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Approximation in weighted Smirnov-Orlicz classes

By

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Abstract

In this work some direct and inverse theorems of approximation theory in the weighted Smirnov-Orlicz classes, defined in the domains with a Dini-smooth boundary, are proved. In particular, a constructive characterization of the generalized Lipschitz classes $Lip^{*}\alpha\,(M,\omega)$, $\alpha>0$, is obtained.

1. Introduction and main results

Let $\Gamma\subset\mathbb{C}$ be a closed bounded rectifiable Jordan curve in the complex plane $\mathbb{C}$. Γ separates the plane $\mathbb{C}$ into two domains $G:=int\Gamma$, $G^{-}:=ext\Gamma$. Without loss of generality we may assume $0\in G$. Let $\mathbb{D}:=\{w\in\mathbb{C}:|w|<1\}$, $\mathbb{T}:=\partial\mathbb{D}$, $\mathbb{D}^{-}:=ext\mathbb{T}$ and $w=\varphi\,(z)$ be the conformal mapping of G^{-} onto $\mathbb{D}^{-}$ normalized by the conditions

$$\varphi\,(\infty)=\infty,\qquad \lim_{z\to\infty}\varphi\,(z)\,/z>0,$$

and let $\psi:=\varphi^{-1}$ be the inverse mapping of φ.

By $E^{p}\,(G)$, $0<p<\infty$, we denote the *Smirnov class* of analytic functions in G. Every function in $E^{p}\,(G)$, $1\leq p<\infty$, has the non-tangential boundary values almost everywhere (a. e.) on Γ and the boundary function belongs to *Lebesgue space* $L^{p}\,(\Gamma)$ [7, p. 438].

Let h be a continuous function on $[0,2\pi]$. Its modulus of continuity is defined by

$$\omega\,(t,h):=\sup\{|h\,(t_1)-h\,(t_2)|:t_1,t_2\in[0,2\pi]\,,\ |t_1-t_2|\leq t\},\qquad t\geq 0.$$

The function h is called *Dini-continuous* if

$$\int_0^{\pi}\frac{\omega\,(t,h)}{t}dt<\infty.$$

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The curve Γ is called *Dini-smooth* if it has a parametrization

$$\Gamma : \varphi_0(\tau), \qquad 0 \leq \tau \leq 2\pi$$

such that $\varphi_0'(\tau)$ is Dini-continuous and $\varphi_0'(\tau) \neq 0$ [22, p. 48].

When Γ is Dini-smooth, [24] asserts that

$$\begin{aligned} 0 < c_1 \leq |\psi'(w)| \leq c_2, &\qquad |w| \geq 1, \\ 0 < c_3 \leq |\varphi'(z)| \leq c_4, &\qquad z \in G^-, \end{aligned} \tag{1.1}$$

for some constants c_1, c_2 and c_3, c_4 independent of w and z, respectively.

A continuous and convex function $M : [0, \infty) \to [0, \infty)$ which satisfies the conditions

$$M(0) = 0; \quad M(x) > 0 \qquad \text{for } x > 0;$$
$$\lim_{x \to 0} (M(x)/x) = 0; \quad \lim_{x \to \infty} (M(x)/x) = \infty,$$

is called an N-function.

The complementary N-function to M is defined by

$$N(y) := \max_{x \geq 0} (xy - M(x)), \qquad y \geq 0.$$

We denote by $L_M(\Gamma)$ the linear space of Lebesgue measurable functions $f : \Gamma \to \mathbb{C}$ satisfying the condition

$$\int_\Gamma M[\alpha |f(z)|] |dz| < \infty$$

for some $\alpha > 0$.

The space $L_M(\Gamma)$ becomes a Banach space with the *Luxemburg norm*

$$\|f\|_{L_{(M)}(\Gamma)} := \inf\{\tau > 0 : \rho(f/\tau; M) \leq 1\},$$

and also with the *Orlicz norm*

$$\|f\|_{L_M(\Gamma)} := \sup \left\{ \int_\Gamma |f(z) g(z)| |dz| : g \in L_N(\Gamma);\ \rho(g; N) \leq 1 \right\},$$

where N is the complementary N-function to M and

$$\rho(g; N) := \int_\Gamma N[|g(z)|] |dz|.$$

The Banach space $L_M(\Gamma)$ is called Orlicz space.

A function ω is called a *weight* on Γ if $\omega : \Gamma \to [0, \infty]$ is measurable and $\omega^{-1}(\{0, \infty\})$ has measure zero (with respect to Lebesgue measure).

The class of measurable functions f defined on Γ and satisfying the condition $\omega f \in L_M(\Gamma)$ is called *weighted Orlicz space* $L_M(\Gamma, \omega)$ with the norm

$$\|f\|_{L_M(\Gamma,\omega)} := \|f\omega\|_{L_M(\Gamma)} .$$

For $z \in \Gamma$ and $\epsilon > 0$ let $\Gamma(z, \epsilon)$ denotes the portion of Γ contained in the open disc of radius ϵ and centered at z, i.e. $\Gamma(z, \epsilon) := \{t \in \Gamma : |t - z| < \epsilon\}$.

