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Uniform convergence of the Bieberbach polynomials in closed smooth domains of bounded boundary rotation

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Abstract

Let G be a Jordan smooth domain of bounded boundary rotation, let $z_0 \in G$, and let $w = \varphi_0(z)$ be the conformal mapping of G onto $D(0, r_0) := \{w : |w| < r_0\}$ with the normalization $\varphi_0(z_0) = 0, \varphi_0'(z_0) = 1$. Let also $\pi_n(z), n = 1, 2, \ldots,$ be the Bieberbach polynomials for the pair (G, z_0). We investigate the uniform convergence of these polynomials on $\bar{G}$ and prove the estimate

$$\|\varphi_0 - \pi_n\|_{\bar{G}} := \max_{z \in \bar{G}} |\varphi_0(z) - \pi_n(z)| \leqslant \frac{c}{n^{1-\varepsilon}},$$

for some constant $c = c(\varepsilon)$ independent of n.

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1. Introduction and new results

Let G be a finite simply connected domain in the complex plane C bounded by rectifiable Jordan curve L, and let $z_0 \in G$. By the Riemann mapping theorem, there exists a unique conformal mapping $w = \varphi_0(z)$ of G onto $D(0, r_0) := \{w : |w| < r_0\}$

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with the normalization $\varphi_0(z_0) = 0, \varphi_0'(z_0) = 1$. The radius r_0 of this disc is called the conformal radius of G with respect to z_0. Let $\psi_0(w)$ be the inverse to $\varphi_0(z)$. Let also $G^- := ext\, L, D := D(0,1) = \{w : |w| < 1\},\ T := \partial D, D^- := \{w : |w| > 1\}$, and let φ be the conformal mapping of G^- onto D^- normalized by

$$\varphi(\infty) = \infty, \lim_{z \to \infty} \varphi(z)/z > 0.$$

We denote by ψ the inverse mappings of φ.

For an arbitrary function f given on G we set

$$||f||^2_{L_2(G)} := \int \int_G |f(z)|^2 d\sigma_z.$$

If the function f has a continuous extension to $\bar{G}$ we use also the uniform norm

$$||f||_{\bar{G}} := \sup\{|f(z)|, z \in \bar{G}\}.$$

It is well known that the function $\varphi_0(z)$ minimizes the integral $||f'||^2_{L_2(G)}$ in the class of all functions analytic in G with the normalization $f(z_0) = 0, f'(z_0) = 1$. On the other hand, let Π_n be the class of all polynomials p_n of degree at most n satisfying the conditions $p_n(z_0) = 0, p_n'(z_0) = 1$. Then the integral $||p_n'||^2_{L_2(G)}$ is minimized in Π_n by an unique polynomial π_n which is called the nth Bieberbach polynomial for the pair (G, z_0).

As follows from the results due to Farrel and Markushevich, if G is a Caratheodory domain, then $||\varphi_0' - \pi_n'||_{L_2(G)} \to 0 (n \to \infty)$ and from this it follows that $\pi_n(z) \to \varphi_0(z) (n \to \infty)$ for $z \in G$, uniformly on compact subsets of G.

First of all, the uniform convergence of the Bieberbach polynomials in the closed domain $\bar{G}$ was investigated by Keldych. He showed [15] that if the boundary L of G is a smooth Jordan curve with bounded curvature then the following estimate holds for every $\varepsilon > 0$:

$$||\varphi_0 - \pi_n||_{\bar{G}} \leq \frac{const}{n^{1-\varepsilon}}.$$

In [15] the author also gives an example of domains G with a Jordan rectifiable boundary L for which the appropriate sequence of the Bieberbach polynomials diverges on a set which is everywhere dense in L.

Furthermore, Mergelyan [16] has shown that the Bieberbach polynomials satisfy

$$||\varphi_0 - \pi_n||_{\bar{G}} \leq \frac{const}{n^{\frac{1}{2}-\varepsilon}}, \quad (1)$$

for every $\varepsilon > 0$, whenever L is a smooth Jordan curve.

Therefore, the uniform convergence of the sequence $\{\pi_n\}_{n=1}^{\infty}$ in $\bar{G}$ and the estimate of the error $||\varphi_0 - \pi_n||_{\bar{G}}$ depend on the geometric properties of boundary L. If L has a certain degree of smoothness, this error tends to zero with a certain speed. In the literature there are sufficiently many results about the uniform convergence of the Bieberbach polynomials in the closed domains $\bar{G}$. In several papers (see, for example, [1–3,9–11,13–16,18,19,21]) various estimates of the error $||\varphi_0 - \pi_n||_{\bar{G}}$ and sufficient conditions on the geometry of the boundary L are given to guarantee the uniform convergence of the Bieberbach polynomials on $\bar{G}$. Recently the important results in this area has been obtained by Andrievskii [2,3] and by Gaier [9–11]. In particular Andrievskii proved the uniform convergence of Bieberbach polynomials in closed domains with quasiconformal and piecewise-quasiconformal boundary, and Gaier obtained the results about the uniform convergence of these polynomials in closed domains with the various boundary constructions and also studied the cases when the rate of this convergence is quite close to the best possible rate in uniform polynomial approximation of the conformal mapping φ_0. It should also be pointed out the recent paper of Andrievskii and Pritsker [4], where they investigated the uniform convergence in closed domains with certain interior zero angles and discussed the critical order of tangency at this interior zero angle, separating the convergent behaviour of Bieberbach polynomials from the divergent one for sufficiently thin cusps.

