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ON THE SEQUENCES RELATED TO FIBONACCI AND LUCAS NUMBERS

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ABSTRACT. In this paper, we obtain some properties of the sequences $\mathcal{U}_n^q$ and $\mathcal{V}_n^q$ introduced in [6]. We find polynomial representations and formulas of them. For $q = 5$, $\mathcal{U}_n^5$ is the Fibonacci sequence F_n and $\mathcal{V}_n^5$ is the Lucas sequence L_n.

1. Introduction

Hecke groups $H(\lambda)$ are the generalizations of the well-known modular group

$$PSL(2,\mathbb{Z}) = \left\{ \frac{az+b}{cz+d} : a,\ b,\ c,\ d \in \mathbb{Z},\ ad-bc=1 \right\}.$$

They are discrete subgroups of $PSL(2,\mathbb{R})$ (the group of orientation preserving isometries of the upper half plane U) generated by two linear fractional transformations

$$R(z) = -\frac{1}{z} \text{ and } T(z) = z+\lambda,$$

where λ is a fixed positive real number, [3]. In studying the principal congruence subgroups of Hecke groups $H(\sqrt{q})$, $q \geq 5$ a prime number, we need the powers of the transformation $S = RT$, [7]. In [6], for each q, two new sequences denoted by $\mathcal{U}_n$ and $\mathcal{V}_n$, were introduced. It has been shown that

$$S^{2n} = \begin{pmatrix} -\mathcal{V}_{2n-1} & -\mathcal{U}_{2n}\sqrt{q} \\ \mathcal{U}_{2n}\sqrt{q} & \mathcal{V}_{2n+1} \end{pmatrix},$$

$$S^{2n+1} = \begin{pmatrix} -\mathcal{U}_{2n}\sqrt{q} & -\mathcal{V}_{2n+1} \\ \mathcal{V}_{2n+1} & \mathcal{U}_{2n+2}\sqrt{q} \end{pmatrix}.$$

Therefore, these sequences play a very important role in this problem.

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The recurrence relation for $\mathcal{U}_n$ is

$$(1) \qquad \mathcal{U}_n = (q-2)\mathcal{U}_{n-2} - \mathcal{U}_{n-4}, \ n \geq 4$$

with the initial conditions: $\mathcal{U}_0 = 0, \mathcal{U}_1 = 1, \mathcal{U}_2 = 1$ and $\mathcal{U}_3 = q-3$. Also the recurrence relation for $\mathcal{V}_n$ is

$$(2) \qquad \mathcal{V}_n = (q-2)\mathcal{V}_{n-2} - \mathcal{V}_{n-4}, \ n \geq 4$$

with the initial conditions: $\mathcal{V}_0 = 2, \mathcal{V}_1 = 1, \mathcal{V}_2 = q-2$ and $\mathcal{V}_3 = q-1$. For $q = 5$, $\mathcal{U}_n$ is the Fibonacci sequence F_n and $\mathcal{V}_n$ is the Lucas sequence L_n. In a sense, $\mathcal{U}_n$ and $\mathcal{V}_n$ are the generalizations of Fibonacci and Lucas sequences. From now on, we denote these sequences by $\mathcal{U}_n^q$ and $\mathcal{V}_n^q$, respectively.

Our problem is to determine for which values of n, the congruence

$$S^n \equiv \pm I(\text{modp}), \text{ p is an odd prime,}$$

holds. For $q = 5$, the well known properties of Fibonacci and Lucas sequences are enough to solve the congruence in $H(\sqrt{5})$. Therefore, for all values $q > 5$ in $H(\sqrt{q})$, we need to obtain such properties and determine formulas of $\mathcal{U}_n^q$ and $\mathcal{V}_n^q$. In this paper, our aim is to give some properties of the new sequences (see Section 2). We obtain polynomial representations and formulas for them (see Section 3). We see that the sequences $\mathcal{U}_n^q$ and $\mathcal{V}_n^q$ contain a wealth of subtle and fascinating properties as Fibonacci and Lucas numbers.

Note that, the sequences $\mathcal{U}_n^q$ and $\mathcal{V}_n^q$ are not generalized Fibonacci sequences except for $q = 5$. For, in [6], it has been shown that

$$(3) \qquad \mathcal{U}_{2n}^q = \mathcal{U}_{2n-1}^q + \mathcal{U}_{2n-2}^q$$

but

$$(4) \qquad \mathcal{U}_{2n+1}^q = (q-4)\mathcal{U}_{2n}^q + \mathcal{U}_{2n-1}^q.$$

Also,

$$(5) \qquad \mathcal{V}_{2n+1}^q = \mathcal{V}_{2n}^q + \mathcal{V}_{2n-1}^q$$

and

$$(6) \qquad \mathcal{V}_{2n}^q = (q-4)\mathcal{V}_{2n-1}^q + \mathcal{V}_{2n-2}^q.$$

In [6], it was also shown that

$$(7) \qquad \mathcal{V}_n^q = \mathcal{U}_{n+1}^q + \mathcal{U}_{n-1}^q$$

and

$$(8) \qquad \mathcal{U}_{2n}^q = \mathcal{U}_n^q \mathcal{V}_n^q$$

which are the same well-known properties of Fibonacci and Lucas numbers (for more details about the Fibonacci and Lucas numbers one can consult [1], [2], [4] and [5]). Throughout this paper, we frequently use the identities given in (3) - (8).

2. Basic properties

In this section, we start a theorem. Afterwards, we develop many identities.

THEOREM 2.1. *Successive terms of* $\mathcal{U}_n^q$ *and* $\mathcal{V}_n^q$ *(except* $\mathcal{V}_0^q = 2$*) are relatively prime.*

Proof. Assume that $\mathcal{U}_n^q$ and $\mathcal{U}_{n+1}^q$ are both divisible by a positive integer d. Without loss of generality, we can suppose that n is odd. From (3) we have $\mathcal{U}_{n+1}^q = \mathcal{U}_n^q + \mathcal{U}_{n-1}^q$. Then the difference $\mathcal{U}_{n+1}^q - \mathcal{U}_n^q = \mathcal{U}_{n-1}^q$ will also be divisible by d. As n is odd, from (4) we have

$$\mathcal{U}_n^q = (q-4)\mathcal{U}_{n-1}^q + \mathcal{U}_{n-2}^q$$

and so $\mathcal{U}_{n-2}^q = \mathcal{U}_n^q - (q-4)\mathcal{U}_{n-1}^q$ will also be divisible by d. Continuing, we see that $d \mid \mathcal{U}_{n-3}^q$, $d \mid \mathcal{U}_{n-4}^q$, and so on. Finally we must have $d \mid \mathcal{U}_2^q$. Since $\mathcal{U}_2^q = 1$, it is clear that $d = 1$. Since the only positive integer which divides successive terms of $\mathcal{U}_n^q$ is 1, the proof is completed.