For fixed $p \in (1, \infty)$, we define $q \in (1, \infty)$ by $p^{-1} + q^{-1} = 1$. The set of all weights $\omega : \Gamma \to [0, \infty]$ satisfying the relation

$$\sup_{t\in\Gamma} \sup_{\epsilon>0} \left(\frac{1}{\epsilon} \int_{\Gamma(z,\epsilon)} \omega(\tau)^p |d\tau| \right)^{1/p} \left(\frac{1}{\epsilon} \int_{\Gamma(z,\epsilon)} \omega(\tau)^{-q} |d\tau| \right)^{1/q} < \infty$$

is denoted by $A_p(\Gamma)$.

We denote by $L^p(\Gamma, \omega)$ the set of all measurable functions $f : \Gamma \to \mathbb{C}$ such that $|f|\omega \in L^p(\Gamma)$, $1 < p < \infty$.

Let $M^{-1} : [0, \infty) \to [0, \infty)$ be the inverse function of the N-function M. The lower and upper *indices* α_M, β_M [3, p. 350]

$$\alpha_M := \lim_{x\to 0} \frac{\log \varrho(x)}{\log x}, \qquad \beta_M := \lim_{x\to\infty} \frac{\log \varrho(x)}{\log x}$$

of the function

$$\varrho : (0, \infty) \to (0, \infty], \quad \varrho(x) := \limsup_{y\to\infty} \frac{M^{-1}(y)}{M^{-1}(y/x)}, \quad x \in (0, \infty),$$

first considered by W. Matuszewska and W. Orlicz [20], are called the *Boyd indices* of the Orlicz space $L_M(\Gamma)$. It is well known that $0 \le \alpha_M \le \beta_M \le 1$. For this and other properties of Boyd indices of Orlicz spaces we refer to [19].

The indices α_M, β_M are called *nontrivial* if $0 < \alpha_M$ and $\beta_M < 1$.

Definition 1. For a weight ω on Γ we denote by $E_M(G, \omega)$ the subclass of analytic functions of $E^1(G)$ whose boundary value functions belong to weighted Orlicz space $L_M(\Gamma, \omega)$.

The weighted Smirnov-Orlicz class $E_M(G, \omega)$ is a generalization of the Smirnov class $E^p(G)$. In particular, if $M(x) := x^p$, $1 < p < \infty$, then the weighted Smirnov-Orlicz class $E_M(G, \omega)$ coincides with the weighted Smirnov class $E^p(G, \omega)$; if $\omega := 1$, then $E_M(G, \omega)$ coincides with the Smirnov-Orlicz class $E_M(G)$, defined in [18].

Let Γ be a rectifiable Jordan curve and $f \in L^1(\Gamma)$. The functions f^+ and f^- defined by

$$f^+(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\varsigma)}{\varsigma - z} d\varsigma, \qquad z \in G,$$

and

$$f^-(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\varsigma)}{\varsigma - z} d\varsigma, \qquad z \in G^-,$$

are analytic in G and G^{-}, respectively and $f^{-}(\infty)=0$.

For $g\in L_M(\mathbb{T},\omega)$ we set

$$\sigma_h(g)(w):=\frac{1}{2h}\int_{-h}^{h} g\left(we^{it}\right)dt,\quad 0<h<\pi,\quad w\in\mathbb{T}.$$

If α_M and β_M are nontrivial and $\omega\in A_{\frac{1}{\alpha_M}}(\mathbb{T})\cap A_{\frac{1}{\beta_M}}(\mathbb{T})$, then by [14] we have

$$\|\sigma_h(g)\|_{L_M(\mathbb{T},\omega)}\leq c_5\|g\|_{L_M(\mathbb{T},\omega)},\tag{1.2}$$

and consequently $\sigma_h(g)\in L_M(\mathbb{T},\omega)$ for any $g\in L_M(\mathbb{T},\omega)$.

Definition 2. Let α_M and β_M be nontrivial and $\omega\in A_{\frac{1}{\alpha_M}}(\mathbb{T})\cap A_{\frac{1}{\beta_M}}(\mathbb{T})$. The function

$$\Omega^r_{M,\omega}(g,\delta):=\sup_{\substack{0<h_i\leq\delta\\ i=1,2,\ldots,r}}\left\|\prod_{i=1}^{r}\left(I-\sigma_{h_i}\right)g\right\|_{L_M(\mathbb{T},\omega)},\quad\delta>0,\quad r=1,2,\ldots$$

is called rth modulus of smoothness of $g\in L_M(\mathbb{T},\omega)$, where I is the identity operator.

Note that in case of weighted Lebesgue spaces $L^p(\mathbb{T},\omega)$ this definition originates from [25] (*see also* [10], [11], [12]).

It is easily verified that the function $\Omega_{M,\omega}(g,\cdot)$ is continuous, non-negative and satisfy

$$\lim_{\delta\to 0}\Omega^r_{M,\omega}(g,\delta)=0,\quad\Omega^r_{M,\omega}(g+g_1,\cdot)\leq\Omega^r_{M,\omega}(g,\cdot)+\Omega^r_{M,\omega}(g_1,\cdot)$$

for $g,g_1\in L_M(\mathbb{T},\omega)$.