But no improvement of the Mergelyan's estimation (1) in the above cited works, when the boundary of G is smooth has been observed. However, Mergelyan [16] stated it as a conjecture that the exponent $\frac{1}{2} - \varepsilon$ in (1) could be replaced by $1 - \varepsilon$.

In [14] it has been possible for us to obtain some improvement of the above cited Mergelyan's estimation (1). From this result in particular it follows that if G is a finite domain with a smooth Jordan boundary, then

$$||\varphi_0 - \pi_n||_{\bar{G}} \leqslant const \left(\frac{\ln n}{n} \right)^{\frac{1}{2}}, \quad n \geqslant 2,$$

which improves estimation (1).

Developing the idea used in [14] we shall prove the above cited Mergelyan's conjecture for a smooth domain of bounded boundary rotation.

Our main result states as

Theorem 1. *If G is a finite smooth domain of bounded boundary rotation, then for every* $\varepsilon > 0$ *there exists a constant* $c = c(\varepsilon)$ *such that*

$$||\varphi_0 - \pi_n||_{\bar{G}} \leqslant \frac{c}{n^{1-\varepsilon}}, \quad n \geqslant 1.$$

We shall use $c, c_1, c_2, \ldots$ to denote constants (in general, different in different relations) depending only on numbers that are not important for the questions of interest.

2. Auxiliary results

We denote by $L^p(L)$ and $E^p(G)$ the set of all measurable complex valued functions such that $|f|^p$ is Lebesgue integrable with respect to arclength, and the Smirnov class of analytic functions in G, respectively. Each function $f \in E^p(G)$ has a nontangential limit almost everywhere (a.e.) on L, and if we use the same notation for the nontangential limit of f, then $f \in L^p(L)$.

For $p \geqslant 1, L^p(L)$ and $E^p(G)$ are Banach spaces with respect to the norm

$$||f||_{E^p(G)} = ||f||_{L^p(L)} := \left(\int_L |f(z)|^p |dz| \right)^{1/p}.$$

For the further fundamental properties, see [6, pp. 168–185]; [12, pp. 438–453].

For a weight function ω given on L, and $p > 1$ we also set

$$L^p(L, \omega) := \{ f \in L^1(L) : |f|^p \omega \in L^1(L) \},$$

$$E^p(G, \omega) := \{ f \in E^1(G) : f \in L^p(L, \omega) \}.$$

We denote by $A_p(L)$ the set of all weight functions ω satisfying the Muckenhoupt condition, i.e.,

$$\sup_{z \in L} \sup_{r>0} \left(\frac{1}{r} \int_{L \cap D(z,r)} \omega(\varsigma) |d\varsigma| \right) \left(\frac{1}{r} \int_{L \cap D(z,r)} [\omega(\varsigma)]^{-1/(p-1)} |d\varsigma| \right)^{p-1} < \infty, 1 < p < \infty.$$

Definition 1. For $g \in L^p = L^p(0, 2\pi), 1 \leqslant p < \infty$, the function

$$\omega_p(\delta) = \omega_p(g, \delta) := \sup_{0 < h \leqslant \delta} \left\{ \int_0^{2\pi} |g(x+t) - g(x)|^p \, dx \right\}^{1/p}$$

is called the integral modulus of continuity of order p for g.

If

$$\omega_p(g, t) = O(t^\alpha), \quad 0 < \alpha \leqslant 1,$$

we say that g belongs to the class Λ_α^p.

Definition 2. Let G be a domain with a smooth boundary L, and let $\Phi(w) := \varphi_0'(\psi(w))$. The function

$$\omega_p^*(\varphi_0', \delta) := \sup_{|h| \leqslant \delta} ||\Phi(we^{ih}) - \Phi(w)||_{L^p(T)} =: \omega_p(\Phi, \delta), \quad p > 1$$

is called the generalized integral modulus of continuity for $\varphi_0' \in E^p(G)$.

This definition is correct. Indeed, if $1/p_0 + 1/q_0 = 1$ and $|h| \geqslant 0$, by virtue of Hölder's inequality we have

$$\begin{aligned} ||\Phi(we^{ih})||_{L^p(T)}^p &= \int_T |(\varphi_0' \circ \psi)(we^{ih})|^p |dw| \\ &= \int_T |(\varphi_0' \circ \psi)(w)|^p |dw| = \int_L |\varphi_0'(z)|^p |\varphi'(z)| |dz| \\ &\leqslant \left(\int_L |\varphi_0'(z)|^{pp_0} |dz| \right)^{1/p_0} \left(\int_L |\varphi'(z)|^{q_0} |dz| \right)^{1/q_0} < \infty, \end{aligned}$$

because for the smooth domains $\varphi_0', \varphi' \in L^p(L)$, for every $p \geqslant 1$ [20].