The same result can be proved similarly for $\mathcal{V}_n^q$ excluding $\mathcal{V}_0^q = 2$. □

It is possible to extend $\mathcal{U}_n^q$ and $\mathcal{V}_n^q$ backward with negative subscripts. Recurrences (1) and (2) allow us to do this. For example, from the equation $\mathcal{U}_2^q = (q-2)\mathcal{U}_0^q - \mathcal{U}_{-2}^q = 1$, we have $\mathcal{U}_{-2}^q = -1$. Similarly we get $\mathcal{U}_{-1}^q = 1$, $\mathcal{U}_{-3}^q = q-3$, and so on. Therefore, we can deduce that

$$\mathcal{U}_{-n}^q = (-1)^{n+1}\mathcal{U}_n^q \tag{9}$$

and similarly

$$\mathcal{V}_{-n}^q = (-1)^n\mathcal{V}_n^q. \tag{10}$$

Some properties of Fibonacci and Lucas numbers are naturally generalized to $\mathcal{U}_n^q$ and $\mathcal{V}_n^q$. We have

$$\mathcal{V}_{n-1}^q + \mathcal{V}_{n+1}^q = q\mathcal{U}_n^q, \tag{11}$$

$$\mathcal{U}_n^q + \mathcal{U}_{n+4}^q = (q-2)\mathcal{U}_{n+2}^q \tag{12}$$

and

$$\mathcal{V}_n^q + \mathcal{V}_{n+4}^q = (q-2)\mathcal{V}_{n+2}^q. \tag{13}$$

For the proof of (11), we use induction. For $n = 1$, we have $\mathcal{V}_0^q + \mathcal{V}_2^q = 2 + q - 2 = q = q\mathcal{U}_1^q$ and for $n = 2$, $\mathcal{V}_1^q + \mathcal{V}_3^q = 1 + q - 1 = q = q\mathcal{U}_2^q$ Suppose that $\mathcal{V}_{n-1}^q + \mathcal{V}_{n+1}^q = q\mathcal{U}_n^q$. We will show that $\mathcal{V}_n^q + \mathcal{V}_{n+2}^q = q\mathcal{U}_{n+1}^q$ From (2) and (1), we have

$$\begin{aligned}
\mathcal{V}_n^q + \mathcal{V}_{n+2}^q &= (q-2)\mathcal{V}_{n-2}^q - \mathcal{V}_{n-4}^q + (q-2)\mathcal{V}_n^q - \mathcal{V}_{n-2}^q \\
&= (q-2)(\mathcal{V}_{n-2}^q + \mathcal{V}_n^q) - (\mathcal{V}_{n-2}^q + \mathcal{V}_{n-4}^q) \\
&= (q-2)q\mathcal{U}_{n-1}^q - q\mathcal{U}_{n-3}^q = q\left((q-2)\mathcal{U}_{n-1}^q - \mathcal{U}_{n-3}^q\right) \\
&= q\mathcal{U}_{n+1}^q.
\end{aligned}$$

The proofs of (12) and (13) are obtained immediately from the recurrence relations (1) and (2).

For $q = 5$, we get the well-known formulas $L_{n-1} + L_{n+1} = 5F_n$ $F_n + F_{n+4} = 3F_{n+2}$ and $L_n + L_{n+4} = 3L_{n+2}$, [5].

An interesting property of $\mathcal{U}_n^q$ and $\mathcal{V}_n^q$ is the following:

$$\mathcal{U}_{n-3}^q + \mathcal{U}_{n+3}^q = (q-3)\mathcal{V}_n^q. \tag{14}$$

For the proof, from (7), we have $\mathcal{U}_{n+1}^q + \mathcal{U}_{n-1}^q = \mathcal{V}_n^q$ and so

$$(q-2)\mathcal{U}_{n+1}^q + (q-2)\mathcal{U}_{n-1}^q = (q-2)\mathcal{V}_n^q. \tag{15}$$

From (1), we can write $\mathcal{U}_{n+3}^q = (q-2)\mathcal{U}_{n+1}^q - \mathcal{U}_{n-1}^q$ and $\mathcal{U}_{n+1}^q = (q-2)\mathcal{U}_{n-1}^q - \mathcal{U}_{n-3}^q$. Hence, we have $(q-2)\mathcal{U}_{n+1}^q = \mathcal{U}_{n+3}^q + \mathcal{U}_{n-1}^q$ and $(q-2)\mathcal{U}_{n-1}^q = \mathcal{U}_{n+1}^q + \mathcal{U}_{n-3}^q$. Putting them in (15), we get

$$\mathcal{U}_{n+3}^q + \mathcal{U}_{n-1}^q + \mathcal{U}_{n+1}^q + \mathcal{U}_{n-3}^q = (q-2)\mathcal{V}_n^q.$$

Using (7), we obtain

$$\mathcal{U}_{n+3}^q + \mathcal{U}_{n-3}^q + \mathcal{V}_n^q = (q-2)\mathcal{V}_n^q$$

and so

$$\mathcal{U}_{n+3}^q + \mathcal{U}_{n-3}^q = (q-3)\mathcal{V}_n^q.$$

For $q = 5$, we get

$$F_{n+3} + F_{n-3} = 2F_n. \tag{16}$$

The following results express properties known for F_n and L_n.

$$\mathcal{U}_{2n+1}^q = \mathcal{U}_{n+1}^q\mathcal{V}_{n+1}^q - \mathcal{U}_n^q\mathcal{V}_n^q \tag{17}$$

and

$$\mathcal{V}_{2n+1}^q = \mathcal{U}_{n+1}^q\mathcal{V}_{n+1}^q + \mathcal{U}_n^q\mathcal{V}_n^q. \tag{18}$$

For the proof of (17), by (8) and (3). it is easily deduced that

$$\begin{aligned}\mathcal{U}^q_{n+1}\mathcal{V}^q_{n+1}-\mathcal{U}^q_n\mathcal{V}^q_n &= \mathcal{U}^q_{2n+2}-\mathcal{U}^q_{2n}\\ &= \mathcal{U}^q_{2n+1}+\mathcal{U}^q_{2n}-\mathcal{U}^q_{2n}=\mathcal{U}^q_{2n+1}.\end{aligned}$$

Now from (17) and (8), we get $\mathcal{U}^q_{2n+1}+2\mathcal{U}^q_n\mathcal{V}^q_n=\mathcal{U}^q_{n+1}\mathcal{V}^q_{n+1}+\mathcal{U}^q_n\mathcal{V}^q_n$ and so

$$\mathcal{U}^q_{2n+1}+2\mathcal{U}^q_{2n}=\mathcal{U}^q_{n+1}\mathcal{V}^q_{n+1}+\mathcal{U}^q_n\mathcal{V}^q_n.$$