Let $\omega_0(w):=\omega(\psi(w))$ and $f_0(w):=f(\psi(w))$ for a weight ω on Γ, $f\in L_M(\Gamma,\omega)$ and $w\in\mathbb{T}$. By (1.1) we have $f_0\in L_M(\mathbb{T},\omega_0)$ for $f\in L_M(\Gamma,\omega)$. Using the nontangential boundary values of f_0^{+} on $\mathbb{T}$ we define the *rth modulus of smoothness* of $f\in L_M(\Gamma,\omega)$ as

$$\Omega^r_{\Gamma,M,\omega}(f,\delta):=\Omega^r_{M,\omega_0}\left(f_0^{+},\delta\right),\qquad\delta>0,$$

for $r=1,2,3,\ldots$.

Let

$$E_n(f,G)_{M,\omega}:=\inf_{P\in\mathcal{P}_n}\|f-P\|_{L_M(\Gamma,\omega)}$$

be the best approximation to $f\in E_M(G,\omega)$ in the class $\mathcal{P}_n$ of algebraic polynomials of degree not greater than n.

When $r=1$ and Γ is a Carleson curve, some direct theorems of the approximation theory in the Smirnov-Orlicz and Orlicz classes are given in [8],

[9]. One direct theorem in the Smirnov-Orlicz classes $E_M(G)$, defined on the domains with a Dini-smooth boundary, is obtained in [15]. The inverse problems of approximation theory in these domains have been investigated by V. M. Kokilashvili [18]. Note that the modulus of smoothness used in these works are constructed by applying the usual shift $f_0\left(e^{i(t+h)}\right)$, $h \in [0, 2\pi]$, for $f_0\left(e^{it}\right)$.

In this work we prove some direct and inverse theorems in the weighted Smirnov-Orlicz classes. In particular, we obtain a constructive characterization of the generalized Lipschitz classes $Lip^{*}\alpha\,(M,\omega)$, $\alpha > 0$. Since the usual shift, in general, is noninvariant in the weighted Orlicz classes, we use the modulus of smoothness $\Omega^{r}_{\Gamma,M,\omega}(f,\cdot)$, constructed with respect to the mean value operator σ_h.

The main results of this work are the following.

Theorem 1. *Let G be a bounded simply connected domain with a Dini-smooth boundary Γ and let $L_M(\Gamma)$ be an Orlicz space with nontrivial indices α_M, β_M and $\omega \in A_{\frac{1}{\alpha_M}}(\Gamma) \cap A_{\frac{1}{\beta_M}}(\Gamma)$. If $f \in E_M(G,\omega)$, then for every natural number n,*

$$E_n(f,G)_{M,\omega} \le c_6\ \Omega^{r}_{\Gamma,M,\omega}\left(f, \frac{1}{n+1}\right), \quad r = 1, 2, 3, \ldots$$

with some constant $c_6 > 0$ independent of n.

Theorem 2. *Let G be a bounded simply connected domain with a Dini-smooth boundary Γ and let $E_M(G,\omega)$ be a weighted Smirnov-Orlicz class with nontrivial indices α_M, β_M. If $\omega \in A_{\frac{1}{\alpha_M}}(\Gamma) \cap A_{\frac{1}{\beta_M}}(\Gamma)$ and $f \in E_M(G,\omega)$, then*

$$\Omega^{r}_{\Gamma,M,\omega}\left(f, \frac{1}{n}\right) \le \frac{c_7}{n^{2r}} \left\{ E_0(f,G)_{M,\omega} + \sum_{k=1}^{n} k^{2r-1} E_k(f,G)_{M,\omega} \right\}, \qquad r = 1, 2, 3, \ldots,$$

with some constant $c_7 > 0$ independent of n.

Corollary 1. *Under the conditions of Theorem 2, if*

$$E_n(f,G)_{M,\omega} = \mathcal{O}\left(n^{-\alpha}\right), \quad \alpha > 0, \quad n = 1, 2, 3, \ldots,$$

then

$$\Omega^{r}_{\Gamma,M,\omega}(f,\delta) = \begin{cases} \mathcal{O}\left(\delta^{\alpha}\right) & ;\ r > \alpha/2 \\ \mathcal{O}\left(\delta^{\alpha} \log \frac{1}{\delta}\right) & ;\ r = \alpha/2 \\ \mathcal{O}\left(\delta^{2r}\right) & ;\ r < \alpha/2 \end{cases}$$

for $f \in L_M(\Gamma,\omega)$.

Definition 3. For $\alpha > 0$ let $r := \left[\frac{\alpha}{2}\right] + 1$. The set of functions $f \in E_M(G,\omega)$ such that

$$\Omega^{r}_{\Gamma,M,\omega}(f,\delta) = \mathcal{O}\left(\delta^{\alpha}\right), \quad \delta > 0$$

is called the generalized Lipschitz class $Lip^{*}\alpha\,(M,\omega)$.

According to Corollary 1 we have the following.

Corollary 2. *Under the conditions of Theorem 2, if*

$$E_n(f,G)_{M,\omega} = \mathcal{O}\left(n^{-\alpha}\right), \quad \alpha > 0, \quad n = 1,2,3,\ldots,$$

then $f \in Lip^{*}\alpha(M,\omega)$.

