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POWER SUBGROUPS OF SOME HECKE GROUPS

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ABSTRACT. Let $q > 3$ be an even integer and let $H(\lambda_q)$ be the Hecke group associated to q. Let m be a positive integer and $H^m(\lambda_q)$ the power subgroup of $H(\lambda_q)$. In this work the power subgroups $H^m(\lambda_q)$ are discussed. The Reidemeister-Schreier method and the permutation method are used to obtain the abstract group structure and generators of $H^m(\lambda_q)$; their signatures are then also determined.

1. Introduction. In [**4**], Erich Hecke introduced the groups $H(\lambda)$ generated by two linear fractional transformations

$$x(z) = -\frac{1}{z} \quad \text{and} \quad u(z) = z + \lambda,$$

where λ is a fixed positive real number. Let $y = xu$, i.e.,

$$y(z) = -\frac{1}{z+\lambda}.$$

E. Hecke showed that $H(\lambda)$ is Fuchsian if and only if $\lambda = \lambda_q = 2\cos(\pi/q)$, for $q = 3, 4, 5, \dots$, or $\lambda \geq 2$. We are going to be interested in the former case. These groups have come to be known as the *Hecke groups*, and we will denote them by $H(\lambda_q)$, for $q \geq 3$. Then the Hecke group $H(\lambda_q)$ is the discrete subgroup of PSL $(2, \mathbf{R})$ generated by x and y, and it is isomorphic to the free product of two finite cyclic groups of orders 2 and q. $H(\lambda_q)$ has a presentation

$$H(\lambda_q) = \langle x, y \mid x^2 = y^q = I \rangle \cong C_2 * C_q, \quad [\mathbf{1}]. \tag{1.1}$$

Also $H(\lambda_q)$ has the signature $(0; 2, q, \infty)$, that is, all the groups $H(\lambda_q)$ are triangle groups. The first several of these groups are $H(\lambda_3) =$

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$\Gamma = \mathrm{PSL}\,(2,\mathbf{Z})$ (the modular group), $H(\lambda_4) = H(\sqrt{2})$, $H(\lambda_5) = H((1+\sqrt{5})/2)$ and $H(\lambda_6) = H(\sqrt{3})$.

Hecke groups $H(\lambda_q)$ and their normal subgroups have been extensively studied for many aspects in the literature, [**5, 10**]. The Hecke group $H(\lambda_3)$, the modular group $\mathrm{PSL}\,(2,\mathbf{Z})$, and its normal subgroups have especially been of great interest in many fields of mathematics, for example number theory, automorphic function theory and group theory.

The power subgroups of the modular group $H(\lambda_3)$ have been studied and classified in [**6, 7**] by Newman. In fact, it is a well-known and important result that the only normal subgroups of $H(\lambda_3)$ with torsion are $H(\lambda_3)$, $H^2(\lambda_3)$ and $H^3(\lambda_3)$ of indices 1, 2, 3, respectively. These results have been generalized to Hecke groups $H(\lambda_q)$, q prime, by Cangül and Singerman in [**1**] and to Picard group $\mathbf{P}$ by Fine and Newman in [**3**], and by Özgür in [**9**]. In [**1**], Cangül and Singerman proved that if q is prime then the only normal subgroups of $H(\lambda_q)$ with torsion are $H(\lambda_q)$, $H^2(\lambda_q)$ and $H^q(\lambda_q)$ of indices 1, 2, p, respectively. Also the power subgroups of the Hecke groups $H(\sqrt{n})$, n square-free integer, were investigated by Özgür and Cangül in [**9**].

In this work we study the power subgroups $H^m(\lambda_q)$ of the Hecke groups $H(\lambda_q)$, $q \geq 4$ an even integer. Using techniques of combinatorial group theory (Reidemeister-Schreier method and permutation method), for each integer m, we determine the abstract group structure and generators of $H^m(\lambda_q)$. Also we give the signatures of $H^m(\lambda_q)$.

2. Power subgroups of $H(\lambda_q)$. Let m be a positive integer. Let us define $H^m(\lambda_q)$ to be the subgroup generated by the m-th powers of all elements of $H(\lambda_q)$. The subgroup $H^m(\lambda_q)$ is called the m-th power subgroup of $H(\lambda_q)$. As fully invariant subgroups, they are normal in $H(\lambda_q)$.