Without loss of generality, we assume that the conformal radius r_0 of G with respect to z_0 equal to 1. Let $\psi_0(e^{it}), 0 \leqslant t \leqslant 2\pi$, be the conformal parametrization of the smooth boundary L and let $\beta(t)$ be its tangent direction angle at the point $\psi_0(e^{it})$.

Definition 3 (See, for example, Pommerenke [17, pp. 63–64]). The domain G is of bounded boundary rotation if $\beta(t)$ has bounded variation, i.e. if

$$\int_0^{2\pi} |d\beta(t)| = \sup_{t_v} \sum_{v=1}^{n} |\beta(t_v) - \beta(t_{v-1})| < \infty$$

for all partitions $0 = t_0 < t_1 < \cdots < t_n = 2\pi$.

The following theorem holds.

Theorem 2. *Let G be a finite smooth domain of bounded boundary rotation, and let $p > 1$. Then*

$$\psi_0'(e^{it}) \in \Lambda^p_{\frac{1}{p}-\varepsilon},$$

for every $\varepsilon > 0$.

Proof. Since L is smooth we have [17, Theorem 3.2, pp. 43–44]

$$\arg \psi_0'(e^{it}) = \beta(t) - t - \frac{\pi}{2}$$

for the conformal parametrization and

$$\log\ \psi_0'(w) = \frac{i}{2\pi} \int_0^{2\pi} \frac{e^{it} + w}{e^{it} - w} \Big(\beta(t) - t - \frac{\pi}{2}\Big) dt, \quad w \in D. \tag{2}$$

It follows from (2) that

$$\psi_0''(w) = \frac{i\psi_0'(w)}{\pi} \int_0^{2\pi} \frac{e^{it}}{(e^{it} - w)^2} \Big(\beta(t) - t - \frac{\pi}{2}\Big)\, dt, \quad w \in D,$$

and also

$$\psi_0''(w) = -\frac{\psi_0'(w)}{\pi} \int_0^{2\pi} \Big(\beta(t) - t - \frac{\pi}{2}\Big) d_t \Big(\frac{1}{e^{it} - w}\Big), \quad w \in D. \tag{3}$$

Since the function

$$\Big(\beta(t) - t - \frac{\pi}{2}\Big) \frac{1}{e^{it} - w}$$

is periodic, an integration by parts gives

$$\psi_0''(w) = \frac{\psi_0'(w)}{\pi} \int_0^{2\pi} \frac{d(\beta(t) - t - \frac{\pi}{2})}{e^{it} - w}, \quad w \in D. \tag{4}$$

Denoting

$$M_p(r,\psi_0'') := \left(\int_0^{2\pi} |\psi_0''(re^{i\theta})|^p d\theta\right)^{1/p}$$

from (4) we have

$$M_p^p(r,\psi_0'') = \frac{1}{\pi^p}\int_0^{2\pi}\left|\psi_0'(re^{i\theta})\int_0^{2\pi}\frac{d(\beta(t)-t-\pi/2)}{e^{it}-re^{i\theta}}\right|^p d\theta$$

and applying Hölder's inequality we find

$$M_p^p(r,\psi_0'') \leqslant \frac{1}{\pi^p}\left(\int_0^{2\pi} |\psi_0'(re^{i\theta})|^{pp_0} d\theta\right)^{1/p_0}\left(\int_0^{2\pi}\left|\int_0^{2\pi}\frac{d(\beta(t)-t-\pi/2)}{e^{it}-re^{i\theta}}\right|^{pq_0} d\theta\right)^{1/q_0},$$

where $1/p_0 + 1/q_0 = 1$. Since L is smooth the first integral is finite and hence

$$M_p^p(r,\psi_0'') \leqslant c_1\left(\int_0^{2\pi}\left|\int_0^{2\pi}\frac{d(\beta(t)-t-\pi/2)}{e^{it}-re^{i\theta}}\right|^{pq_0} d\theta\right)^{1/q_0}$$

or

$$M_p(r,\psi_0'') \leqslant c_2\left(\int_0^{2\pi}\left|\int_0^{2\pi}\frac{d(\beta(t)-t-\pi/2)}{e^{it}-re^{i\theta}}\right|^{pq_0} d\theta\right)^{1/(pq_0)}.$$

Applying Minkowski's inequality to the right side we obtain that

$$M_p(r,\psi_0'') \leqslant c_2\int_0^{2\pi}\left(\int_0^{2\pi}\frac{d\theta}{|e^{it}-re^{i\theta}|^{pq_0}}\right)^{1/(pq_0)} |d(\beta(t)-t-\pi/2)|. \tag{5}$$