Theorem 1 and Corollary 2 imply the following.

Theorem 3. *If* $\alpha > 0$*, then under the conditions of Theorem 2, the following conditions are equivalent*:
(a) $f \in Lip^{*}\alpha(M,\omega)$
(b) $E_n(f) = \mathcal{O}(n^{-\alpha}), \quad n = 1,2,3,\ldots.$

In the case of weighted Smirnov classes $E^p(G,\omega)$ the analogues results are proved in the papers [11], [13].

Throughout this work by c, c_1, $c_2,\ldots,$ we denote the constants which are different in different places.

2. Auxiliary results

Let Γ be a rectifiable Jordan curve, $f \in L^1(\Gamma)$ and let

$$(S_\Gamma f)(t) := \lim_{\varepsilon\to 0} \frac{1}{2\pi i} \int\limits_{\Gamma\setminus\Gamma(t,\epsilon)} \frac{f(\varsigma)}{\varsigma - t} d\varsigma, \qquad t \in \Gamma$$

be Cauchy's singular integral of f. The linear operator $S_\Gamma : f \to S_\Gamma f$ is called the Cauchy singular operator.

If one of the functions f^+ or f^- has the non-tangential limits a. e. on Γ, then $S_\Gamma f(z)$ exist a. e. on Γ and also the other one has non-tangential limits a. e. on Γ. Conversely, if $S_\Gamma f(z)$ exist a. e. on Γ, then both functions f^+ and f^- have non-tangential limits a. e. on Γ. In both cases, the formulae

$$\begin{aligned} f^+(z) &= (S_\Gamma f)(z) + f(z)/2, \\ f^-(z) &= (S_\Gamma f)(z) - f(z)/2, \end{aligned} \tag{2.1}$$

and hence

$$f = f^+ - f^-$$

holds a. e. on Γ (*see, e.g.,* [7, p. 431]).

Lemma 1. *Let* $0 < \alpha_M,\ \beta_M < 1$, $\omega \in A_{\frac{1}{\alpha_M}}(\Gamma) \cap A_{\frac{1}{\beta_M}}(\Gamma)$ *and* $f \in L_M(\Gamma,\omega)$. *Then* $f^+ \in E_M(G,\omega)$ *and* $f^- \in E_M(G^-,\omega)$.

Proof. Let $f \in L_M(\Gamma, \omega)$. By [3, p. 58, Th. 2.31] there exist $p, q \in (1, \infty)$ such that $1 < p < 1/\beta_M \leq 1/\alpha_M < q < \infty$, and $\omega \in A_p(\Gamma) \cap A_q(\Gamma)$. Then [16, Th. 2.5] we have

$$L^q(\Gamma) \subset L_M(\Gamma) \subset L^p(\Gamma),$$

where the inclusion maps being continuous, and therefore $f \in L^p(\Gamma, \omega)$. Now using Lemmas 2 and 3 of [11] we get

$$f^+ \in E^1(G) \text{ and } f^- \in E^1\left(G^-\right).$$

Hence, using the relations (2.1) which hold a. e. on Γ, and the boundedness of the singular operator S_Γ in weighted Orlicz spaces [17, Th. 4.5], we conclude that

$$f^+ \in L_M(\Gamma, \omega), \quad f^- \in L_M(\Gamma, \omega)$$

and the assertion follows. □

Lemma 2. *Let* $0 < \alpha_M$, $\beta_M < 1$, $\omega \in A_{\frac{1}{\alpha_M}}(\mathbb{T}) \cap A_{\frac{1}{\beta_M}}(\mathbb{T})$ *and* $g \in E_M(\mathbb{D}, \omega)$. *If* $\sum_{k=0}^{n} \alpha_k w^k$ *is the nth partial sum of the Taylor series of the function g at the origin, then there exists a constant* $c_8 > 0$ *such that*

$$\left\| g(w) - \sum\nolimits_{k=0}^{n} \alpha_k w^k \right\|_{L_M(\mathbb{T}, \omega)} \leq c_8 \; \Omega_{M,\omega}^r\left(g, \frac{1}{n+1}\right)$$

for every natural number n.

This result was proved in [14, Theorem 3].

The Faber polynomials $\Phi_k(z)$, $k = 0, 1, 2, 3, \ldots$, associated with $G \cup \Gamma$, are defined through the expansion

$$\frac{\psi'(w)}{\psi(w) - z} = \sum_{k=0}^{\infty} \frac{\Phi_k(z)}{w^{k+1}}, \qquad z \in G, \quad w \in \mathbb{D}^-, \tag{2.2}$$

and the equalities

$$\Phi_k(z) = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{w^k \psi'(w)}{\psi(w) - z} dw, \quad z \in G, \tag{2.3}$$

$$\Phi_k(z) = \varphi^k(z) + \frac{1}{2\pi i} \int_{\Gamma} \frac{\varphi^k(\varsigma)}{\varsigma - z} d\varsigma, \quad z \in G^-, \tag{2.4}$$

hold [23, p. 34].