Using (4), we have $(q-4)\mathcal{U}^q_{2n}+\mathcal{U}^q_{2n-1}+2\mathcal{U}^q_{2n}=\mathcal{U}^q_{n+1}\mathcal{V}^q_{n+1}+\mathcal{U}^q_n\mathcal{V}^q_n$. So we can write

$$(q-2)\mathcal{U}^q_{2n}+\mathcal{U}^q_{2n-1}-\mathcal{U}^q_{2n-2}+\mathcal{U}^q_{2n-2}=\mathcal{U}^q_{n+1}\mathcal{V}^q_{n+1}+\mathcal{U}^q_n\mathcal{V}^q_n.$$

Finally applying (1) and (4), we deduce that

$$\mathcal{U}^q_{2n+2}+\mathcal{U}^q_{2n}=\mathcal{U}^q_{n+1}\mathcal{V}^q_{n+1}+\mathcal{U}^q_n\mathcal{V}^q_n$$

and hence using (7) we get

$$\mathcal{V}^q_{2n+1}=\mathcal{U}^q_{n+1}\mathcal{V}^q_{n+1}+\mathcal{U}^q_n\mathcal{V}^q_n.$$

Another interesting properties of $\mathcal{U}^q_n$ and $\mathcal{V}^q_n$ are the following:

(19) $$\mathcal{U}^q_n\mathcal{U}^q_{n+3}-\mathcal{U}^q_{n+1}\mathcal{U}^q_{n+2}=(-1)^{n+1}$$

and

(20) $$\mathcal{V}^q_n\mathcal{V}^q_{n+3}-\mathcal{V}^q_{n+1}\mathcal{V}^q_{n+2}=(-1)^n q.$$

For the proof of (19), without loss of generality, we can assume that n is odd. Then using (3) we have

$$\begin{aligned}\mathcal{U}^q_n\mathcal{U}^q_{n+3}-\mathcal{U}^q_{n+1}\mathcal{U}^q_{n+2} &= \mathcal{U}^q_n(\mathcal{U}^q_{n+2}+\mathcal{U}^q_{n+1})-\mathcal{U}^q_{n+2}(\mathcal{U}^q_n+\mathcal{U}^q_{n-1})\\ &= -(\mathcal{U}^q_{n-1}\mathcal{U}^q_{n+2}-\mathcal{U}^q_n\mathcal{U}^q_{n+1}).\end{aligned}$$

Using (4), we can repeat the above process on the last line to attain

$$\begin{aligned}-(\mathcal{U}^q_{n-1}\mathcal{U}^q_{n+2}-\mathcal{U}^q_n\mathcal{U}^q_{n+1}) &= -\left[\mathcal{U}^q_{n-1}\left((q-4)\mathcal{U}^q_{n+1}+\mathcal{U}^q_n\right)\right.\\ &\qquad \left.-\mathcal{U}^q_{n+1}\left((q-4)\mathcal{U}^q_{n-1}+\mathcal{U}^q_{n-2}\right)\right]\\ &= (-1)^2(\mathcal{U}^q_{n-2}\mathcal{U}^q_{n+1}-\mathcal{U}^q_{n-1}\mathcal{U}^q_n)\\ &= \cdots\\ &= (-1)^{n+1}(\mathcal{U}^q_{-1}\mathcal{U}^q_2-\mathcal{U}^q_0\mathcal{U}^q_1)=(-1)^{n+1}.\end{aligned}$$

For the proof of (20), assume that n is even. Similar to the proof of (19), using (5) and (6), we get

$$\begin{aligned}
\mathcal{V}_n^q\mathcal{V}_{n+3}^q - \mathcal{V}_{n+1}^q\mathcal{V}_{n+2}^q &= \mathcal{V}_n^q(\mathcal{V}_{n+2}^q + \mathcal{V}_{n+1}^q) - \mathcal{V}_{n+2}^q(\mathcal{V}_n^q + \mathcal{V}_{n-1}^q) \\
&= -(\mathcal{V}_{n-1}^q\mathcal{V}_{n+2}^q - \mathcal{V}_n^q\mathcal{V}_{n+1}^q) \\
&= -\left[\mathcal{V}_{n-1}^q\left((q-4)\mathcal{V}_{n+1}^q + \mathcal{V}_n^q\right)\right. \\
&\quad \left.-\mathcal{V}_{n+1}^q\left((q-4)\mathcal{V}_{n-1}^q + \mathcal{V}_{n-2}^q\right)\right] \\
&= -\left[-(\mathcal{V}_{n-2}^q\mathcal{V}_{n+1}^q - \mathcal{V}_{n-1}^q\mathcal{V}_n^q)\right] \\
&= (-1)^2(\mathcal{V}_{n-2}^q\mathcal{V}_{n+1}^q - \mathcal{V}_{n-1}^q\mathcal{V}_n^q) \\
&= \cdots = (-1)^n(\mathcal{V}_0^q\mathcal{V}_3^q - \mathcal{V}_1^q\mathcal{V}_2^q) \\
&= (-1)^n\left(2(q-1) - (q-2)\right) = (-1)^n q.
\end{aligned}$$

We have the following property for $\mathcal{U}_n^q$:

(21) $$\mathcal{U}_{2m+n}^q = \mathcal{U}_{2m+2}^q\mathcal{U}_n^q - \mathcal{U}_{2m}^q\mathcal{U}_{n-2}^q.$$

For $n = 0$, we have $\mathcal{U}_{2m}^q = \mathcal{U}_{2m}^q$, since $\mathcal{U}_0^q = 0$ and $\mathcal{U}_{-2}^q = -1$. For $n = 1$, we get $\mathcal{U}_{2m+1}^q = \mathcal{U}_{2m+2}^q - \mathcal{U}_{2m}^q$ which is deduced from (3) since $\mathcal{U}_1^q = 1$ and $\mathcal{U}_{-1}^q = 1$.

