

# Approximation properties of the Bieberbach polynomials in closed Dini-smooth domains

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## Abstract

Let  $G$  be a finite Dini-smooth domain and  $w = \varphi_0(z)$  be the conformal mapping of  $G$  onto  $D(0, r_0) := \{w : |w| < r_0\}$  with the normalization  $\varphi_0(z_0) = 0$ ,  $\varphi_0'(z_0) = 1$ , where  $z_0 \in G$ . We investigate the approximation properties of the Bieberbach polynomials  $\pi_n(z)$ ,  $n = 1, 2, 3, \dots$  for the pair  $(G, z_0)$  and estimate the error

$$\|\varphi_0 - \pi_n\|_{\overline{G}} := \max\{|\varphi_0(z) - \pi_n(z)| : z \in \overline{G}\}$$

in accordance with the geometric parameters of  $\overline{G}$ .

## 1 Introduction

In the following  $G$  is a domain in the complex plane  $\mathbb{C}$ , bounded by a Jordan curve  $L$ , and  $w = \varphi_0(z)$  is the conformal mapping of  $G$  onto the disk  $D(0, r_0) := \{w : |w| < r_0, r_0 > 0\}$  normalized by

$$\varphi_0(z_0) = 0, \varphi_0'(z_0) = 1$$

with some fixed  $z_0 \in G$ .

Without loss of generality, we may assume that the conformal radius of  $G$  with respect to  $z_0$  equals 1. Let  $\psi_0(w)$  be the inverse to  $\varphi_0(z)$ . Let also

$$G^- := \mathbb{C} \setminus \overline{G}, \quad D := D(0, 1), \quad T = \partial D, \quad D^- := \mathbb{C} \setminus \overline{D}.$$

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We denote by  $\varphi$  the conformal mapping of  $G^-$  onto  $D^-$ , with the normalization

$$\varphi(\infty) = \infty, \quad \lim_{z \rightarrow \infty} \frac{\varphi(z)}{z} > 0$$

and we also set  $\psi := \varphi^{-1}$ .

The conformal mappings play an important role in many areas of applied mathematics and mechanics. But the determination of these mappings is very difficult; only for some special domains the conformal mappings have an explicit analytical expression. Therefore it is actually the approximation of the conformal mappings by polynomials, expressed in terms of some other data, which can be found easily.

One elegant polynomial to achieve this is the Bieberbach polynomial  $\pi_n$ ,  $n = 1, 2, \dots$ , for the pair  $(G, z_0)$ , which admits the representation

$$\frac{\int_{z_0}^z \sum_{j=0}^{n-1} \overline{P_j(z_0)} P_j(\zeta) d\zeta}{\sum_{j=0}^{n-1} |P_j(z_0)|^2}, \quad z \in G$$

with respect to the orthogonal polynomials  $P_j(z)$ ,  $j = 0, 1, 2, \dots, n-1$ , over  $G$ .

Given a domain  $G$ , our problem is to find an estimate for

$$\|\varphi_0 - \pi_n\|_{\overline{G}} := \max_{z \in \overline{G}} |\varphi_0(z) - \pi_n(z)|,$$

depending on the geometric properties of the boundary  $L$ .

If  $L$  has a certain degree of smoothness, this error tends to zero with a certain rate. In several papers (see, for example [5], [6], [3], [14], [13], [7], [8], [9], [10]) various estimates of the error  $\|\varphi_0 - \pi_n\|_{\overline{G}}$  and sufficient conditions on the geometry of the boundary  $L$  are given to guarantee the uniform convergence of the Bieberbach polynomials on  $\overline{G}$ . In particular if  $G$  is a Lyapunov domain, i.e. if the tangent direction angle  $\theta(s)$  of  $L$ , expressed as a function of the arclength  $s$ , satisfies the condition

$$|\theta(s_1) - \theta(s_2)| \leq c |s_1 - s_2|^\alpha, \quad \alpha \in (0, 1)$$

with  $c = c(L) > 0$ , then by [14]

$$\|\varphi_0 - \pi_n\|_{\overline{G}} \leq \frac{c \ln n}{n^{\frac{1}{2} + \alpha}}, \quad 0 < \alpha < 1 \quad (1)$$

for some constant  $c$  which is independent of  $n$ . According to [13] and [9], for the same domain there is also the estimate

$$\|\varphi_0 - \pi_n\|_{\overline{G}} \leq \frac{c \sqrt{\ln n}}{n^{\frac{1}{2} + \alpha}} \quad (2)$$

which improves the Wu estimate (1).