If $f \in E_M(G, \omega)$, then by definition $f \in E^1(G)$ and hence

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\varsigma)}{\varsigma - z} d\varsigma \\ &= \frac{1}{2\pi i} \int_{\mathbb{T}} f(\psi(w)) \frac{\psi'(w)}{\psi(w) - z} dw, \quad z \in G. \end{aligned}$$

Here, taking the relation (2.2) into account, we have

$$f(z) \sim \sum_{k=0}^{\infty} a_k \Phi_k(z), \quad z \in G$$

where

$$a_k := a_k(f) := \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{f(\psi(w))}{w^{k+1}} dw, \quad k = 0, 1, 2, \ldots .$$

This series is called the Faber series of $f \in E_M(G, \omega)$ and the values a_k, $k = 0, 1, 2, \ldots$ are called the Faber coefficients of f. Let $S_n(f, \cdot) := \sum_{k=0}^{n} a_k \Phi_k$ be the nth partial sum of the Faber expansion of the function $f \in E_M(G, \omega)$.

Let $\mathcal{P} :=$ {all polynomials (with no restriction on the degree)}, $\mathcal{P}(\mathbb{D}) :=$ {traces of all members of $\mathcal{P}$ on $\mathbb{D}$} and let

$$T(P)(z) := \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{P(w)\psi'(w)}{\psi(w) - z} dw, \quad z \in G$$

be an operator T defined on $\mathcal{P}(\mathbb{D})$.

Then by (2.3)

$$T\left(\sum_{k=0}^{n} b_k w^k\right) = \sum_{k=0}^{n} b_k \Phi_k(z), \quad z \in G.$$

If $z' \in G$, then

$$T(P)(z') = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{P(w)\psi'(w)}{\psi(w) - z'} dw = \frac{1}{2\pi i} \int_{\Gamma} \frac{(P \circ \varphi)(\varsigma)}{\varsigma - z'} d\varsigma$$
$$= (P \circ \varphi)^{+}(z'),$$

which by (2.1) implies that

$$T(P)(z) = S_{\Gamma}(P \circ \varphi)(z) + \frac{1}{2}(P \circ \varphi)(z) \tag{2.5}$$

a. e. on Γ.

As in the proof of Lemma 1, there exist $p, q \in (1, \infty)$ such that $1 < p < 1/\beta_M \leq 1/\alpha_M < q < \infty$, $\omega \in A_p(\Gamma) \cap A_q(\Gamma)$ and the inclusions

$$L^q(\Gamma) \subset L_M(\Gamma) \subset L^p(\Gamma)$$

hold. Then $P \circ \varphi \in L^q(\Gamma, \omega)$, for any polynomial P, and hence $P \circ \varphi \in L_M(\Gamma, \omega)$. Since S_{Γ} is bounded [17, Th. 4.5] in $L_M(\Gamma, \omega)$, from (2.5) we have that $T(P) \in L_M(\Gamma, \omega)$ for every $P \in \mathcal{P}(\mathbb{D})$. The property $T(P) \in E^1(G)$ can be obtained from continuity of $P \circ \varphi$. Hence we obtain $T(P) \in E_M(G, \omega)$ for every $P \in \mathcal{P}(\mathbb{D})$.

Therefore, we get the following result.

Lemma 3. *If Γ is a Dini-smooth curve, $0 < \alpha_M$, $\beta_M < 1$ and $\omega \in A_{\frac{1}{\alpha_M}}(\Gamma) \cap A_{\frac{1}{\beta_M}}(\Gamma)$, then the linear operator*

$$T: \ \mathcal{P}(\mathbb{D}) \to E_M(G, \omega)$$

is bounded.

Extending the operator T from $\mathcal{P}(\mathbb{D})$ to the space $E_M(\mathbb{D}, \omega_0)$ as a linear and bounded operator, for the extension $T: E_M(\mathbb{D}, \omega_0) \to E_M(G, \omega)$, we have the representation

$$T(g)(z) := \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{g(w)\psi'(w)}{\psi(w) - z} dw, \quad z \in G, \quad g \in E_M(\mathbb{D}, \omega_0).$$

Theorem 4. *If Γ is a Dini-smooth curve, $0 < \alpha_M$, $\beta_M < 1$ and $\omega \in A_{\frac{1}{\alpha_M}}(\Gamma) \cap A_{\frac{1}{\beta_M}}(\Gamma)$, then the operator*

$$T: \ E_M(\mathbb{D}, \omega_0) \to E_M(G, \omega)$$

is one-to-one and onto.

Proof. Let $g \in E_M(\mathbb{D}, \omega_0)$ with the Taylor expansion

$$g(w) := \sum_{k=0}^{\infty} \alpha_k w^k, \quad w \in \mathbb{D}.$$

It is easily seen that if Γ is Dini-smooth, then the conditions $\omega \in A_{\frac{1}{\alpha_M}}(\Gamma)$, $\omega_0 \in A_{\frac{1}{\alpha_M}}(\mathbb{T})$ and also $\omega \in A_{\frac{1}{\beta_M}}(\Gamma)$, $\omega_0 \in A_{\frac{1}{\beta_M}}(\mathbb{T})$ are equivalent. Since $\omega_0 \in A_{\frac{1}{\alpha_M}}(\mathbb{T}) \cap A_{\frac{1}{\beta_M}}(\mathbb{T})$, by the proof of Theorem 4.5 of [17] there exist $p, q \in (1, \infty)$ such that

$$1 < p < 1/\beta_M \le 1/\alpha_M < q < \infty \text{ and } \omega_0 \in A_p(\mathbb{T}) \cap A_q(\mathbb{T}),$$

and then, by [16, Th. 2.5],

$$L^q(\mathbb{T}) \subset L_M(\mathbb{T}) \subset L^p(\mathbb{T}),$$

where inclusion maps being continuous.