Now let us assume the identity is true for $n = 1, 2, 3, \ldots, k$ and we will show that it holds for $n = k+1$. By assumption

$$\mathcal{U}_{2m+k}^q = \mathcal{U}_{2m+2}^q\mathcal{U}_k^q - \mathcal{U}_{2m}^q\mathcal{U}_{k-2}^q$$

and

$$\mathcal{U}_{2m+k-1}^q = \mathcal{U}_{2m+2}^q\mathcal{U}_{k-1}^q - \mathcal{U}_{2m}^q\mathcal{U}_{k-3}^q.$$

By (3), if k is odd, the summation of the last two equation gives us

$$\mathcal{U}_{2m+k+1}^q = \mathcal{U}_{2m+2}^q(\mathcal{U}_k^q + \mathcal{U}_{k-1}^q) - \mathcal{U}_{2m}^q(\mathcal{U}_{k-2}^q + \mathcal{U}_{k-3}^q)$$

which implies

$$\mathcal{U}_{2m+k+1}^q = \mathcal{U}_{2m+2}^q\mathcal{U}_{k+1}^q - \mathcal{U}_{2m}^q\mathcal{U}_{k-1}^q.$$

If k is even, multiplying $\mathcal{U}_{2m+k}^q$ by $(q-4)$ and then summing with $\mathcal{U}_{2m+k-1}^q$, we obtain

$$\begin{aligned}
(q-4)\mathcal{U}_{2m+k}^q + \mathcal{U}_{2m+k-1}^q &= (q-4)\mathcal{U}_{2m+2}^q\mathcal{U}_k^q - (q-4)\mathcal{U}_{2m}^q\mathcal{U}_{k-2}^q \\
&\quad +\mathcal{U}_{2m+2}^q\mathcal{U}_{k-1}^q - \mathcal{U}_{2m}^q\mathcal{U}_{k-3}^q \\
&= \mathcal{U}_{2m+2}^q((q-4)\mathcal{U}_k^q + \mathcal{U}_{k-1}^q) \\
&\quad -\mathcal{U}_{2m}^q((q-4)\mathcal{U}_{k-2}^q + \mathcal{U}_{k-3}^q)
\end{aligned}$$

and by (4) we find

$$\mathcal{U}_{2m+k+1}^q = \mathcal{U}_{2m+2}^q\mathcal{U}_{k+1}^q - \mathcal{U}_{2m}^q\mathcal{U}_{k-1}^q.$$

Using (21), we get the following result which is well-known for F_n and L_n.

$$\mathcal{U}^q_{2n}((\mathcal{V}^q_{2n})^2-1)=\mathcal{U}^q_{6n}. \tag{22}$$

From (21), we have

$$\mathcal{U}^q_{6n}=\mathcal{U}^q_{4n+2n}=\mathcal{U}^q_{4n+2}\mathcal{U}^q_{2n}-\mathcal{U}^q_{4n}\mathcal{U}^q_{2n-2}.$$

By (8), we get

$$\mathcal{U}^q_{6n}=\mathcal{U}^q_{2n}(\mathcal{U}^q_{2n+1}\mathcal{V}^q_{2n+1}-\mathcal{V}^q_{2n}\mathcal{U}^q_{2n-2})$$

and by (7), we find

$$\mathcal{U}^q_{6n}=\mathcal{U}^q_{2n}\left[\mathcal{U}^q_{2n+1}(\mathcal{U}^q_{2n+2}+\mathcal{U}^q_{2n})-(\mathcal{U}^q_{2n+1}+\mathcal{U}^q_{2n-1})\mathcal{U}^q_{2n-2}\right].$$

Using (7) and (3), we obtain

$$\begin{aligned}\mathcal{U}^q_{6n} &= \mathcal{U}^q_{2n}\left[\mathcal{U}^q_{2n+1}(\mathcal{U}^q_{2n+1}+2\mathcal{U}^q_{2n})\right.\\ &\quad \left.-(\mathcal{U}^q_{2n+1}+\mathcal{U}^q_{2n-1})(\mathcal{U}^q_{2n}-\mathcal{U}^q_{2n-1})\right]\\ &= \mathcal{U}^q_{2n}\left[(\mathcal{U}^q_{2n+1})^2+2\mathcal{U}^q_{2n}\mathcal{U}^q_{2n+1}-\mathcal{U}^q_{2n+1}\mathcal{U}^q_{2n}\right.\\ &\quad \left.+\mathcal{U}^q_{2n+1}\mathcal{U}^q_{2n-1}-\mathcal{U}^q_{2n-1}\mathcal{U}^q_{2n}+(\mathcal{U}^q_{2n-1})^2\right]\\ &= \mathcal{U}^q_{2n}\left[(\mathcal{U}^q_{2n+1})^2+(\mathcal{U}^q_{2n-1})^2+\mathcal{U}^q_{2n+1}\mathcal{U}^q_{2n-1}\right.\\ &\quad \left.+\mathcal{U}^q_{2n}\mathcal{U}^q_{2n+1}-\mathcal{U}^q_{2n-1}\mathcal{U}^q_{2n}\right]\\ &= \mathcal{U}^q_{2n}\left[(\mathcal{U}^q_{2n+1})^2+(\mathcal{U}^q_{2n-1})^2+\mathcal{U}^q_{2n+1}\mathcal{U}^q_{2n-1}\right.\\ &\quad \left.+(\mathcal{U}^q_{2n-1}+\mathcal{U}^q_{2n-2})\mathcal{U}^q_{2n+1}-\mathcal{U}^q_{2n-1}\mathcal{U}^q_{2n}\right]\\ &= \mathcal{U}^q_{2n}\left[(\mathcal{U}^q_{2n+1}+\mathcal{U}^q_{2n-1})^2+\mathcal{U}^q_{2n-2}\mathcal{U}^q_{2n+1}-\mathcal{U}^q_{2n-1}\mathcal{U}^q_{2n}\right].\end{aligned}$$

Finally by (7) and (19), we get

$$\mathcal{U}^q_{6n}=\mathcal{U}^q_{2n}\left[(\mathcal{V}^q_{2n})^2-1\right].$$

THEOREM 2.2. *$\mathcal{U}^q_{2m}\mid\mathcal{U}^q_{2mn}$ for all integers m,n.*

Proof. Let m be fixed and we will induct on n. If either m or n equals zero, then the theorem is true by easy inspection. For $n=1$, it is clear that $\mathcal{U}^q_{2m}\mid\mathcal{U}^q_{2m}$.

Let us assume that the theorem holds for $n=1,2,\ldots,k$. Using (21), we see that $\mathcal{U}^q_{2m(k+1)}=\mathcal{U}^q_{2mk+2m}=\mathcal{U}^q_{2mk+2}\mathcal{U}^q_{2m}-\mathcal{U}^q_{2mk}\mathcal{U}^q_{2m-2}$. By assumption $\mathcal{U}^q_{2m}\mid\mathcal{U}^q_{2mk}$ and so $\mathcal{U}^q_{2m}$ divides the entire right side of the equation. Hence $\mathcal{U}^q_{2m}$ divides $\mathcal{U}^q_{2m(k+1)}$ and the theorem is proved for $n\geq 1$. Since $\mathcal{U}^q_{2mn}$ differs from $\mathcal{U}^q_{-2mn}$ by at most a factor of -1, then $\mathcal{U}^q_{2m}\mid\mathcal{U}^q_{2mn}$ for $n\leq -1$ as well. □

3. Polynomial representations and formulas

In this section, we find polynomial representations of $\mathcal{U}_n^q$ and $\mathcal{V}_n^q$. In particular, for $q = 5$, we get the polynomial representations of Fibonacci and Lucas numbers. Next, we obtain formulas for $\mathcal{U}_n^q$ and $\mathcal{V}_n^q$.