From the definition one can easily deduce that

$$H^m(\lambda_q) > H^{mk}(\lambda_q).$$

Now we consider the presentation of the Hecke group $H(\lambda_q)$ given in (1.1):

$$H(\lambda_q) = \langle x, y \mid x^2 = y^q = I \rangle.$$

First we find a presentation for the quotient $H(\lambda_q)/H^m(\lambda_q)$ by adding the relation $T^m = I$ to the presentation of $H(\lambda_q)$. The order of $H(\lambda_q)/H^m(\lambda_q)$ gives us the index. We have

(2.1)
$$H(\lambda_q)/H^m(\lambda_q) \cong \langle x, y \mid x^2 = y^q = x^m = y^m = (xy)^m = \cdots = I\rangle.$$

Thus we use the Reidemeister-Schreier process to find the presentation of the power subgroups $H^m(\lambda_q)$. The idea is as follows: We first choose (not uniquely) a Schreier transversal Σ for $H^m(\lambda_q)$. (This method, in general, applies to all normal subgroups.) Σ consists of certain words in x and y. Then we take all possible products in the following order:

(An element of Σ)$\times$(A generator of $H(\lambda_q)$)
$\times$(coset representative of the preceding product)$^{-1}$.

We now discuss the group theoretical structure of these subgroups. First we begin with the case $m = 2$:

Theorem 2.1. *Let $q > 3$ be an even integer. The normal subgroup $H^2(\lambda_q)$ is a free product of the infinite cyclic group and two finite cyclic groups of order $q/2$. Also*

$$H(\lambda_q)/H^2(\lambda_q) \cong C_2 \times C_2,$$
$$H(\lambda_q) = H^2(\lambda_q) \cup xH^2(\lambda_q) \cup yH^2(\lambda_q) \cup xyH^2(\lambda_q)$$

and

$$H^2(\lambda_q) = \langle y^2\rangle \star \langle xy^2x\rangle \star \langle xyxy^{q-1}\rangle.$$

The elements of $H^2(\lambda_q)$ can be characterized by the requirement that the sums of the exponents of x and y are both even.

Proof. By (2.1), we have

$$H(\lambda_q)/H^2(\lambda_q) \cong \langle x, y \mid x^2 = y^q = x^2 = y^2 = (xy)^2 = \cdots = I\rangle.$$

Thus we use the Reidemeister-Schreier process to find the presentation of the power subgroups $H^2(\lambda_q)$. We have

$$H(\lambda_q)/H^2(\lambda_q) \cong \langle x, y \mid x^2 = y^2 = (xy)^2 = I\rangle,$$

since $x^2 = y^2 = I$. Thus we get

$$H(\lambda_q)/H^2(\lambda_q) \cong C_2 \times C_2,$$

and

$$\left|H(\lambda_q) : H^2(\lambda_q)\right| = 4.$$

Now we choose I, x, y, xy. Hence, all possible products are

$$\begin{array}{ll} I.x.(x)^{-1} = I & I.y.(y)^{-1} = I \\ x.x.(I)^{-1} = I & x.y.(xy)^{-1} = I \\ y.x.(xy)^{-1} = yxy^{-1}x & y.y.(I)^{-1} = y^2 \\ xy.x.(y)^{-1} = xyxy^{-1} & xy.y.(x)^{-1} = xy^2x. \end{array}$$

Since $(yxy^{-1}x)^{-1} = xyxy^{-1}$, the generators of $H^2(\lambda_q)$ are y^2, xy^2x, $xyxy^{q-1}$. Thus $H^2(\lambda_q)$ has a presentation

$$H^2(\lambda_q) = \langle y^2 \rangle \star \langle xy^2x \rangle \star \langle xyxy^{q-1} \rangle,$$

and we get

$$H(\lambda_q) = H^2(\lambda_q) \cup xH^2(\lambda_q) \cup yH^2(\lambda_q) \cup xyH^2(\lambda_q).$$

Let us now use the permutation method, see [**11**], to find the signature of $H^2(\lambda_q)$. We consider the homomorphism

$$H(\lambda_q) \longrightarrow H(\lambda_q)/H^2(\lambda_q) \cong C_2 \times C_2.$$

As each of x, y and xy goes to elements of order 2, they have the following permutation representations:

$$\begin{aligned} x &\longrightarrow (1\ 2)\ (3\ 4), \\ y &\longrightarrow (1\ 3)\ (2\ 4), \\ xy &\longrightarrow (1\ 4)\ (2\ 3). \end{aligned}$$