Take into account the inequality

$$\int_0^{2\pi}\frac{d\theta}{|e^{it}-re^{i\theta}|^{pq_0}} \leqslant \frac{c_3}{(1-r)^{pq_0-1}},$$

which can be verified easily, from relation (5) we get

$$M_p(r,\psi_0'') \leqslant \frac{c_4}{(1-r)^{\frac{pq_0-1}{pq_0}}}\int_0^{2\pi} |d(\beta(t)-t-\pi/2)|.$$

Since G is a domain of bounded boundary rotation, the function $\beta(t)-t-\pi/2$ has bounded variation. This property implies that the last integral is also finite and then

$$M_p(r,\psi_0'') \leqslant \frac{c_5}{(1-r)^{1-\frac{1}{pq_0}}}.$$

Choosing the number $q_0 > 1$ sufficiently close to 1 we have

$$M_p(r, \psi_0'') \leqslant \frac{c_5}{(1-r)^{1-(\frac{1}{p}-\varepsilon)}},$$

for every $\varepsilon > 0$.

Now applying the well-known Hardy–Littlewood theorem (see for example [6, p. 78]) from the last inequality we deduce that $\psi_0'(e^{it}) \in \Lambda^p_{\frac{1}{p}-\varepsilon}$.

Remark 1. Note that for the smooth domains the statement of Theorem in general is false.

Indeed, consider the function

$$\psi(w) = 6w + \sum_{k=1}^{\infty} \frac{w^{2^k+1}}{k^2(2^k+1)}, \quad w \in D.$$

Then

$$\psi'(w) = 6 + \sum_{k=1}^{\infty} \frac{w^{2^k}}{k^2}.$$

Hence

$$Re\, \psi'(w) \geqslant 6 - \sum_{k=1}^{\infty} \frac{1}{k^2} > 1 \quad \text{for } w \in D.$$

Thus ψ is univalent. Furthermore, ψ' is continuous in $\bar{D}$ and $\psi'(w) \neq 0$. It follows that the image domain is smoothly bounded.

Now take $p = 2$. We have

$$\begin{aligned} A &:= \frac{1}{2\pi} \int_0^{2\pi} |\psi'(e^{it+ih}) - \psi'(e^{it})|^2\, dt \\ &= \sum_{k=1}^{\infty} \frac{1}{k^4} |e^{i2^k h} - 1|^2 = 4 \sum_{k=1}^{\infty} \frac{1}{k^4} \sin^2(2^{k-1}h). \end{aligned}$$

We choose $h = \pi/2^m,\ m = 1, 2, \ldots$. Then

$$A \geqslant \frac{4}{m^4},$$

which is not $O(h^\alpha) = O(\frac{1}{2^{m\alpha}})$ for any $\alpha > 0$. □

Theorem 3. *Let G be a domain with a smooth boundary L, and let $p > 1$. Then*

$$||\varphi_0' - S_n(\varphi_0', \cdot)||_{L^p(L)} \leqslant c\omega_{p+\varepsilon}(\Phi, 1/n),$$

for every $\varepsilon > 0$, *where*

$$S_n(\varphi_0', z) := \sum_{k=0}^{n} a_k(\varphi_0')F_k(z), \quad n = 0, 1, 2, \dots$$

are the nth partial sums of the Faber series of φ_0'.

Proof. As we showed after definition 2, $\Phi \in L^p(T)$ for every $p \geq 1$. Let us consider the functions Φ^+ and Φ^+ defined by

$$\Phi^+(w) := \frac{1}{2\pi i}\int_T \frac{\Phi(\tau)}{\tau - w}\, d\tau, \quad w \in D$$

and

$$\Phi^-(w) := \frac{1}{2\pi i}\int_T \frac{\Phi(\tau)}{\tau - w}\, d\tau, \quad w \in D^-.$$

Since $\varphi_0' \in E^p(G)$ for every $p \geq 1$, we can associate a formal Faber series

$$\sum_{k=0}^{\infty} a_k(\varphi_0')F_k(z),$$

with the function φ_0, i.e.,

$$\varphi_0'(z) \sim \sum_{k=0}^{\infty} a_k(\varphi_0')F_k(z),$$

where

$$a_k(\varphi_0') := \frac{1}{2\pi i}\int_T \frac{\Phi(\tau)}{\tau^{k+1}}\, d\tau, \quad k = 0, 1, 2, \dots, \tag{6}$$

are the Faber coefficients of φ_0'.