Let us write out the first 11 terms of $\mathcal{U}_n^q$ and $\mathcal{V}_n^q$.

$\underline{\mathcal{U}_n^q}$	$\underline{\mathcal{V}_n^q}$
$\mathcal{U}_0^q = 0$	$\mathcal{V}_0^q = 2$
$\mathcal{U}_1^q = 1$	$\mathcal{V}_1^q = 1$
$\mathcal{U}_2^q = 1$	$\mathcal{V}_2^q = q - 2$
$\mathcal{U}_3^q = q - 3$	$\mathcal{V}_3^q = q - 1$
$\mathcal{U}_4^q = q - 2$	$\mathcal{V}_4^q = q^2 - 4q + 2$
$\mathcal{U}_5^q = q^2 - 5q + 5$	$\mathcal{V}_5^q = q^2 - 3q + 1$
$\mathcal{U}_6^q = q^2 - 4q + 3$	$\mathcal{V}_6^q = q^3 - 6q^2 + 9q - 2$
$\mathcal{U}_7^q = q^3 - 7q^2 + 14q - 7$	$\mathcal{V}_7^q = q^3 - 5q^2 + 6q - 1$
$\mathcal{U}_8^q = q^3 - 6q^2 + 10q - 4$	$\mathcal{V}_8^q = q^4 - 8q^3 + 20q^2 - 16q + 2$
$\mathcal{U}_9^q = q^4 - 9q^3 + 27q^2 - 30q + 9$	$\mathcal{V}_9^q = q^4 - 7q^3 + 15q^2 - 10q + 1$
$\mathcal{U}_{10}^q = q^4 - 8q^3 + 21q^2 - 20q + 5$	$\mathcal{V}_{10}^q = q^5 - 10q^4 + 35q^3 - 50q^2 + 25q - 2$

Before we find the polynomial representations of $\mathcal{U}_n^q$ and $\mathcal{V}_n^q$, note that it is easy to see the following identities by straightforward computations:

$$\binom{n}{p} + 2\binom{n+1}{p-1} - \binom{n}{p-2} = \binom{n+2}{p} \tag{23}$$

and

$$\binom{n}{p} + \binom{n-1}{p-1} = \binom{n-1}{p-1}\frac{p+n}{p}. \tag{24}$$

THEOREM 3.1. *(i) The polynomial representations of* $\mathcal{U}_{2n}^q$ *and* $\mathcal{U}_{2n+1}^q$ *are*

$$\begin{aligned} \mathcal{U}_{2n}^q = q^{n-1} - \binom{2n-2}{1} q^{n-2} + \binom{2n-3}{2} q^{n-3} \\ - \cdots + (-1)^{n-2}\binom{n+1}{n-2} q + (-1)^{n-1} n \end{aligned} \tag{25}$$

and

$$\begin{aligned} \mathcal{U}_{2n+1}^q = q^n - (2n+1)q^{n-1} + \binom{2n-2}{1}\tfrac{2n+1}{2} q^{n-2} \\ - \binom{2n-3}{2}\tfrac{2n+1}{3} q^{n-3} \\ + \cdots + (-1)^{n-1}\binom{n+1}{n-2}\tfrac{2n+1}{n-1} q + (-1)^n (2n+1). \end{aligned} \tag{26}$$

(ii) The polynomial representations of $\mathcal{V}^q_{2n+1}$ *and* $\mathcal{V}^q_{2n}$ *are*

$$\begin{aligned}\mathcal{V}^q_{2n+1} &= q^n - \binom{2n-1}{1} q^{n-1} + \binom{2n-2}{2} q^{n-2} \\ &\quad - \cdots + (-1)^{n-1} \binom{n+1}{n-1} q + (-1)^n\end{aligned} \tag{27}$$

and

$$\begin{aligned}\mathcal{V}^q_{2n} &= q^n - 2nq^{n-1} + \binom{2n-3}{1} \tfrac{2n}{2} q^{n-2} \\ &\quad - \binom{2n-4}{2} \tfrac{2n}{3} q^{n-3} + \cdots \\ &\quad + (-1)^{n-1} \binom{n}{n-2} \tfrac{2n}{n-1} q + 2(-1)^n.\end{aligned} \tag{28}$$

Proof. (i) For $n = 1$, we have $\mathcal{U}^q_2 = 1$ and for $n = 2$ we have $\mathcal{U}^q_4 = q-2$. Let us assume that the conclusion is true for $n = 1, 2, \ldots, k$ and show that it holds for $n = k+1$. By assumption,

$$\begin{aligned}\mathcal{U}^q_{2k} &= q^{k-1} - \binom{2k-2}{1} q^{k-2} + \binom{2k-3}{2} q^{k-3} \\ &\quad - \cdots + (-1)^{k-2} \binom{k+1}{k-2} q + (-1)^{k-1} k\end{aligned}$$

and

$$\begin{aligned}\mathcal{U}^q_{2k-2} &= q^{k-2} - \binom{2k-4}{1} q^{k-3} + \binom{2k-5}{2} q^{k-4} \\ &\quad - \cdots + (-1)^{k-3} \binom{k}{k-3} q + (-1)^{k-2}(k-1).\end{aligned}$$

By the definition, we get

$$\begin{aligned}&\mathcal{U}^q_{2(k+1)} \\ &= (q-2)\mathcal{U}^q_{2k} - \mathcal{U}^q_{2k-2} \\ &= (q-2)\Big[q^{k-1} - \binom{2k-2}{1} q^{k-2} + \cdots + (-1)^{k-2} \binom{k+1}{k-2} q \\ &\quad + (-1)^{k-1} k\Big] - \Big[q^{k-2} - \binom{2k-4}{1} q^{k-3} + \cdots \\ &\quad + (-1)^{k-3} \binom{k}{k-3} q + (-1)^{k-2}(k-1)\Big] \\ &= q^k - \Big[\binom{2k-2}{1} + 2\Big] q^{k-1} + \Big[\binom{2k-3}{2} + 2\binom{2k-2}{1}\end{aligned}$$

$$- 1\Big] q^{k-2} - \cdots + \Big[(-1)^{k-1}k + 2(-1)^{k-1}\binom{k+1}{k-2} + (-1)^{k-2}\binom{k}{k-3}\Big] q + \Big[2(-1)^k k + (-1)^{k-1}(k-1)\Big].$$