Therefore the signature of $H^2(\lambda_q)$ is $(g;\ q/2,\ q/2,\ \infty,\ \infty) = (g;\ (q/2)^{(2)},\ \infty^{(2)})$. Now, by the Riemann-Hurwitz formula, $g = 0$. Thus we obtain $H^2(\lambda_q) = (0;\ (q/2)^{(2)},\ \infty^{(2)})$. $\square$

Theorem 2.2. *Let $q > 3$ be an even integer, and let m be a positive integer such that $(m, q) = 2$. The normal subgroup $H^m(\lambda_q)$ is the free product of m finite cyclic groups of order $q/2$ and the infinite cyclic group $\mathbf{Z}$. Also*

$$H(\lambda_q)/H^m(\lambda_q) \cong D_m,$$

$$\begin{aligned} H(\lambda_q) = {} & H^m(\lambda_q) \cup xH^m(\lambda_q) \cup xyH^m(\lambda_q) \cup xyxH^m(\lambda_q) \cup \cdots \\ & \cup \underbrace{(xy)(xy)\cdots(xy)}_{(m/2)\ \textit{times}} H^m(\lambda_q) \cup yH^m(\lambda_q) \cup yxH^m(\lambda_q) \\ & \cup yxyH^m(\lambda_q) \cup \cdots \cup \underbrace{(yx)(yx)\cdots(yx)}_{(m/2)-1\ \textit{times}} yH^m(\lambda_q) \end{aligned}$$

and

$$\begin{aligned} H^m(\lambda_q) = {} & \langle \underbrace{(xy)(xy)\cdots(xy)}_{(m/2)\ \textit{times}} x \underbrace{(y^{-1}x)(y^{-1}x)\cdots(y^{-1}x)}_{(m/2)-1\ \textit{times}} y^{-1} \rangle \\ & \star \langle y^2 \rangle \star \langle xy^2x \rangle \star \langle yxy^2xy^{-1} \rangle \star \cdots \\ & \star \langle \underbrace{(yx)(yx)\cdots(yx)}_{(m/2)-1\ \textit{times}} y^2 \underbrace{(xy^{-1})(xy^{-1})\cdots(xy^{-1})}_{(m/2)-1\ \textit{times}} \rangle \\ & \star \langle \underbrace{(xy)(xy)\cdots(xy)}_{(m/2)-1\ \textit{times}} xy^2 \underbrace{(xy^{-1})(xy^{-1})\cdots(xy^{-1})}_{(m/2)-1\ \textit{times}} x \rangle. \end{aligned}$$

The elements of $H^m(\lambda_q)$ can be characterized by the requirement that the sums of the exponents of x and y are both even.

Proof. If $(m, q) = 2$, then by (2.1), we obtain

$$H(\lambda_q)/H^m(\lambda_q) \cong \langle x, y \mid x^2 = y^2 = (xy)^m = I \rangle \cong D_m,$$

from the relations $y^m = y^q = y^2 = I$. Thus

$$|H(\lambda_q)/H^m(\lambda_q)| = 2m.$$

Therefore we choose $\{I,\ x,\ xy,\ xyx,\ xyxy,\ \ldots,\ \underbrace{(xy)(xy)\cdots(xy)}_{(m/2)\ \text{times}},\ y,$ $yx,\ yxy,\ \ldots,\ \underbrace{(yx)(yx)\cdots(yx)}_{(m/2)-1\ \text{times}} y\}$ as a Schreier transversal for $H^m(\lambda_q)$.

According to the Reidemeister-Schreier method, we can form all possible products:

$$
\begin{aligned}
&I.x.(x)^{-1} = I,\\
&x.x.(I)^{-1} = I,\\
&xy.x.(xyx)^{-1} = I,\\
&xyx.x.(xy)^{-1} = I\\
&xyxy.x.(xyxyx)^{-1} = I,\\
&\vdots\\
&\underbrace{(xy)(xy)\cdots(xy)}_{(m/2)\text{ times}}.x.(\underbrace{(yx)(yx)\cdots(yx)}_{(m/2)-1\text{ times}}y)^{-1}\\
&\quad = \underbrace{(xy)(xy)\cdots(xy)}_{(m/2)\text{ times}}x\underbrace{(y^{-1}x)(y^{-1}x)\cdots(y^{-1}x)}_{(m/2)-1\text{ times}}y^{-1}\\
&y.x.(yx)^{-1} = I,\\
&yx.x.(y)^{-1} = I,\\
&yxy.x.(yxyx)^{-1} = I,\\
&\vdots\\
&\underbrace{(yx)(yx)\cdots(yx)}_{(m/2)-1\text{ times}}y.x.(\underbrace{(xy)(xy)\cdots(xy)}_{(m/2)\text{ times}})^{-1} = I,\\
&I.y.(y)^{-1} = I,\\
&x.y.(xy)^{-1} = I,\\
&xy.y.(x)^{-1} = xy^2x,\\
&xyx.y.(xyxy)^{-1} = I,\\
&xyxy.y.(xyx)^{-1} = xyxy^2xy^{-1}x,\\
&\vdots\\
&\underbrace{(xy)(xy)\cdots(xy)}_{(m/2)\text{ times}}.y.(\underbrace{(xy)(xy)\cdots(xy)}_{(m/2)-1\text{ times}}x)^{-1}\\
&\quad = \underbrace{(xy)(xy)\cdots(xy)}_{(m/2)-1\text{ times}}xy^2\underbrace{(xy^{-1})(xy^{-1})\cdots(xy^{-1})}_{(m/2)-1\text{ times}}x,\\
&y.y.(I)^{-1} = y^2,
\end{aligned}
$$

$$\begin{aligned}
&yx.y.(yxy)^{-1} = I,\\
&yxy.y.(yx)^{-1} = yxy^2xy^{-1},\\
&\vdots\\
&\underbrace{(yx)(yx)\cdots(yx)}_{(m/2)-1 \text{ times}}y.y.(\underbrace{(yx)(yx)\cdots(yx)}_{(m/2)-1 \text{ times}})^{-1}\\
&\quad = \underbrace{(yx)(yx)\cdots(yx)}_{(m/2)-1 \text{ times}}y^2\underbrace{(xy^{-1})(xy^{-1})\cdots(xy^{-1})}_{(m/2)-1 \text{ times}}.
\end{aligned}$$

The generators are $\underbrace{(xy)(xy)\cdots(xy)}_{(m/2) \text{ times}}x\underbrace{(y^{-1}x)(y^{-1}x)\cdots(y^{-1}x)}_{(m/2)-1 \text{ times}}y^{-1}$, y^2, xy^2x, yxy^2xy^{-1}, $\ldots$, $\underbrace{(yx)(yx)\cdots(yx)}_{(m/2)-1 \text{ times}}y^2\underbrace{(xy^{-1})(xy^{-1})\cdots(xy^{-1})}_{(m/2)-1 \text{ times}}$, $\underbrace{(xy)(xy)\cdots(xy)}_{(m/2)-1 \text{ times}}xy^2\underbrace{(xy^{-1})(xy^{-1})\cdots(xy^{-1})}_{(m/2)-1 \text{ times}}x$. Thus $H^m(\lambda_q)$ has a presentation

$$\begin{aligned}
H^m(\lambda_q) = \langle&\underbrace{(xy)(xy)\cdots(xy)}_{(m/2) \text{ times}}x\underbrace{(y^{-1}x)(y^{-1}x)\cdots(y^{-1}x)}_{(m/2)-1 \text{ times}}y^{-1}\rangle\\
&\star\langle y^2\rangle\star\langle xy^2x\rangle\star\langle yxy^2xy^{-1}\rangle\star\cdots\\
&\star\langle\underbrace{(yx)(yx)\cdots(yx)}_{(m/2)-1 \text{ times}}y^2\underbrace{(xy^{-1})(xy^{-1})\cdots(xy^{-1})}_{(m/2)-1 \text{ times}}\rangle\\
&\star\langle\underbrace{(xy)(xy)\cdots(xy)}_{(m/2)-1 \text{ times}}xy^2\underbrace{(xy^{-1})(xy^{-1})\cdots(xy^{-1})}_{(m/2)-1 \text{ times}}x\rangle.
\end{aligned}$$

Now consider the homomorphism

$$H(\lambda_q) \longrightarrow H(\lambda_q)/H^m(\lambda_q) \cong D_m.$$

The permutation representations of x, y and xy are

$$\begin{aligned}
x &\longrightarrow (1\ 2)(3\ 4)\ldots(2m-1\ 2m),\\
y &\longrightarrow (2\ 3)(4\ 5)\ldots(2m\ 1),\\
xy &\longrightarrow (1\ 3\ 5\ldots 2m-1)(2m\ 2m-2\ldots 4\ 2).
\end{aligned}$$

Thus $H^m(\lambda_q)$ has the signature $(0; (q/2)^{(m)}, \infty^{(2)})$ similar to the previous cases. □