In this paper we give a generalization of the estimate (2) for a new class of domains containing the Lyapunov domains as a particular case. As we shall see in the next section, if  $G$  is a Lyapunov domain, then this estimate coincides with (2).

We shall use  $c, c_1, \dots$  to denote general constants not depending on specific data of the problems posed. For  $a > 0$  and  $b > 0$  we say that  $a \asymp b$  (equivalence relation) if  $c_1 b \leq a \leq c_2 b$  for some constants  $c_1, c_2 > 0$ .

## 2 New results

A curve  $L$  is called smooth if there is a parametrization of  $L : w(\tau)$ ,  $0 \leq \tau \leq 2\pi$ , such that  $w'(\tau)$  is continuous and  $w'(\tau) \neq 0$ . On the other hand the curve is smooth if and only if it has a continuously varying tangent. Since the characterization of smoothness in terms of tangents does not depend on the parametrization considered, we may choose for convenience the conformal parametrization, i.e.,  $w(t) = \psi_0(e^{it})$ ,  $0 \leq t \leq 2\pi$ .

Let  $L$  be a smooth Jordan curve and let  $\theta(t)$  be the tangent direction angle of  $L$  at  $w(t) = \psi_0(e^{it})$ .

**Definition 1** We say that  $L \in \mathfrak{B}(\alpha, \beta)$  if

$$\omega(\theta, \delta) := \sup_{|h| \leq \delta} \|\theta(\cdot) - \theta(\cdot + h)\|_{[0, 2\pi]} \leq c\delta^\alpha \ln^\beta \frac{4}{\delta}, \quad \delta \in (0, \pi]$$

for some parameters  $\alpha \in (0, 1]$  and  $\beta \in [0, \infty)$  and for a positive constant  $c$  independent of  $\delta$ .

In particular the class  $\mathfrak{B}(\alpha, 0)$ ,  $0 < \alpha < 1$ , coincides with the class of Lyapunov curves. Furthermore, it is easy to verify that if  $0 < \alpha_1 < \alpha_2 \leq 1$ , then

$$\mathfrak{B}(\alpha_1, \beta) \supset \mathfrak{B}(\alpha_2, \beta), \quad \beta \in [0, \infty)$$

and also

$$\mathfrak{B}(\alpha, \beta_1) \subset \mathfrak{B}(\alpha, \beta_2), \quad \alpha \in (0, 1]$$

for  $0 \leq \beta_1 < \beta_2 < \infty$ .

Our main result is presented in the following theorem and proved in section 4.

**Theorem 1** If  $L \in \mathfrak{B}(\alpha, \beta)$  with  $\alpha \in (0, 1]$  and  $\beta \in [0, \infty)$ , then

$$\|\varphi_0 - \pi_n\|_{\overline{G}} \leq \begin{cases} \frac{c \ln^{\beta+1/2} n}{n^{\alpha+1/2}}, & \alpha \in (0, 1); \\ \frac{c \ln^{\beta+3/2} n}{n^{3/2}}, & \alpha = 1, \end{cases}$$

with a constant  $c > 0$  independent of  $n$ .

The estimate given in Theorem 1 is a generalization of (2) to a wider class of domains than the Lyapunov domains. In case of Lyapunov domains  $\mathfrak{B}(\alpha, 0)$  with  $\alpha \in (0, 1)$  we have the following corollary.

**Corollary 1** If  $L \in \mathfrak{B}(\alpha, 0)$ , then

$$\|\varphi_0 - \pi_n\|_{\overline{G}} \leq \frac{c\sqrt{\ln n}}{n^{\frac{1}{2}+\alpha}}, \quad \alpha \in (0, 1).$$

The estimate given in the above Corollary 1 was already proved in [13] and [9]. It improves the Wu estimate (1) in [14].