By well-known Privalov's Lemma $\Phi = \Phi^+ - \Phi^-$ a.e. on T. Moreover, $\Phi^+ \in E^p(D), \Phi^- \in E^p(D^-)$ and $\Phi^-(\infty) = 0$. Then from (6) we find

$$a_k(\varphi_0') = \frac{1}{2\pi i}\int_T \frac{\Phi(\tau)}{\tau^{k+1}}\, d\tau = \frac{1}{2\pi i}\int_T \frac{\Phi^+(\tau) - \Phi^-(\tau)}{\tau^{k+1}}\, d\tau = a_k(\Phi^+).$$

Namely, the kth Faber coefficient of $\varphi_0' \in E^p(G)$ is the kth-Taylor's coefficient of $\Phi^+ \in E^p(D)$ at the origin. On the other hand, the relation $\varphi_0' \in E^p(G)$ implies

$$\int_L \frac{\varphi_0'(\varsigma)}{\varsigma - z'}\, d\varsigma = 0, \quad z' \in G^-,$$

and considering the relation $\Phi = \Phi^+ - \Phi^-$ which holds a.e. on T we have the equality

$$\varphi_0'(\varsigma) = \Phi^+(\varphi(\varsigma)) - \Phi^-(\varphi(\varsigma)) \tag{7}$$

a.e. on L.

Let us take a $z' \in G^-$. Since $\varphi_0' \in E^p(G)$ for $p \geqslant 1$, using the well-known integral representation for the Faber polynomials $F_k(z)$,

$$F_k(z') = \varphi^k(z') + \frac{1}{2\pi i}\int_L \frac{\varphi^k(\varsigma)}{\varsigma - z'}\,d\varsigma,$$

and (7) we have

$$\begin{aligned} S_n(\varphi_0', z') &= \sum_{k=0}^{n} a_k(\varphi_0')F_k(z') \\ &= \sum_{k=0}^{n} a_k(\varphi_0')\varphi^k(z') + \frac{1}{2\pi i}\int_L \frac{\sum_{k=0}^n a_k(\varphi_0')\varphi^k(\varsigma)}{\varsigma - z'}\,d\varsigma - \frac{1}{2\pi i}\int_L \frac{\varphi_0'(\varsigma)}{\varsigma - z'}\,d\varsigma \\ &= \sum_{k=0}^{n} a_k(\varphi_0')\varphi^k(z') + \frac{1}{2\pi i}\int_L \frac{\sum_{k=0}^n a_k(\varphi_0')\varphi^k(\varsigma)}{\varsigma - z'}\,d\varsigma \\ &\quad - \frac{1}{2\pi i}\int_L \frac{\Phi^+(\varphi(\varsigma))}{\varsigma - z'}\,d\varsigma + \frac{1}{2\pi i}\int_L \frac{\Phi^-(\varphi(\varsigma))}{\varsigma - z'}\,d\varsigma. \end{aligned}$$

It is easy to verify that $\Phi^-(\varphi(\varsigma)) \in E^p(G^-)$ for $p \geqslant 1$ and $\Phi^-(\varphi(\infty)) = 0$. Then

$$\frac{1}{2\pi i}\int_L \frac{\Phi^-(\varphi(\varsigma))}{\varsigma - z'}\,d\varsigma = -\Phi^-(\varphi(z'))$$

and we get

$$\begin{aligned} S_n(\varphi_0', z') &= \sum_{k=0}^{n} a_k(\varphi_0')\varphi^k(z') \\ &\quad + \frac{1}{2\pi i}\int_L \frac{\left[\sum_{k=0}^n a_k(\varphi_0')\varphi^k(\varsigma) - \Phi^+(\varphi(\varsigma))\right]}{\varsigma - z'}\,d\varsigma - \Phi^-(\varphi(z')). \end{aligned}$$

Taking limit as $z' \to z$ along all nontangential paths outside of L,

$$\begin{aligned} S_n(\varphi_0', z) &= \tfrac{1}{2}\left[\sum_{k=0}^{n} a_k(\varphi_0')\varphi^k(z) - \Phi^+(\varphi(z))\right] \\ &\quad + [\Phi^+(\varphi(z)) - \Phi^-(\varphi(z))] + S_L\left(\sum_{k=0}^{n} a_k(\varphi_0')\varphi^k - \Phi^+ \circ \varphi\right)(z) \end{aligned}$$

holds a.e. on L. Further, taking relation (7) into account and applying the boundedness of the singular operator from $L^p(L), p > 1$, into itself and Hölder's inequality, respectively, from the last equality we obtain

$$
\begin{aligned}
\|\varphi_0' - S_n(\varphi_0', \cdot)\|_{L^p(L)} &\leqslant c_6 \left\| \Phi^+(\varphi(z)) - \sum_{k=0}^{n} a_k(\varphi_0')\varphi^k(z) \right\|_{L^p(L)} \\
&\leqslant c_6 \left\| \Phi^+(w) - \sum_{k=0}^{n} a_k(\varphi_0') w^k \right\|_{L^p(T,|\psi'|)} \\
&\leqslant c_7 \left\| \Phi^+(w) - \sum_{k=0}^{n} a_k(\Phi^+) w^k \right\|_{L^{pp_0}(T)},
\end{aligned}
$$

for every $p_0 > 1$. Now applying the appropriate result from L^p approximation (see for example [5, Theorem 2.3, formula (2.11), p. 205] due to Stechkin) we get

$$
\left\| \Phi^+(w) - \sum_{k=0}^{n} a_k(\Phi^+) w^k \right\|_{L^{pp_0}(T)} \leqslant c\omega_{pp_0}(\Phi^+, 1/n),
$$

where

$$
\omega_{pp_0}(\Phi^+, 1/n) = \sup_{|h| \leqslant 1/n} \|\Phi^+(we^{ih}) - \Phi^+(w)\|_{L^{pp_0}(T)},
$$

and find that

$$
\|\varphi_0' - S_n(\varphi_0', \cdot)\|_{L^p(L)} \leqslant c_8 \omega_{pp_0}(\Phi^+, 1/n). \tag{8}
$$