Let $g_r(w) := g(rw)$, $0 < r < 1$. Since $g \in E^1(\mathbb{D})$ is the Poisson integral of its boundary function [5, p. 41], using [21, Th. 10] and Boyd interpolation theorem [2], we get

$$\|g_r - g\|_{L_M(\mathbb{T}, \omega_0)} = \left\|g\left(re^{i\theta}\right) - g\left(e^{i\theta}\right)\right\|_{L_M([0,2\pi], \omega_0)} \to 0, \quad \text{as } r \to 1^-.$$

Therefore, the boundedness of the operator T implies that

$$\|T(g_r) - T(g)\|_{L_M(\Gamma,\omega)} \to 0, \quad \text{as } r \to 1^-. \tag{2.6}$$

Since the series $\sum_{k=0}^{\infty} \alpha_k w^k$ is uniformly convergent for $|w| = r < 1$, the series $\sum_{k=0}^{\infty} \alpha_k r^k w^k$ is uniformly convergent on $\mathbb{T}$, and hence

$$\begin{aligned} T(g_r)(z') &= \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{g_r(w)\,\psi'(w)}{\psi(w) - z'} dw = \sum\nolimits_{m=0}^{\infty} \alpha_m r^m \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{w^m \psi'(w)}{\psi(w) - z'} dw \\ &= \sum_{m=0}^{\infty} \alpha_m r^m \Phi_m(z'), \quad z' \in G. \end{aligned}$$

Now, taking the limit as $z' \to z \in \Gamma$ along all non-tangential paths inside Γ, we obtain

$$T(g_r)(z) = \sum_{m=0}^{\infty} \alpha_m r^m \Phi_m(z), \quad z \in \Gamma.$$

From the last equality and Lemma 3 of [6, p. 43] for the Faber coefficients $a_k(T(g_r))$ we have

$$\begin{aligned} a_k(T(g_r)) &= \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{T(g_r)(\psi(w))}{w^{k+1}} dw \\ &= \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{\sum_{m=0}^{\infty} \alpha_m r^m \Phi_m(\psi(w))}{w^{k+1}} dw \\ &= \sum_{m=0}^{\infty} \alpha_m r^m \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{\Phi_m(\psi(w))}{w^{k+1}} dw = \alpha_k r^k \end{aligned}$$

and therefore

$$a_k(T(g_r)) \to \alpha_k, \quad \text{as } r \to 1^-. \tag{2.7}$$

Now applying (1.1), Hölder's inequality and Theorem 2.1 of [17], respectively, we obtain

$$
\begin{aligned}
|a_k(T(g_r)) - a_k(T(g))| &= \left| \frac{1}{2\pi i} \int\limits_{\mathbb{T}} \frac{[T(g_r) - T(g)](\psi(w))}{w^{k+1}} dw \right| \\
&\le \frac{1}{2\pi} \int\limits_{\mathbb{T}} |[T(g_r) - T(g)](\psi(w))| \, |dw| \\
&= \frac{1}{2\pi} \int\limits_{\Gamma} |[T(g_r) - T(g)](z)| \, |\varphi'(z)| \, |dz| \\
&\le \frac{c_{11}}{2\pi} \int\limits_{\Gamma} |[T(g_r) - T(g)](z)| \, |dz| \\
&= \frac{c_{11}}{2\pi} \int\limits_{\Gamma} |[T(g_r) - T(g)](z)| \, \omega(z) \, \omega^{-1}(z) \, |dz| \\
&\le \frac{c_{11}}{2\pi} \|(T(g_r) - T(g)) \, \omega(z)\|_{L_M(\Gamma)} \left\|\omega^{-1}(\cdot)\right\|_{L_N(\Gamma)} \\
&\le \frac{c_{12}}{2\pi} \|T(g_r) - T(g)\|_{L_M(\Gamma,\omega)}.
\end{aligned}
$$

From the last inequality and (2.6) we get

$$
a_k(T(g_r)) \to a_k(T(g)), \quad \text{as } r \to 1^-,
$$

and then by (2.7) $a_k(T(g)) = \alpha_k$, $k = 0, 1, 2, \ldots$. If $T(g) = 0$, then $\alpha_k = a_k(T(g)) = 0$, $k = 0, 1, 2, \ldots$, and therefore $g = 0$. This means that the operator T is one-to-one.