By (23) and straightforward computations, we obtain

$$\mathcal{U}^q_{2(k+1)} = q^k - \binom{2k}{1} q^{k-1} + \binom{2k-1}{2} q^{k-2} - \cdots + (-1)^{k-1}\binom{k+2}{k-1} q + (-1)^k (k+1).$$

Now it is easy to find the polynomial representation of $\mathcal{U}^q_{2n+1}$. Indeed, by (3) we get

$$\begin{aligned}\mathcal{U}^q_{2n+1} &= \mathcal{U}^q_{2n+2} - \mathcal{U}^q_{2n} \\ &= q^n - \left[\binom{2n}{1} + 1\right] q^{n-1} + \left[\binom{2n-1}{2} + \binom{2n-2}{1}\right] q^{n-2} \\ &\quad - \cdots + \left[(-1)^{n-1}\binom{n+2}{n-1} + (-1)^{n-2}\binom{n+1}{n-2}\right] q \\ &\quad + \left[(-1)^n (n+1) + (-1)^n n\right].\end{aligned}$$

So using (24) we obtain the polynomial representation of $\mathcal{U}^q_{2n+1}$ as the statement of this theorem.

(*ii*) By (7) it is easy to obtain the polynomial representations of $\mathcal{V}^q_{2n+1}$ and $\mathcal{V}^q_{2n}$. □

Now we want to find a formula for $\mathcal{U}^q_n$, so we need not have to compute all the preceding terms. Let

$$g_q(x) = \sum_{i=0}^{\infty} \mathcal{U}^q_i x^i = \mathcal{U}^q_0 x^0 + \mathcal{U}^q_1 x + \mathcal{U}^q_2 x^2 + \mathcal{U}^q_3 x^3 + \cdots .$$

It follows that

$$\begin{aligned} & g_q(x) - x - x^2 - (q-3)x^3 \\ = \ & \sum_{i=4}^{\infty} \mathcal{U}^q_i x^i = \sum_{i=4}^{\infty} \left[(q-2)\mathcal{U}^q_{i-2} - \mathcal{U}^q_{i-4}\right] x^i \\ = \ & (q-2)\sum_{i=4}^{\infty} \mathcal{U}^q_{i-2} x^i - \sum_{i=4}^{\infty} \mathcal{U}^q_{i-4} x^i \end{aligned}$$

$$\begin{aligned} &= (q-2)x^2\sum_{i=0}^{\infty}\mathcal{U}_i^q x^i - (q-2)x^3 - x^4\sum_{i=0}^{\infty}\mathcal{U}_i^q x^i \\ &= (q-2)x^2 g_q(x) - x^4 g_q(x) - (q-2)x^3. \end{aligned}$$

Then, we have $g_q(x)\left(1-(q-2)x^2+x^4\right) = x + x^2 - x^3$. That is,

$$g_q(x) = \frac{x+x^2-x^3}{1-(q-2)x^2+x^4}. \tag{29}$$

An easy calculation shows that

$$1-(q-2)x^2+x^4 = (x-\sqrt{\tau})(x+\sqrt{\tau})(x-\sqrt{\sigma})(x+\sqrt{\sigma})$$

where $\tau = \frac{q-2+\sqrt{q(q-4)}}{2}$ and $\sigma = \frac{q-2-\sqrt{q(q-4)}}{2}$. τ and σ have the following properties:

$$\tau+\sigma = q-2,\ \tau-\sigma = \sqrt{q(q-4)},\ \tau\sigma = 1. \tag{30}$$

So we have

$$g_q(x) = \frac{A}{x-\sqrt{\tau}} + \frac{B}{x+\sqrt{\tau}} + \frac{C}{x-\sqrt{\sigma}} + \frac{D}{x+\sqrt{\sigma}}.$$

By straightforward computations, we find

$$\begin{aligned} A &= \frac{\sqrt{\tau}(1-\tau)+\tau}{2\sqrt{\tau}(\tau-\sigma)},\ B = \frac{\sqrt{\tau}(1-\tau)-\tau}{2\sqrt{\tau}(\tau-\sigma)}, \\ C &= \frac{\sqrt{\sigma}(\sigma-1)-\sigma}{2\sqrt{\sigma}(\tau-\sigma)},\ D = \frac{\sqrt{\sigma}(\sigma-1)+\sigma}{2\sqrt{\sigma}(\tau-\sigma)}. \end{aligned} \tag{31}$$

Thus we obtain

$$\begin{aligned} g_q(x) = &-\frac{A}{\sqrt{\tau}}\cdot\frac{1}{1-\frac{x}{\sqrt{\tau}}} + \frac{B}{\sqrt{\tau}}\cdot\frac{1}{1+\frac{x}{\sqrt{\tau}}} \\ &-\frac{C}{\sqrt{\sigma}}\cdot\frac{1}{1-\frac{x}{\sqrt{\sigma}}} + \frac{D}{\sqrt{\sigma}}\cdot\frac{1}{1+\frac{x}{\sqrt{\sigma}}}. \end{aligned}$$

Expressing $\frac{1}{1\pm\frac{x}{\sqrt{\tau}}}$ and $\frac{1}{1\pm\frac{x}{\sqrt{\sigma}}}$ as the sums of geometric series, we get

$$g_q(x) = -\frac{A}{\sqrt{\tau}}\left(1+\frac{x}{\sqrt{\tau}}+\frac{x^2}{(\sqrt{\tau})^2}+\frac{x^3}{(\sqrt{\tau})^3}+\cdots\right)$$

$$+\frac{B}{\sqrt{\tau}}\left(1-\frac{x}{\sqrt{\tau}}+\frac{x^2}{(\sqrt{\tau})^2}-\frac{x^3}{(\sqrt{\tau})^3}+\cdots\right)$$
$$-\frac{C}{\sqrt{\sigma}}\left(1+\frac{x}{\sqrt{\sigma}}+\frac{x^2}{(\sqrt{\sigma})^2}+\frac{x^3}{(\sqrt{\sigma})^3}+\cdots\right)$$
$$+\frac{D}{\sqrt{\sigma}}\left(1-\frac{x}{\sqrt{\sigma}}+\frac{x^2}{(\sqrt{\sigma})^2}-\frac{x^3}{(\sqrt{\sigma})^3}+\cdots\right)$$
$$=\left[\frac{B-A}{\sqrt{\tau}}+\frac{D-C}{\sqrt{\sigma}}\right]-\left[\frac{A+B}{(\sqrt{\tau})^2}+\frac{C+D}{(\sqrt{\sigma})^2}\right]x$$
$$+\left[\frac{B-A}{(\sqrt{\tau})^3}+\frac{D-C}{(\sqrt{\sigma})^3}\right]x^2-\left[\frac{A+B}{(\sqrt{\tau})^4}+\frac{C+D}{(\sqrt{\sigma})^4}\right]x^3$$
$$+\cdots .$$