### 3 Auxiliary results

We start with the following definition.

**Definition 2** *The smooth curve  $L$  is called Dini-smooth if*

$$\int_0^c \frac{\omega(\theta, t)}{t} dt < \infty$$

for some  $c > 0$ .

The following lemma holds.

**Lemma 1** *If  $L \in \mathfrak{B}(\alpha, \beta)$  with  $\alpha \in (0, 1]$  and  $\beta \in [0, \infty)$ , then  $L$  is Dini-smooth.*

*Proof* If  $L \in \mathfrak{B}(\alpha, \beta)$  then by definition 1

$$\omega(\theta, t) \leq ct^\alpha \ln^\beta \frac{4}{t}$$

and hence

$$\int_0^c \frac{\omega(\theta, t)}{t} dt \leq c \int_0^c \frac{t^\alpha \ln^\beta \frac{4}{t}}{t} dt \leq c_1 \int_0^c t^{\alpha-1-\varepsilon} dt < \infty,$$

since  $\lim_{t \rightarrow 0} t^\varepsilon \ln^\beta \frac{4}{t} = 0$  for every fixed  $\varepsilon > 0$  and  $\beta \geq 0$ . ■

As every curve  $L \in \mathfrak{B}(\alpha, \beta)$  with  $\alpha \in (0, 1]$  and  $\beta \in [0, \infty)$  is Dini-smooth, the derivative  $\varphi'_0$  is continuous on  $L$  and hence the modulus of continuity given below is well-defined.

**Definition 3** *Let  $\Phi(w) := \varphi'_0[\psi(w)]$  and  $L \in \mathfrak{B}(\alpha, \beta)$  with  $\alpha \in (0, 1]$  and  $\beta \in [0, \infty)$ . The function*

$$\tilde{\omega}(\varphi'_0, \delta) := \sup_{|h| \leq \delta} \left\| (\varphi'_0 \circ \psi)(we^{ih}) - (\varphi'_0 \circ \psi)(w) \right\|_T := \omega(\Phi, \delta)$$

is called the generalized modulus of continuity for  $\varphi'_0$ .

**Lemma 2** *If  $L \in \mathfrak{B}(\alpha, \beta)$  with  $\alpha \in (0, 1]$  and  $\beta \in [0, \infty)$ , then*

$$\omega(\psi'_0, \delta) := \sup_{|h| \leq \delta} \left\| \psi'_0(we^{ih}) - \psi'_0(w) \right\|_T \leq \begin{cases} c\delta^\alpha \ln^\beta \frac{4}{\delta}, & \alpha \in (0, 1); \\ c\delta \ln^{\beta+1} \frac{4}{\delta}, & \alpha = 1. \end{cases}$$

*Proof.* As  $L$  is Dini-smooth we have [11, p.44]

$$\arg \psi'_0(e^{it}) = \theta(t) - t - \frac{\pi}{2} \tag{3}$$

and

$$\log \psi'_0(w) = \frac{i}{2\pi} \int_0^{2\pi} \frac{e^{it} + w}{e^{it} - w} \left( \theta(t) - t - \frac{\pi}{2} \right) dt. \quad (4)$$

Moreover, if we denote  $g(t) := \arg \psi'_0(e^{it})$ , then by (3)

$$\omega(g, \delta) := \sup_{|h| \leq \delta} \|g(t+h) - g(t)\|_{[0, 2\pi]} \leq \omega(\theta, \delta) + \omega(t, \delta) \leq c_2 \delta^\alpha \ln^\beta \frac{4}{\delta}. \quad (5)$$

On the other hand, for

$$\omega^*(g, \delta) := \int_0^\delta \frac{\omega(g, t)}{t} dt + \delta \int_\delta^\pi \frac{\omega(g, t)}{t^2} dt$$

we have

$$\omega^*(g, \delta) \leq \begin{cases} c\delta^\alpha \ln^\beta \frac{4}{\delta}, & \alpha \in (0, 1); \\ c\delta \ln^{\beta+1} \frac{4}{\delta}, & \alpha = 1. \end{cases} \quad (6)$$