Theorem 2.3. *Let $q > 3$ be an even integer, and let m be a positive integer such that $(m, 2) = 1$ and $(m, q) = d$. The normal subgroup $H^m(\lambda_q)$ is the free product of d finite cyclic groups of order two and the finite cyclic group of order q/d. Also,*

$$H(\lambda_q)/H^m(\lambda_q) \cong C_d,$$
$$H(\lambda_q) = H^m(\lambda_q) \cup yH^m(\lambda_q) \cup y^2 H^m(\lambda_q) \cup \cdots \cup y^{d-1} H^m(\lambda_q),$$

and

$$H^m(\lambda_q) = \langle x \rangle \star \langle yxy^{q-1} \rangle \star \langle y^2 xy^{q-2} \rangle \star \cdots \star \langle y^{d-1} xy^{q-d+1} \rangle \star \langle y^d \rangle.$$

Proof. If $(m, 2) = 1$ and $(m, q) = d$, then by (2.1), we find

$$H(\lambda_q)/H^m(\lambda_q) \cong \langle y \mid y^d = I \rangle \cong C_d$$

from the relations $x^2 = x^m = I$ and $y^q = y^m = I$. Thus

$$|H(\lambda_q) : H^m(\lambda_q)| = d.$$

Therefore, we choose $\{I, y, y^2, \ldots, y^{d-1}\}$ as a Schreier transversal for $H^m(\lambda_q)$. According to the Reidemeister-Schreier method, we can form all possible products:

$$\begin{array}{ll}
I.x.(I)^{-1} = x, & I.y.(y)^{-1} = I, \\
y.x.(y)^{-1} = yxy^{q-1}, & y.y.(y^2)^{-1} = I, \\
y^2.x.(y^2)^{-1} = y^2xy^{q-2}, & y^2.y.(y^3)^{-1} = I, \\
\vdots & \vdots \\
y^{d-1}.x.(y^{d-1})^{-1} = y^{d-1}xy^{q-d+1}, & y^{d-1}.y.(I)^{-1} = y^d.
\end{array}$$

The generators are x, y^d, yxy^{q-1}, y^2xy^{q-2}, $\ldots$, $y^{d-1}xy^{q-d+1}$. Thus $H^m(\lambda_q)$ has a presentation

$$H^m(\lambda_q) = \langle x \rangle \star \langle yxy^{q-1} \rangle \star \langle y^2 xy^{q-2} \rangle \star \cdots \star \langle y^{d-1} xy^{q-d+1} \rangle \star \langle y^d \rangle,$$

and we get

$$H(\lambda_q) = H^m(\lambda_q) \cup yH^m(\lambda_q) \cup y^2 H^m(\lambda_q) \cup \cdots \cup y^{d-1} H^m(\lambda_q).$$

Also as the quotient is isomorphic to C_d and, by the permutation method, $H^m(\lambda_q)$ has the signature $(0; 2^{(d)}, q/d, \infty)$. □

Corollary 2.4. *Let $q > 3$ be an even integer, and let m be a positive odd integer such that $(m, q) = 1$. Then*

$$H^m(\lambda_q) = H(\lambda_q).$$

Now we are only left to consider the case where $(m, 2) = 2$ and $(m, q) = d > 2$. Then in $H(\lambda_q)/H^m(\lambda_q)$ we have the relations $X^2 = Y^d = (XY)^m$, where X, Y and XY are the images of x, y and xy, respectively, under the homomorphism of $H(\lambda_q)$ to $H(\lambda_q)/H^m(\lambda_q)$. Then the factor group is the group whose signature $(2, d, m)$. Coxeter and Moser [**2**] have shown that the triangle group (k, l, m) is finite when $(1/k + 1/l + 1/m) > 1$ and infinite when $(1/k + 1/l + 1/m) \leq 1$. Then the factor group $(2, d, m)$ is a group of infinite order. Therefore the above techniques do not say much about $H^m(\lambda_q)$ in this case apart from the fact they are all normal subgroups with torsion.