Now we take a function $f \in E_M(G, \omega)$ and consider the function $f_0 = f \circ \psi \in L_M(\mathbb{T}, \omega_0)$. The Cauchy type integral

$$
\frac{1}{2\pi i} \int\limits_{\mathbb{T}} \frac{f_0(\tau)}{\tau - w} d\tau
$$

represents analytic functions f_0^+ and f_0^- in $\mathbb{D}$ and $\mathbb{D}^-$, respectively. Since $\omega_0 \in A_{\frac{1}{\alpha_M}}(\mathbb{T}) \cap A_{\frac{1}{\beta_M}}(\mathbb{T})$, by Lemma 1, we have

$$
f_0^+ \in E_M(\mathbb{D}, \omega_0) \text{ and } f_0^- \in E_M\left(\mathbb{D}^-, \omega_0\right),
$$

and for the non-tangential boundary values we get

$$
\begin{aligned}
f_0^+(w) &= S_{\mathbb{T}}(f_0)(w) + \frac{1}{2} f_0(w), \\
f_0^-(w) &= S_{\mathbb{T}}(f_0)(w) - \frac{1}{2} f_0(w).
\end{aligned}
$$

Therefore

$$
f_0(w) = f_0^+(w) - f_0^-(w) \tag{2.8}
$$

holds a. e. on $\mathbb{T}$ and $f_0^-(\infty)=0$. For the Faber coefficients a_k of f we get

$$\begin{aligned} a_k &= \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{f_0(w)}{w^{k+1}}dw \\ &= \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{f_0^+(w)}{w^{k+1}}dw - \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{f_0^-(w)}{w^{k+1}}dw. \end{aligned}$$

Since the function f_0^- belongs to $E^1(\mathbb{D}^-)$, the second integral vanishes and hence the values $\{a_k\}_{k=0}^{\infty}$ also become the Taylor coefficients of the function f_0^+ at the origin, namely,

$$f_0^+(w) = \sum_{k=0}^{\infty} a_k w^k, \quad w \in \mathbb{D}.$$

From the first part of the proof we get

$$T\left(f_0^+\right) \backsim \sum_{k=0}^{\infty} a_k \Phi_k.$$

Since there is no two different functions in $E_M(G,\omega)$ that have the same Faber coefficients [1], we conclude that $T\left(f_0^+\right)=f$. Therefore, the operator T is onto. □

3. Proofs of main results

Proof of Theorem 1. Let $f \in E_M(G,\omega)$. Then $f_0 \in L_M(\mathbb{T},\omega_0)$. According to (2.8)

$$f(\varsigma) = f_0^+(\varphi(\varsigma)) - f_0^-(\varphi(\varsigma)) \tag{3.1}$$

a. e. on Γ and

$$\int_{\Gamma} \frac{f(\varsigma)}{\varsigma - z'}d\varsigma = 0, \qquad z' \in G^-$$

because $f \in E^1(G)$.

Now let $z' \in G^-$. Using (2.4) we have

$$\begin{aligned} \sum_{k=0}^{n} a_k \Phi_k(z') &= \sum_{k=0}^{n} a_k \varphi^k(z') + \frac{1}{2\pi i}\int_{\Gamma} \frac{\sum_{k=0}^{n} a_k \varphi^k(\varsigma)}{\varsigma - z'}d\varsigma \\ &= \sum_{k=0}^{n} a_k \varphi^k(z') + \frac{1}{2\pi i}\int_{\Gamma} \frac{\sum_{k=0}^{n} a_k \varphi^k(\varsigma)}{\varsigma - z'}d\varsigma - \frac{1}{2\pi i}\int_{\Gamma} \frac{f(\varsigma)}{\varsigma - z'}d\varsigma \\ &= \sum_{k=0}^{n} a_k \varphi^k(z') + \frac{1}{2\pi i}\int_{\Gamma} \frac{\sum_{k=0}^{n} a_k \varphi^k(\varsigma)}{\varsigma - z'}d\varsigma \\ &\quad - \frac{1}{2\pi i}\int_{\Gamma} \frac{f_0^+(\varphi(\varsigma))}{\varsigma - z'}d\varsigma + \frac{1}{2\pi i}\int_{\Gamma} \frac{f_0^-(\varphi(\varsigma))}{\varsigma - z'}d\varsigma. \end{aligned}$$

Since

$$\frac{1}{2\pi i}\int\limits_{\Gamma}\frac{f_0^-\left(\varphi\left(\varsigma\right)\right)}{\varsigma-z'}d\varsigma=-f_0^-\left(\varphi\left(z'\right)\right),$$

we get

$$\sum_{k=0}^{n}a_k\Phi_k\left(z'\right)=\frac{1}{2\pi i}\int\limits_{\Gamma}\frac{\sum_{k=0}^{n}a_k\varphi^k\left(\varsigma\right)-f_0^+\left(\varphi\left(\varsigma\right)\right)}{\varsigma-z'}d\varsigma$$
$$+\sum_{k=0}^{n}a_k\varphi^k\left(z'\right)-f_0^-\left(\varphi\left(z'\right)\right).$$