Indeed, if  $\alpha \in (0, 1)$ , then for a sufficiently small  $\varepsilon > 0$

$$t^{\alpha-\varepsilon} \ln^\beta \frac{4}{t} \leq c_3 \delta^{\alpha-\varepsilon} \ln^\beta \frac{4}{\delta}, \quad t \in (0, \delta]$$

with a constant  $c_3$  independent of  $\delta$ . On the other hand

$$\ln^\beta \frac{4}{t} \leq \ln^\beta \frac{4}{\delta}, \quad t \in [\delta, \pi]$$

for every  $\beta \in [0, \infty)$ . Hence by relation (5) we have

$$\begin{aligned} \omega^*(g, \delta) &\leq c_2 \int_0^\delta \frac{t^{\alpha-\varepsilon} \ln^\beta \frac{4}{t}}{t^{1-\varepsilon}} dt + c_2 \delta \int_\delta^\pi \frac{t^\alpha \ln^\beta \frac{4}{t}}{t^2} dt \\ &\leq c_4 \delta^{\alpha-\varepsilon} \ln^\beta \frac{4}{\delta} \int_0^\delta t^{\varepsilon-1} dt + c_5 \delta \ln^\beta \frac{4}{\delta} \int_\delta^\pi t^{\alpha-2} dt \leq c\delta^\alpha \ln^\beta \frac{4}{\delta}. \end{aligned}$$

If  $\alpha = 1$ , then by similar arguments we get

$$\begin{aligned} \omega^*(g, \delta) &\leq c_6 \int_0^\delta \frac{t^\varepsilon \ln^\beta \frac{4}{t}}{t^\varepsilon} dt + c_7 \delta \int_\delta^\pi \frac{\ln^\beta \frac{4}{t}}{t} dt \\ &\leq c_8 \delta^\varepsilon \ln^\beta \frac{4}{\delta} \int_0^\delta t^{-\varepsilon} dt - c_7 \delta \int_\delta^\pi \ln^\beta \frac{4}{t} d \ln \frac{4}{t} \\ &\leq c_9 \delta \ln^\beta \frac{4}{\delta} + c_{10} \delta \ln^{\beta+1} \frac{4}{\delta} \leq c\delta \ln^{\beta+1} \frac{4}{\delta}. \end{aligned}$$

Now as  $L$  is Dini-smooth, the  $2\pi$  periodic function  $g(t) = \arg \psi'_0(e^{it}) = \theta(t) - t - \frac{\pi}{2}$  is Dini-continuous on the real line. Hence starting from (3), (4) and applying *Proposition 3.4* from [11, p.47] we conclude that the function  $\log \psi'_0(w)$  has a continuous extension to  $\overline{D}$  and furthermore

$$\left| \log \psi'_0(w_1) - \log \psi'_0(w_2) \right| \leq c\omega^*(g, \delta)$$

for  $w_1, w_2 \in \overline{D}$  with  $|w_1 - w_2| \leq \delta < 1$ .

The last inequality implies that

$$\left| \psi'_0(w_1) - \psi'_0(w_2) \right| \leq c_{11}\omega^*(g, \delta), \quad |w_1 - w_2| \leq \delta < 1. \quad (7)$$

Now combining the relations (6) and (7) results into

$$\omega(\psi'_0, \delta) \leq \begin{cases} c\delta^\alpha \ln^{\beta+1} \frac{4}{\delta}, & \alpha \in (0, 1); \\ c\delta \ln^{\beta+1} \frac{4}{\delta}, & \alpha = 1. \end{cases}$$

which completes the proof. ■

**Lemma 3** *If  $L \in \mathfrak{B}(\alpha, \beta)$  with  $\alpha \in (0, 1]$  and  $\beta \in [0, \infty)$ , then*

$$\omega(\Phi, \delta) \leq \begin{cases} c\delta^\alpha \ln^{\beta+1} \frac{4}{\delta}, & \alpha \in (0, 1); \\ c\delta \ln^{\beta+1} \frac{4}{\delta}, & \alpha = 1. \end{cases}$$