Since

$$
\Phi^+ = \tfrac{1}{2}\Phi + S_T(\Phi),
$$

a.e. on T, from the last two inequality we conclude that

$$
\begin{aligned}
\omega_{pp_0}(\Phi^+, 1/n) \leqslant{}& \tfrac{1}{2} \sup_{|h| \leqslant 1/n} \|\Phi(we^{ih}) - \Phi(w)\|_{L^{pp_0}(T)} \\
&+ \sup_{|h| \leqslant 1/n} \|S_T(\Phi)(we^{ih}) - S_T(\Phi)(w)\|_{L^{pp_0}(T)}.
\end{aligned} \tag{9}
$$

On the other hand, since

$$
S_T(\Phi)(w) := (P.V)\frac{1}{2\pi i}\int_T \frac{\Phi(\tau)}{\tau - w}\, d\tau, \quad |w| = 1,
$$

and therefore

$$
S_T(\Phi)(we^{ih}) := (P.V)\frac{1}{2\pi i}\int_T \frac{\Phi(\tau e^{ih})}{\tau - w}\, d\tau, \quad |w| = 1,
$$

we have

$$
S_T(\Phi)(we^{ih}) - S_T(\Phi)(w) = (P.V)\frac{1}{2\pi i}\int_T \frac{\Phi(\tau e^{ih}) - \Phi(\tau)}{\tau - w}\, d\tau, \quad |w| = 1.
$$

Now applying the boundedness of the singular operator from $L^p(T), p>1$, into itself we conclude that

$$\sup_{|h|\leqslant 1/n} ||S_T(\Phi)(we^{ih}) - S_T(\Phi)(w)||_{L^{pp_0}(T)} \leqslant c_9 \sup_{|h|\leqslant 1/n} ||\Phi(we^{ih}) - \Phi(w)||_{L^{pp_0}(T)}$$
$$= c_9 \omega_{pp_0}\left(\Phi, \frac{1}{n}\right). \tag{10}$$

Then from (8) to (10) we derive the inequality

$$||\varphi_0' - S_n(\varphi_0', \cdot)||_{L^p(L)} \leqslant c\omega_{pp_0}(\Phi, 1/n).$$

Choosing the number $p_0 > 1$ sufficiently close to 1 we finally from here have

$$||\varphi_0' - S_n(\varphi_0', \cdot)||_{L^p(L)} \leqslant c\omega_{p+\varepsilon}(\Phi, 1/n). \qquad \square$$

Lemma 1. *If $p>1$ and G is a smooth domain of bounded boundary rotation, then*

$$\omega_p(\Phi, 1/n) \leqslant \frac{c}{n^{\frac{1}{p}-\varepsilon}}$$

for every $\varepsilon > 0$.

Proof. In fact, by Hölder's inequality

$$\begin{aligned}
||\Phi(we^{ih}) - \Phi(w)||_{L^p(T)} &= \left(\int_T |\varphi_0'[\psi(we^{ih})] - \varphi_0'[\psi(w)]|^p |dw|\right)^{1/p} \\
&= \left(\int_T \left|\frac{1}{\psi_0'[\varphi_0(\psi(we^{ih}))]} - \frac{1}{\psi_0'[\varphi_0(\psi(w))]}\right|^p |dw|\right)^{1/p} \\
&= \left(\int_T \left|\frac{\psi_0'[\varphi_0(\psi(we^{ih}))] - \psi_0'[\varphi_0(\psi(w))]}{\psi_0'[\varphi_0(\psi(we^{ih}))]\psi_0'[\varphi_0(\psi(w))]}\right|^p |dw|\right)^{1/p} \\
&\leqslant \left(\int_T |\psi_0'[\varphi_0(\psi(we^{ih}))] - \psi_0'[\varphi_0(\psi(w))]|^{pp_0} |dw|\right)^{1/(pp_0)} \\
&\quad \times \left(\int_T \frac{|dw|}{|\psi_0'[\varphi_0(\psi(we^{ih}))]\psi_0'[\varphi_0(\psi(w))]|^{pq_0}}\right)^{1/(pq_0)} \\
&= A_1 B_1,
\end{aligned} \tag{11}$$

where $1/p_0 + 1/q_0 = 1$. Later if $1/p_1 + 1/q_1 = 1$, then applying again Hölder's inequality we get

$$B_1 := \left(\int_T \frac{1}{|\psi_0'[\varphi_0(\psi(we^{ih}))] \cdot \psi_0'[\varphi_0(\psi(w))]|^{pq_0}} |dw|\right)^{1/(pq_0)}$$

$$\leqslant \left(\int_T \frac{1}{|\psi_0'[\varphi_0(\psi(w))]|^{pq_0p_1}} |dw| \right)^{1/(pq_0p_1)}$$
$$\times \left(\int_T \frac{1}{|\psi_0'[\varphi_0(\psi(we^{ih}))]|^{pq_0q_1}} |dw| \right)^{1/(pq_0q_1)} =: B_{11}B_{12}$$