Example 2.1. For $q = 6$, let us consider the Hecke group $H(\lambda_6)$.

i) If $(m, 6) = 1$, then $H^m(\lambda_6) = H(\lambda_6)$.

ii) If $(m, 6) = 3$, then

$$H(\lambda_6)/H^m(\lambda_6) \cong \langle y \mid y^3 = I \rangle \cong C_3.$$

We choose $\{I, y, y^2\}$ as a Schreier transversal for $H^m(\lambda_6)$. Using the Reidemeister-Schreier method we obtain

$$H^m(\lambda_6) \cong \langle x \rangle \star \langle yxy^5 \rangle \star \langle y^2 xy^4 \rangle \star \langle y^3 \rangle.$$

We have

$$H(\lambda_6) = H^m(\lambda_6) \cup yH^m(\lambda_6) \cup y^2 H^m(\lambda_6).$$

iii) If $(m,6)=2$, then

$$H(\lambda_6)/H^m(\lambda_6) \cong \langle x, y \mid x^2 = y^2 = (xy)^m = I\rangle$$

so

$$|H(\lambda_6) : H^m(\lambda_6)| \cong 2m.$$

We choose $\{I, x, xy, xyx, xyxy, \dots, \underbrace{(xy)(xy)\cdots(xy)}_{(m/2) \text{ times}}, y, yx, yxy, \dots, \underbrace{(yx)(yx)\cdots(yx)}_{(m/2)-1 \text{ times}} y\}$ as a Schreier transversal for $H^m(\lambda_6)$. Using the Reidemeister-Schreier method, we get

$$\begin{aligned} H^m(\lambda_6) = \langle & \underbrace{(xy)(xy)\cdots(xy)}_{(m/2) \text{ times}} x \underbrace{(y^5x)(y^5x)\cdots(y^5x)}_{(m/2)-1 \text{ times}} y^5\rangle \\ & \star \langle y^2\rangle \star \langle xy^2x\rangle \star \langle yxy^2xy^5\rangle \star \cdots \\ & \star \langle \underbrace{(yx)(yx)\cdots(yx)}_{(m/2)-1 \text{ times}} y^2 \underbrace{(xy^5)(xy^5)\cdots(xy^5)}_{(m/2)-1 \text{ times}}\rangle \\ & \star \langle \underbrace{(xy)(xy)\cdots(xy)}_{(m/2)-1 \text{ times}} xy^2 \underbrace{(xy^5)(xy^5)\cdots(xy^5)}_{(m/2)-1 \text{ times}} x\rangle. \end{aligned}$$

Then we have

$$\begin{aligned} H(\lambda_q) = & H^m(\lambda_q) \cup xH^m(\lambda_q) \cup xyH^m(\lambda_q) \cup xyxH^m(\lambda_q) \cup \cdots \\ & \cup \underbrace{(xy)(xy)\cdots(xy)}_{(m/2) \text{ times}} H^m(\lambda_q) \cup yH^m(\lambda_q) \cup yxH^m(\lambda_q) \\ & \cup yxyH^m(\lambda_q) \cup \cdots \cup \underbrace{(yx)(yx)\cdots(yx)}_{(m/2)-1 \text{ times}} yH^m(\lambda_q). \end{aligned}$$

For example, if $m=2$, then

$$H(\lambda_6)/H^2(\lambda_6) \cong \langle x, y \mid x^2 = y^2 = (xy)^2 = I\rangle \cong D_2$$

so

$$\left|H(\lambda_6) : H^2(\lambda_6)\right| \cong 4.$$

We choose $\{I, x, y, xy\}$ as a Schreier transversal for $H^2(\lambda_6)$. According to the Reidemeister-Schreier method, we can form all possible products:

$$\begin{array}{ll} I.x.(x)^{-1} = I, & I.y.(y)^{-1} = I, \\ x.x.(I)^{-1} = I, & x.y.(xy)^{-1} = I, \\ y.x.(xy)^{-1} = yxy^5x, & y.y.(I)^{-1} = y^2, \\ xy.x.(y)^{-1} = xyxy^5, & xy.y.(x)^{-1} = xy^2x. \end{array}$$

Since $(yxy^5x)^{-1} = xyxy^5$, the generators are y^2, xy^2x, $xyxy^5$. Then we get

$$H^2(\lambda_6) = \langle y^2 \rangle \star \langle xy^2x \rangle \star \langle xyxy^5 \rangle$$

and

$$H(\lambda_6) = H^2(\lambda_6) \cup xH^2(\lambda_6) \cup yH^2(\lambda_6) \cup xyH^2(\lambda_6).$$

iv) The cases $(m, 6) = 6$, that is, $m = 6k$, $k = 1, 2, 3, \ldots$, the factor group $(2, 6, 6k)$ is a group of infinite order since $(1/2 + 1/6 + 1/6k) = (4k+1)/6k < 1$. In these cases, $H^{6k}(\lambda_6)$ are all normal subgroups with torsion.

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