*Proof.* As  $L$  is Dini-smooth, the functions  $\psi'_0$  and  $\varphi'_0$ , are continuous on  $\overline{D}$  and on  $\overline{G}$  respectively. The same properties are valid also for the functions  $\psi'$  and  $\varphi'$ , respectively on  $\overline{D^-}$  and on  $\overline{G^-}$ . Moreover, the relations  $|\psi'_0| \asymp |\psi'| \asymp 1$  on  $|w| = 1$ , and  $|\varphi'_0| \asymp |\varphi'| \asymp 1$  on  $L$  hold. Hence

$$\begin{aligned} \left\| \varphi_0[\psi(we^{ih})] - \varphi_0[\psi(w)] \right\|_T &\asymp \left\| \psi(we^{ih}) - \psi(w) \right\|_T \\ &\asymp \left\| we^{ih} - w \right\|_T = |e^{ih} - 1| \asymp |h| \end{aligned}$$

and by Lemma 2 we have

$$\begin{aligned} \omega(\Phi, \delta) &= \sup_{|h| \leq \delta} \left\| \Phi(we^{ih}) - \Phi(w) \right\|_T = \sup_{|h| \leq \delta} \left\| \varphi'_0[\psi(we^{ih})] - \varphi'_0[\psi(w)] \right\|_T \\ &= \sup_{|h| \leq \delta} \left\| \frac{1}{\psi'_0[\varphi_0[\psi(we^{ih})]]} - \frac{1}{\psi'_0[\varphi_0[\psi(w)]]} \right\|_T \\ &\leq c_{12} \sup_{|h| \leq \delta} \left\| \psi'_0[\varphi_0[\psi(we^{ih})]] - \psi'_0[\varphi_0[\psi(w)]] \right\|_T \\ &\leq \begin{cases} c\delta^\alpha \ln^{\beta+1} \frac{4}{\delta}, & \alpha \in (0, 1); \\ c\delta \ln^{\beta+1} \frac{4}{\delta}, & \alpha = 1. \end{cases} \end{aligned}$$
■

We denote by  $L^p(L)$ ,  $1 \leq p < \infty$ , the set of all measurable complex valued functions  $f$  on  $L$  such that  $|f|^p$  is Lebesgue integrable with respect to the arclength, and by  $E^p(G)$ ,  $1 \leq p < \infty$  the Smirnov class of analytic functions in  $G$ .

We shall use the following approximation theorem by polynomials in  $E^p(G)$  (see [1]).

**Theorem 2** *Let  $f \in E^p(G)$ ,  $1 < p < \infty$ . If  $L$  is Dini-smooth, then for every natural number  $n$  there are a constant  $c > 0$ , independent of  $n$ , and a polynomial  $P_n(z, f)$  of degree  $n$  such that*

$$\|f - P_n(z, f)\|_{L^p(L)} \leq c\tilde{\omega}\left(f, \frac{1}{n}\right).$$

For the mapping  $\varphi_0$  and a weight function  $\omega$  we put

$$\varepsilon_n(\varphi'_0)_2 := \inf_{P_n} \|\varphi'_0 - P_n\|_{L^2(G)}, \quad E_n^0(\varphi'_0, \omega)_2 := \inf_{P_n} \|\varphi'_0 - P_n\|_{L^2(L, \omega)},$$

where the infimum is taken over all polynomials  $P_n$  of degree at most  $n$  and

$$L^2(L, \omega) := \left\{f \in L^1(L) : |f|^2 \omega \in L^1(L)\right\}$$

for a weight function  $\omega$  given on  $L$ .

## 4 Proof of the main result

Developing for Dini-smooth domains the idea used in [7], [8], and [9], we apply a classical method based on the extremal property of the Bieberbach polynomials, and also the inequality connecting the values  $\varepsilon_n(\varphi'_0)_2$  and  $E_n^0(\varphi'_0, \omega)_2$  established in [4].