Hence, taking the limit as $z'\to z$ along all non-tangential paths outside Γ, we obtain

$$\sum_{k=0}^{n}a_k\Phi_k\left(z\right)=-\frac{1}{2}\left(\sum_{k=0}^{n}a_k\varphi^k\left(z\right)-f_0^+\left(\varphi\left(z\right)\right)\right)+S_\Gamma\left[\sum_{k=0}^{n}a_k\varphi^k-\left(f_0^+\circ\varphi\right)\right]$$
$$+\sum_{k=0}^{n}a_k\varphi^k\left(z\right)-f_0^-\left(\varphi\left(z\right)\right)$$
$$=\frac{1}{2}\left(\sum_{k=0}^{n}a_k\varphi^k\left(z\right)-f_0^+\left(\varphi\left(z\right)\right)\right)+\left[f_0^+\left(\varphi\left(z\right)\right)-f_0^-\left(\varphi\left(z\right)\right)\right]$$
$$+S_\Gamma\left[\sum_{k=0}^{n}a_k\varphi^k-\left(f_0^+\circ\varphi\right)\right]$$

a. e. on Γ. Using (3.1), (1.1), Minkowski's inequality and the boundedness of S_Γ we get

$$\left\|f-S_n\left(f,\cdot\right)\right\|_{L_M(\Gamma,\ \omega)}$$
$$=\left\|\frac{1}{2}\left(\sum_{k=0}^{n}a_k\varphi^k\left(z\right)-f_0^+\left(\varphi\left(z\right)\right)\right)+S_\Gamma\left[\sum_{k=0}^{n}a_k\varphi^k-\left(f_0^+\circ\varphi\right)\right]\right\|_{L_M(\Gamma,\omega)}$$
$$\leq c_{13}\left\|\sum_{k=0}^{n}a_k\varphi^k\left(z\right)-f_0^+\left(\varphi\left(z\right)\right)\right\|_{L_M(\Gamma,\omega)}\leq c_{14}\left\|f_0^+\left(w\right)-\sum_{k=0}^{n}a_kw^k\right\|_{L_M(\mathbb{T},\omega_0)}.$$

On the other hand, from the proof of Theorem 4 we know that the Faber coefficients of the function f and the Taylor coefficients of the function f_0^+ at the origin are the same. Then taking Lemma 2 into account, we conclude that

$$E_n\left(f,G\right)_{M,\omega}\leq\left\|f-S_n\left(f,\cdot\right)\right\|_{L_M(\Gamma,\omega)}\leq c_{15}\ \Omega^r_{\Gamma,M,\omega}\left(f,\frac{1}{n+1}\right).$$

□

Proof of Theorem 2. Let $f \in E_M(G,\omega)$. Then by the proof of Theorem 4 we have $T\left(f_0^+\right) = f$. Since the operator $T : E_M(\mathbb{D},\omega_0) \to E_M(G,\omega)$ is linear, bounded, one-to-one and onto, the operator $T^{-1} : E_M(G,\omega) \to E_M(\mathbb{D},\omega_0)$ is linear and bounded. We take a $p_n^* \in \mathcal{P}_n$ as the best approximating algebraic polynomial to f in $E_M(G,\omega)$, i.e.,

$$E_n(f,G)_{M,\omega} = \|f - p_n^*\|_{L_M(\Gamma,\omega)}.$$

(There exists such a unique polynomial p_n^* of $\mathcal{P}_n$, *see*, *for example*, [4, p. 59]). Then $T^{-1}(p_n^*) \in \mathcal{P}_n(\mathbb{D})$ and therefore

(3.2)

$$\begin{aligned} E_n\left(f_0^+,\mathbb{D}\right)_{M,\omega_0} &\leq \left\|f_0^+ - T^{-1}(p_n^*)\right\|_{L_M(\mathbb{T},\omega_0)} = \left\|T^{-1}(f) - T^{-1}(p_n^*)\right\|_{L_M(\mathbb{T},\omega_0)} \\ &= \left\|T^{-1}(f - p_n^*)\right\|_{L_M(\mathbb{T},\omega_0)} \leq \left\|T^{-1}\right\| \|f - p_n^*\|_{L_M(\Gamma,\omega)} \\ &= \left\|T^{-1}\right\| E_n(f,G)_{M,\omega}, \end{aligned}$$

because the operator T^{-1} is bounded.

On the other hand, from [14] we have

$$\Omega^r_{M,\omega_0}\left(f_0^+,\frac{1}{n}\right) \leq \frac{c_{16}}{n^{2r}}\left\{E_0\left(f_0^+,\mathbb{D}\right)_{M,\omega_0} + \sum_{k=1}^{n} k^{2r-1} E_k\left(f_0^+,\mathbb{D}\right)_{M,\omega_0}\right\}$$

$r = 1, 2, \ldots.$

The last inequality and (3.2) imply that

$$\begin{aligned} \Omega^r_{\Gamma,M,\omega}\left(f,\frac{1}{n}\right) &= \Omega^r_{M,\omega_0}\left(f_0^+,\frac{1}{n}\right) \\ &\leq \frac{c_{16}}{n^{2r}}\left\{E_0\left(f_0^+,\mathbb{D}\right)_{M,\omega_0} + \sum_{k=1}^{n} k^{2r-1} E_k\left(f_0^+,\mathbb{D}\right)_{M,\omega_0}\right\} \\ &\leq \frac{c_{16}\left\|T^{-1}\right\|}{n^{2r}}\left\{E_0(f,G)_{M,\omega} + \sum_{k=1}^{n} k^{2r-1} E_k(f,G)_{M,\omega}\right\}, \end{aligned}$$

$r = 1, 2, \ldots.$ □

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