As $B-A=\frac{-\tau}{\sqrt{\tau}(\tau-\sigma)}, D-C=\frac{\sigma}{\sqrt{\sigma}(\tau-\sigma)}, A+B=\frac{1-\tau}{\tau-\sigma}$ and $C+D=\frac{\sigma-1}{\tau-\sigma}$ we find

$$g_q(x) = \left[\frac{-1}{\tau-\sigma}+\frac{1}{\tau-\sigma}\right]+\left[\frac{-1+\tau}{\tau(\tau-\sigma)}+\frac{1-\sigma}{\sigma(\tau-\sigma)}\right]x$$
$$+\left[\frac{-1}{\tau(\tau-\sigma)}+\frac{1}{\sigma(\tau-\sigma)}\right]x^2+\left[\frac{\tau-1}{\tau^2(\tau-\sigma)}+\frac{1-\sigma}{\sigma^2(\tau-\sigma)}\right]x^3$$
$$+\cdots$$

and so
(32)

$$g_q(x)=\sum_{n=0}^{\infty}\frac{1}{\tau-\sigma}\left[\frac{-1}{\tau^n}+\frac{1}{\sigma^n}\right]x^{2n}+\sum_{n=0}^{\infty}\frac{1}{\tau-\sigma}\left[\frac{\tau-1}{\tau^{n+1}}+\frac{1-\sigma}{\sigma^{n+1}}\right]x^{2n+1}.$$

From this, we have

$$\mathcal{U}_{2n}^q=\frac{1}{\tau-\sigma}\left[\frac{-1}{\tau^n}+\frac{1}{\sigma^n}\right]$$

and $\mathcal{U}_{2n+1}^q=\frac{1}{\tau-\sigma}\left[\frac{\tau-1}{\tau^{n+1}}+\frac{1-\sigma}{\sigma^{n+1}}\right]$. Furthermore, by (30), we find

$$\mathcal{U}_{2n}^q=\frac{1}{\tau-\sigma}\left[\tau^n-\sigma^n\right] \tag{33}$$

and

$$\mathcal{U}_{2n+1}^q=\frac{1}{\tau-\sigma}\left[\sigma^{n+1}(\tau-1)+\tau^{n+1}(1-\sigma)\right]. \tag{34}$$

Therefore, we have proved the following theorem since

$$\tau-1=\frac{q-4+\sqrt{q(q-4)}}{2}, 1-\sigma=\frac{4-q+\sqrt{q(q-4)}}{2},$$

$$\tau = \left(\frac{\sqrt{q-4}+\sqrt{q}}{2}\right)^2$$

and

$$\sigma = \left(\frac{\sqrt{q-4}-\sqrt{q}}{2}\right)^2.$$

THEOREM 3.2. *The sequence $\mathcal{U}_n^q$ satisfies the following formulas:*

$$\mathcal{U}_{2n}^q = \frac{1}{\sqrt{q(q-4)}}\left[\left(\frac{\sqrt{q-4}+\sqrt{q}}{2}\right)^{2n} - \left(\frac{\sqrt{q-4}-\sqrt{q}}{2}\right)^{2n}\right] \tag{35}$$

and

$$\mathcal{U}_{2n+1}^q = \frac{1}{\sqrt{q}}\left[\left(\frac{\sqrt{q-4}+\sqrt{q}}{2}\right)^{2n+1} - \left(\frac{\sqrt{q-4}-\sqrt{q}}{2}\right)^{2n+1}\right]. \tag{36}$$

Note that for $q = 5$, we have

$$g_5(x) = \frac{x+x^2-x^3}{1-3x^2+x^4} = \frac{x}{1-x-x^2}$$

which is the generating function of F_n and

$$\mathcal{U}_n^5 = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

which is the well-known formula for F_n, [5].

Similarly, the corresponding formula for Lucas numbers and their generalizations $\mathcal{V}_n^q$ can be found from the generating function

$$h_q(x) = \sum_{i=0}^{\infty}\mathcal{V}_i^q x^i = \mathcal{V}_0^q x^0 + \mathcal{V}_1^q x + \mathcal{V}_2^q x^2 + \mathcal{V}_3^q x^3 + \cdots .$$

From this equation, we get

$$h_q(x) = \frac{x^3-(q-2)x^2+x+2}{1-(q-2)x^2+x^4}. \tag{37}$$

By straightforward computations, it is easy to deduce that

$$\begin{aligned} h_q(x) &= \sum_{n=0}^{\infty}\frac{1}{\tau-\sigma}\left[\frac{(q-2)\tau-2}{\tau^{n+1}} - \frac{(q-2)\sigma-2}{\sigma^{n+1}}\right]x^{2n} \\ &+\sum_{n=0}^{\infty}\frac{1}{\tau-\sigma}\left[\frac{1+\sigma}{\sigma^{n+1}} - \frac{1+\tau}{\tau^{n+1}}\right]x^{2n+1}. \end{aligned} \tag{38}$$

Then by (30), we have

$$\mathcal{V}^q_{2n+1} = \frac{1}{\tau - \sigma}\left[\tau^n(1+\tau) - \sigma^n(1+\sigma)\right]. \tag{39}$$

Also by (30), we find

$$\mathcal{V}^q_{2n} = \frac{1}{\tau - \sigma}\left[\sigma^n(q-2-2\sigma) + \tau^n(2-q+2\tau)\right].$$

As $q-2-2\sigma = 2-q+2\tau = \tau - \sigma$, we obtain

$$\mathcal{V}^q_{2n} = \sigma^n + \tau^n. \tag{40}$$

Since $\tau - \sigma = \sqrt{q(q-4)}$, $1+\tau = \frac{q+\sqrt{q(q-4)}}{2}$, $1+\sigma = \frac{q-\sqrt{q(q-4)}}{2}$, $\tau = \left(\frac{\sqrt{q-4}+\sqrt{q}}{2}\right)^2$ and $\sigma = \left(\frac{\sqrt{q-4}-\sqrt{q}}{2}\right)^2$, we get the following theorem:

THEOREM 3.3. *The formulas of $\mathcal{V}^q_n$ are the following:*

$$\mathcal{V}^q_{2n+1} = \frac{1}{\sqrt{q-4}}\left[\left(\frac{\sqrt{q-4}+\sqrt{q}}{2}\right)^{2n+1} + \left(\frac{\sqrt{q-4}-\sqrt{q}}{2}\right)^{2n+1}\right] \tag{41}$$

and

$$\mathcal{V}^q_{2n} = \left(\frac{\sqrt{q-4}+\sqrt{q}}{2}\right)^{2n} + \left(\frac{\sqrt{q-4}-\sqrt{q}}{2}\right)^{2n}. \tag{42}$$

For $q = 5$, again note that we get the well-known formula of L_n

$$\mathcal{V}^5_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n$$

and $h_5(x) = \frac{2-x}{1-x-x^2}$, [5].