*Proof of Theorem 1.* If  $q_n(z)$  is the polynomial of degree at most  $n$ , best approximating  $\varphi'_0$  in the norm  $\|\cdot\|_{L^2(G)}$ , i.e.

$$\|\varphi'_0 - q_n\|_{L^2(G)} = \inf_{P_n} \|\varphi'_0 - P_n\|_{L^2(G)},$$

the infimum being taken over all polynomials  $P_n$  of degree at most  $n$ , then putting

$$Q_n(z) := \int_{z_0}^z q_n(t) dt, \quad t_n(z) := Q_n(z) + [1 - q_n(z_0)](z - z_0)$$

we have  $t_n(z_0) = 0$ ,  $t'_n(z_0) = 1$ . From the inequality

$$\varepsilon_n(\varphi'_0)_2 \leq cn^{-\frac{1}{2}} E_n^0\left(\varphi'_0, \frac{1}{|\varphi'|}\right)_2 \tag{8}$$

established in [4], we obtain

$$\|\varphi'_0 - t'_n\|_{L^2(G)} = \|\varphi'_0 - q_n - 1 + q_n(z_0)\|_{L^2(G)}$$

$$\begin{aligned}
&\leq \varepsilon_n \left(\varphi'_0\right)_2 + \|1 - q_n(z_0)\|_{L_2(G)} \\
&\leq cn^{-\frac{1}{2}} E_n^0 \left(\varphi'_0, \frac{1}{|\varphi'|}\right)_2 + \|\varphi'_0(z_0) - q_n(z_0)\|_{L_2(G)}. \tag{9}
\end{aligned}$$

Applying the inequality

$$|f(z_0)| \leq \frac{\|f\|_{L_2(G)}}{\text{dist}(z_0, L)}$$

which holds for every analytic function  $f$  with  $\|f\|_{L_2(G)} < \infty$ , (8) and (9) yield

$$\|\varphi'_0 - t'_n\|_{L_2(G)} \leq cn^{-\frac{1}{2}} E_n^0 \left(\varphi'_0, \frac{1}{|\varphi'|}\right)_2 + \frac{\varepsilon_n \left(\varphi'_0\right)_2}{\text{dist}(z_0, L)} \leq c_{14} n^{-\frac{1}{2}} E_n^0 \left(\varphi'_0, \frac{1}{|\varphi'|}\right)_2.$$

According to the extremal property of  $\pi_n$  we get

$$\|\varphi'_0 - \pi'_n\|_{L_2(G)} \leq c_{14} n^{-\frac{1}{2}} E_n^0 \left(\varphi'_0, \frac{1}{|\varphi'|}\right)_2. \tag{10}$$

Now applying Andrievskii's polynomial lemma (see [2])

$$\|P_n\|_{\overline{G}} \leq c (\ln n)^{\frac{1}{2}} \|P'_n\|_{L_2(G)}$$

which holds for every polynomial  $P_n$  of degree at most  $n$  with  $P_n(z_0) = 0$ , and using the method of Simenonko [12] and Andrievskii [2], relation (10) leads to

$$\|\varphi_0 - \pi_n\|_{\overline{G}} \leq c_{15} \left(\frac{\ln n}{n}\right)^{\frac{1}{2}} E_n^0 \left(\varphi'_0, \frac{1}{|\varphi'|}\right)_2.$$

As  $L$  is Dini-smooth we have

$$|\varphi'(z)| \asymp 1, \quad z \in L$$

and hence

$$\begin{aligned}
\|\varphi_0 - \pi_n\|_{\overline{G}} &\leq c_{15} \left(\frac{\ln n}{n}\right)^{\frac{1}{2}} \inf_{P_n} \|\varphi'_0 - P_n\|_{L^2(L, 1/|\varphi'|)} \\
&\leq c_{16} \left(\frac{\ln n}{n}\right)^{\frac{1}{2}} \inf_{P_n} \|\varphi'_0 - P_n\|_{L^2(L)}. \tag{11}
\end{aligned}$$

From (11) by Theorem 2 and Lemma 3 we finally get the inequality

$$\|\varphi_0 - \pi_n\|_{\overline{G}} \leq c_{17} \left(\frac{\ln n}{n}\right)^{\frac{1}{2}} \omega\left(\Phi, \frac{1}{n}\right) \leq \begin{cases} \frac{c \ln^{\beta+1/2} n}{n^{\alpha+1/2}}, & \alpha \in (0, 1); \\ \frac{c \ln^{\beta+3/2} n}{n^{3/2}}, & \alpha = 1, \end{cases}$$

which completes the proof. ■



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