If $1/p_2 + 1/q_2 = 1$, then by Hölder's inequality

$$B_{11} := \left(\int_T \frac{1}{|\psi_0'[\varphi_0(\psi(w))]|^{pq_0p_1}} |dw| \right)^{1/(pq_0p_1)}$$
$$= \left(\int_L \frac{|\varphi'(z)|}{|\psi_0'[\varphi_0(z)]|^{pq_0p_1}} |dz| \right)^{1/(pq_0p_1)} \leqslant \left(\int_L |\varphi'(z)|^{p_2} |dz| \right)^{1/(pq_0p_1p_2)}$$
$$\times \left(\int_L \frac{1}{|\psi_0'[\varphi_0(z)]|^{pq_0p_1q_2}} |dz| \right)^{1/(pq_0p_1q_2)}$$
$$\leqslant c_{10} \left(\int_L |\varphi_0'(z)|^{pq_0p_1q_2} |dz| \right)^{1/(pq_0p_1q_2)} < \infty, \tag{12}$$

because

$$\varphi_0', \varphi' \in L^p(L)$$

for every $p > 1$ [20]. The finiteness of B_{12} may be proved similarly. Finally, from (11) and (12) we conclude that

$$||\Phi(we^{ih}) - \Phi(w)||_{L^p(T)} \leqslant c_{11}A_1.$$

Hence

$$\omega_p(\Phi, 1/n) = \sup_{|h| \leqslant 1/n} ||\Phi(we^{ih}) - \Phi(w)||_{L^p(T)}$$
$$\leqslant c_{11} \sup_{|h| \leqslant 1/n} \left(\int_T |\psi_0'[\varphi_0(\psi(we^{ih}))] - \psi_0'[\varphi_0(\psi(w))]|^{pp_0} |dw| \right)^{1/(pp_0)},$$

and by virtue of Theorem 2 we have

$$\omega_p(\Phi, 1/n) \leqslant c_{12} \sup_{|h| \leqslant 1/n} |\varphi_0(\psi(we^{ih})) - \varphi_0(\psi(w))|^{\frac{1}{pp_0} - \varepsilon}.$$

Since for a smooth boundary L, the mapping functions φ_0 and ψ belong to the Hölder class on L and on T, respectively, with exponent $1 - \varepsilon$, for every $\varepsilon > 0$, from the last inequality we derive

$$\omega_p(\Phi, 1/n) \leqslant \frac{c}{n^{\frac{1}{pp_0} - \varepsilon}}.$$

Choosing here the number $p_0 > 1$ sufficiently close to 1 we get

$$\omega_p(\Phi, 1/n) \leqslant \frac{c}{n^{\frac{1}{p} - \varepsilon}},$$

for every $\varepsilon > 0$. □

3. Proof of main result

For the mapping φ_0 and a weight function ω we set

$$\varepsilon_n(\varphi_0')_2 := \inf_{p_n} ||\varphi_0' - p_n||_{L_2(G)}, \quad E_n^{\circ}(\varphi_0')_2 := \inf_{p_n} ||\varphi_0' - p_n||_{L^2(L)},$$

$$E_n^{\circ}(\varphi_0', \omega)_2 := \inf_{p_n} ||\varphi_0' - p_n||_{L^2(L,\omega)},$$

where inf is taken over all polynomials p_n of degree at most n.

Developing the idea used in [14] we apply a traditional method based on the extremal property of Bieberbach polynomials and also the inequality connecting the values $\varepsilon_n(\varphi_0')_2$ and $E_n^{\circ}(\varphi_0', \omega)_2$ established in [7].