Finally, we list a group of formulas with binomial coefficients. By (33), we have

$$\sum_{i=0}^{2n}\binom{2n}{i}\mathcal{U}^q_{2i} = \sum_{i=0}^{2n}\binom{2n}{i}\frac{1}{\tau-\sigma}\left[\tau^i - \sigma^i\right]$$
$$= \frac{1}{\tau-\sigma}\left[(1+\tau)^{2n} - (1+\sigma)^{2n}\right].$$

Using

$$1+\tau = \frac{q+\sqrt{q(q-4)}}{2} = \sqrt{q}\frac{\sqrt{q-4}+\sqrt{q}}{2} = \sqrt{q}\sqrt{\tau}$$

and

$$1+\sigma = \frac{q-\sqrt{q(q-4)}}{2} = -\sqrt{q}\frac{\sqrt{q-4}-\sqrt{q}}{2} = -\sqrt{q}\sqrt{\sigma},$$

we obtain

$$\frac{1}{\tau-\sigma}\left[(\sqrt{q}\sqrt{\tau})^{2n}-(-\sqrt{q}\sqrt{\sigma})^{2n}\right]=\frac{q^n}{\tau-\sigma}\left[\tau^n-\sigma^n\right]=q^n\mathcal{U}_{2n}^q,$$

that is we have

$$\sum_{i=0}^{2n}\binom{2n}{i}\mathcal{U}_{2i}^q=q^n\mathcal{U}_{2n}^q. \tag{43}$$

By (40), we obtain

$$\begin{aligned}\sum_{i=0}^{2n}\binom{2n}{i}\mathcal{V}_{2i}^q&=\sum_{i=0}^{2n}\binom{2n}{i}\left[\tau^i+\sigma^i\right]\\&=(1+\tau)^{2n}+(1+\sigma)^{2n}=q^n(\tau^n+\sigma^n),\end{aligned}$$

so

$$\sum_{i=0}^{2n}\binom{2n}{i}\mathcal{V}_{2i}^q=q^n\mathcal{V}_{2n}^q. \tag{44}$$

In a similar way, from (40) and (39), we have

$$\sum_{i=0}^{2n+1}\binom{2n+1}{i}\mathcal{V}_{2i}^q=q^{n+1}\sqrt{q-4}\mathcal{V}_{2n+1}^q. \tag{45}$$

Furthermore, (33), (34) and (40) enable us to derive further relationships. For example, using (33) and (40), we get

$$\begin{aligned}&\mathcal{V}_{2n}^q\mathcal{U}_{2m}^q-\mathcal{U}_{2(m-n)}^q\\=\;&(\tau^n+\sigma^n)\cdot\frac{1}{\tau-\sigma}(\tau^m-\sigma^m)-\frac{1}{\tau-\sigma}(\tau^{m-n}-\sigma^{m-n})\\=\;&\frac{1}{\tau-\sigma}\left[\sigma^n\tau^m-\sigma^{m+n}+\tau^{m+n}-\tau^n\sigma^m-\tau^{m-n}+\sigma^{m-n}\right]\\=\;&\frac{1}{\tau-\sigma}\left[\tau^{m-n}-\sigma^{m+n}+\tau^{m+n}-\sigma^{m-n}-\tau^{m-n}+\sigma^{m-n}\right]\\=\;&\frac{1}{\tau-\sigma}\left[\tau^{m+n}-\sigma^{m+n}\right]=\mathcal{U}_{2(m+n)}^q\end{aligned}$$

and so

$$\mathcal{U}_{2(m+n)}^q=\mathcal{V}_{2n}^q\mathcal{U}_{2m}^q-\mathcal{U}_{2(m-n)}^q. \tag{46}$$

In (46), if we take $m=2n$, we have $\mathcal{U}_{6n}^q=\mathcal{U}_{2n}^q\left((\mathcal{V}_{2n}^q)^2-1\right)$.

Similarly, by (34) and (40), we find

$$\begin{aligned}
&\mathcal{V}_{2n}^q\mathcal{U}_{2m+1}^q - \mathcal{U}_{2(m-n)+1}^q \\
&= (\tau^n + \sigma^n)\frac{1}{\tau-\sigma}\left[\sigma^{m+1}(\tau-1) + \tau^{m+1}(1-\sigma)\right] \\
&\quad - \frac{1}{\tau-\sigma}\left[\sigma^{m-n+1}(\tau-1) + \tau^{m-n+1}(1-\sigma)\right] \\
&= \frac{1}{\tau-\sigma}[\sigma^{m-n} - \sigma^{m-n+1} + \tau^{m+n+1}(1-\sigma) \\
&\quad + \sigma^{m+n+1}(\tau-1) + \tau^{m-n+1} - \tau^{m-n} - \sigma^{m-n} \\
&\quad + \sigma^{m-n+1} - \tau^{m-n+1} + \tau^{m-n} \\
&= \frac{1}{\tau-\sigma}\left[\sigma^{m+n+1}(\tau-1) + \tau^{m+n+1}(1-\sigma)\right] \\
&= \mathcal{U}_{2(m+n)+1}^q
\end{aligned}$$

and hence

$$\mathcal{U}_{2(m+n)+1}^q = \mathcal{V}_{2n}^q\mathcal{U}_{2m+1}^q - \mathcal{U}_{2(m-n)+1}^q. \tag{47}$$

By (40), we get

$$(\mathcal{V}_{2n}^q)^2 = (\tau^n + \sigma^n)^2 = \sigma^{2n} + \tau^{2n} + 2$$

and so

$$(\mathcal{V}_{2n}^q)^2 = \mathcal{V}_{4n}^q + 2. \tag{48}$$

Also by (33) and (40), we obtain

$$\begin{aligned}
(\mathcal{U}_{2n}^q)^2 &= \frac{1}{(\tau-\sigma)^2}\left((\tau^n - \sigma^n)^2\right) = \frac{1}{(\tau-\sigma)^2}\left(\sigma^{2n} + \tau^{2n} - 2\right) \\
&= \frac{1}{(\tau-\sigma)^2}(\mathcal{V}_{4n}^q - 2)
\end{aligned}$$

and so

$$\mathcal{V}_{4n}^q = q(q-4)(\mathcal{U}_{2n}^q)^2 + 2. \tag{49}$$

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