. According to the Reidemeister–Schreier method, we find S and TST as the generators. Therefore, $\overline{H}^m(\lambda_5) = \overline{H}^2(\lambda_5)$. $\square$

Now we have only left the subgroups $\overline{H}^{10k}(\lambda_5)$ to consider. To this end, we need to consider the commutator subgroups of the extended Hecke group $\overline{H}(\lambda_5)$.

Theorem 3.4.
(i) $\overline{H}(\lambda_5)/\overline{H}'(\lambda_5) \cong V_4 \cong C_2 \times C_2$.
(ii) $\overline{H}'(\lambda_5) = \langle S, TST \mid S^5 = (TST)^5 = I\rangle \cong C_5 * C_5$.
(iii) $\overline{H}'(\lambda_5)/\overline{H}''(\lambda_5) \cong V_{25}$, *where* V_{25} *denotes an elementary abelian group of order* 25.

(iv) $\overline{H}''(\lambda_5)$ *is a free group with basis* $[S, TST]$, $[S, TS^2T]$, $[S, TS^3T]$, $[S, TS^4T]$, $[S^2, TST]$, $[S^2, TS^2T]$, $[S^2, TS^3T]$, $[S^2, TS^4T]$, $[S^3, TST]$, $[S^3, TS^2T]$, $[S^3, TS^3T]$, $[S^3, TS^4T]$, $[S^4, TST]$, $[S^4, TS^2T]$, $[S^4, TS^3T]$, $[S^4, TS^4T]$.

Proof. Refer to [12, 13]. □

Notice that $\overline{H}'(\lambda_5)$ is a subgroup of $H(\lambda_5)$, consisting of the words in T and S for which T has even exponent-sum.

It is easy to see the following results from Theorems 3.1 and 3.4:

Theorem 3.5.

(i) $\overline{H}^2(\lambda_5) = H^2(\lambda_5) = \overline{H}'(\lambda_5) = \overline{H}^2(\lambda_5) \cap \overline{H}^5(\lambda_5)$.

(ii) $\overline{H}^{10}(\lambda_5) \subset \overline{H}''(\lambda_5)$.

(iii) $\overline{H}^{10k}(\lambda_5) = H^{10k}(\lambda_5)$.

By means of these results, we are able to investigate the subgroups $\overline{H}^{10k}(\lambda_5)$. Now because $\overline{H}''(\lambda_5)$ is a free group and $\overline{H}^{10k}(\lambda_5) \subset \overline{H}^{10}(\lambda_5) \subset \overline{H}''(\lambda_5)$, by Schreier's theorem, we have:

Theorem 3.6. *The subgroups* $\overline{H}^{10k}(\lambda_5)$ *are free.*

Corollary 3.7. $\overline{H}'(\lambda_5) = H(\lambda_5) \cap H(\lambda_5)\alpha$.

Proof. Both $H(\lambda_5)$ and $H(\lambda_5)\alpha$ have index 2 in $\overline{H}(\lambda_5)$, so $H(\lambda_5) \cap H(\lambda_5)\alpha$ has index 4, and hence, $\overline{H}'(\lambda_5) = H(\lambda_5) \cap H(\lambda_5)\alpha$ by Theorem 3.4(i). □

We can give the following diagram which is the relation between power subgroups and commutator subgroups of the extended Hecke group $\overline{H}(\lambda_5)$:

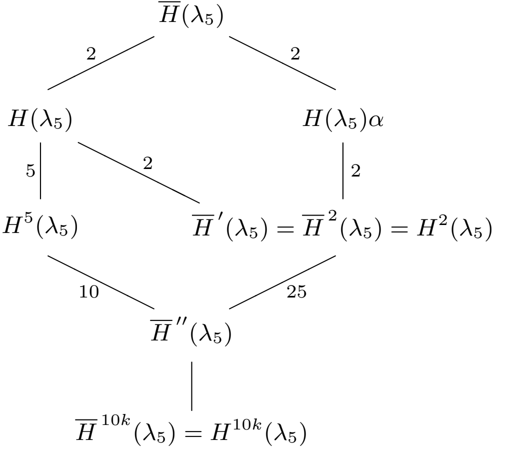

Fig. 1. Some subgroups of $\overline{H}(\lambda_5)$

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