Proof of Theorem 1. Since G is a smooth domain the functions $|\varphi_0'|$ and $1/|\varphi'|$ belong to $L^p(L)$ for every $p>1$ by Warschawski and Schober [20, Theorem 3]. Hölder's inequality then gives $\varphi_0' \in L^2(L, 1/|\varphi'|)$. Hence by definition we have $\varphi_0' \in E^2(G, 1/|\varphi'|)$. On the other hand by Israfilov [14, Lemma 12], $1/|\varphi'| \in A_p(L)$ for every $p>1$. Result [7, Theorem 11, Remark (ii)] now implies that, for $\varphi_0', \omega := 1/|\varphi'|$ and $p=2$,

$$\varepsilon_n(\varphi_0')_2 \leqslant c_{13} n^{-\frac{1}{2}} E_n^{\circ}\left(\varphi_0', \frac{1}{|\varphi'|}\right)_2. \tag{13}$$

For the polynomials $q_n(z)$, best approximating φ_0' in the norm $||\cdot||_{L_2(G)}$, we set

$$Q_n(z) := \int_{z_0}^{z} q_n(t)\,dt, \quad t_n(z) := Q_n(z) + [1 - q_n(z_0)](z - z_0).$$

Then $t_n(z_0) = 0, t_n'(z_0) = 1$ and from (13) we obtain

$$\begin{aligned} &||\varphi_0' - t_n'||_{L_2(G)} \\ &\quad = ||\varphi_0' - q_n - 1 + q_n(z_0)||_{L_2(G)} \leqslant \varepsilon_n(\varphi_0')_2 + ||1 - q_n(z_0)||_{L_2(G)} \\ &\quad \leqslant c_{13} n^{-\frac{1}{2}} E_n^{\circ}\left(\varphi_0', \frac{1}{|\varphi'|}\right)_2 + ||\varphi_0'(z_0) - q_n(z_0)||_{L_2(G)}. \end{aligned} \tag{14}$$

On the other hand, by the inequality

$$|f(z_0)| \leqslant \frac{||f||_{L_2(G)}}{dist(z_0, L)},$$

which holds for every analytic function f with $||f||_{L_2(G)} < \infty$, from (14) and (13), we get

$$||\varphi_0' - t_n'||_{L_2(G)} \leqslant c_{13} n^{-\frac{1}{2}} E_n^{\circ}\left(\varphi_0', \frac{1}{|\varphi'|}\right)_2 + \frac{\varepsilon_n(\varphi_0')_2}{dist(z_0, L)} \leqslant c_{14} n^{-\frac{1}{2}} E_n^{\circ}\left(\varphi_0', \frac{1}{|\varphi'|}\right)_2.$$

According to the extremal property of the polynomials π_n we have

$$||\varphi_0' - \pi_n'||_{L_2(G)} \leqslant c_{15} n^{-\frac{1}{2}} E_n^{\circ}\left(\varphi_0', \frac{1}{|\varphi'|}\right)_2. \tag{15}$$

Further applying Andrievskii's [2] polynomial lemma (see also [8], for a simpler proof and more general result),

$$||p_n||_{\bar{G}} \leqslant c(\ln n)^{\frac{1}{2}} ||p_n'||_{L_2(G)},$$

which holds for every polynomial p_n of degree $\leqslant n$ with $p_n(z_0) = 0$, and using the familiar method of Simonenko [18] and Andrievskii [2] (described in detail in [9]), from (15) we get

$$||\varphi_0 - \pi_n||_{\bar{G}} \leqslant c_{16} \left(\frac{\ln n}{n}\right)^{\frac{1}{2}} E_n^{\circ} \left(\varphi_0', \frac{1}{|\varphi'|}\right)_2,$$

and later by Hölder's inequality

$$\begin{aligned} ||\varphi_0 - \pi_n||_{\bar{G}} &\leqslant c_{16} \left(\frac{\ln n}{n}\right)^{\frac{1}{2}} \inf_{p_n} ||\varphi_0' - p_n||_{L^2(L,1/|\varphi'|)} \\ &\leqslant c_{16} \left(\frac{\ln n}{n}\right)^{\frac{1}{2}} ||\varphi_0' - S_n||_{L^2(L,1/|\varphi'|)} \\ &\leqslant c_{16} \left(\frac{\ln n}{n}\right)^{\frac{1}{2}} ||\varphi_0' - S_n||_{L^{2p_0}(L)} ||1/\varphi'||_{L^{q_0}(L)}^{1/2} \\ &\leqslant c_{17} \left(\frac{\ln n}{n}\right)^{\frac{1}{2}} ||\varphi_0' - S_n||_{L^{2p_0}(L)}, \end{aligned}$$

where $1/p_0 + 1/q_0 = 1$.

Then by virtue of Theorem 3 (in the case of $p \coloneqq 2p_0$) we have

$$||\varphi_0 - \pi_n||_{\bar{G}} \leqslant c_{17} \left(\frac{\ln n}{n}\right)^{\frac{1}{2}} \omega_{2p_0+\varepsilon}(\Phi, 1/n), \quad n \geqslant 2$$

for every $p_0 > 1$ and $\varepsilon > 0$. Now applying Lemma 1 (in the case of $p \coloneqq 2p_0$) and choosing the number p_0 sufficiently close to 1 we get

$$||\varphi_0 - \pi_n||_{\bar{G}} \leqslant c \left(\frac{\ln n}{n}\right)^{\frac{1}{2}} \frac{1}{n^{\frac{1}{2}-\varepsilon}} \leqslant \frac{c}{n^{1-\varepsilon}}. \qquad \square$$

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Further reading

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