Proceedings of the 39th IEEE
Conference on Decision and Control
Sydney, Australia • December, 2000

Variable Sampling Integral Control of Infinite-Dimensional Systems

Necati ÖZDEMİR,
Department of Mathematics, Balikesir University, BALIKESİR, TURKEY
and
Stuart TOWNLEY
School of Mathematical Sciences, University of Exeter EXETER EX4 4QE, UK

Abstract

In this paper we present sampled-data low-gain I-control algorithms for infinite-dimensional systems in which the sampling period is not constant. The system is assumed to be exponentially stable with invertible steady state gain. The choice of the integrator gain is based on steady state gain information. In one algorithm the sampling time is divergent and in the other it increases adaptively.

1 Introduction

The design of low-gain integral (I) and proportional-plus-integral (PI) controllers for uncertain stable plants has been studied extensively during the last 20 years. More recently there has been considerable interest in low-gain integral control for infinite-dimensional systems.

The following principle of low-gain integral control is well known: Closing the loop around a stable, finite-dimensional, continuous-time, single-input, single-output plant, with transfer function $\mathbf{G}(s)$, precompensated by an integral controller k/s leads to a stable closed-loop system which achieves asymptotic tracking of constant reference signals, provided that $|k|$ is sufficiently small and $k\mathbf{G}(0) > 0$. This principle has been extended in various directions to encompass multivariable systems Davison [3], Lunze [12] and Logemann and Townley [11], input and output nonlinearities Logemann et al [6, 8], Logemann and Mawby [18]. Of particular relevance here are the results on sampled-data low-gain integral control of infinite-dimensional systems, see Logemann and Townley [10, 11], Özdemir and Townley [17]. Note that no matter what the context, it is a necessary, in achieving tracking of constant reference signal, that $\mathbf{G}(0)$ is invertible.

The main issue in the design of low-gain integral controllers is the tuning of the gain. In the literature, there have been essentially two approaches to the tuning of the integrator gain:

(i) Based on step-response data and individual tuning of the gain for each I/O channel. For results in this direction see Davison [3], Lunze [12] and Åström [1]. For example, Lunze [12] Section 7 (50) and (51) gives complicated techniques for choosing $\mathbf{\Gamma}$ and estimating k in the integral controller $\frac{k\mathbf{\Gamma}}{s}$ in terms of an approximate step response matrix, an upper bound of the approximation error, and various time constants.

(ii) By choosing $\mathbf{\Gamma}$ so that $\mathbf{\Gamma G}(0)$ has eigenvalues in $\mathbb{C}_+$ and then using error-based adaptive tuning of a scalar gain k in the I-controller $\dot{x} = k\mathbf{\Gamma}e$. Such adaptive tuning has been addressed in a number of papers, see Cook [2] and Miller and Davison [14] for results in the finite-dimensional case and Logemann and Townley [9, 10, 11] for the infinite-dimensional case.

Now the first approach, whilst making use of a variable data, is quite complicated, whilst the second is limited in design. Indeed, for multivariable systems an adaptive approach ought to adapt on whole gain $k\mathbf{\Gamma}$. Note that this involves m^2 parameters. One obvious possibility would be to use searching algorithms for adapting these m^2 parameters, in the spirit of Mårtensson [13]. However, such algorithms would tend to be slow and they are not really appropriate in this context. Inspired to some extent by the following result due to Åström [1] we adopt an alternative approach.

Proposition 1.1 *(Åström [1]) Let a stable single-input, single-output (infinite dimensional) system have a monotone increasing step response $t \mapsto H(t)$. Choose a fixed sampling period τ so that $2H(\tau) > \mathbf{G}(0)$ and a fixed integrator gain k so that $k\mathbf{G}(0) < 2$. Then the sampled-data integral controller, with current error integrator,*

$$u(t) = u_n \text{ for } t \in [n\tau, (n+1)\tau)$$
$$u_{n+1} = u_n + k(r - y((n+1)\tau))\,.$$

achieves tracking of constants r.

In this result we see that simple estimates for the gain and sampling period are derived easily from step-response data. Note, this result uses a current error integrator and only applies in the SISO case. For MIMO systems the relationship between appropriate choices of integrator gain and sampling period is rather complicated. Our aim is to derive simple criteria for choosing the integrator gain matrix based on steady-state data similar to Åström's results above. To do so we introduce the novel idea of using the sampling period as a control parameter. We consider sampled-data low-gain control of continuous-time infinite-dimensional systems of the form

$$\dot{x}(t) = Ax(t) + Bu(t), x(0) \in X, \tag{1a}$$

$$y(t) = Cx(t). \tag{1b}$$

In (1), X is a Hilbert space, A is the generator of an exponentially stable semigroup $T(t)$, $t \geq 0$ on X so that $\|T(t)\| \leq Me^{-wt}$ for some $M \geq 1$ and $w > 0$. The input operator B is unbounded but we assume $B \in \mathcal{L}(\mathbb{R}^m, X_{-1})$ (where X_{-1} is the completion of X with respect to the norm $\|x\|_{-1} := \|A^{-1}x\|_X$) and the output operator C is bounded so that $C \in \mathcal{L}(X, \mathbb{R}^m)$.

Remark 1.2

(a) The class of systems encompassed by (1) is large. Note that because we use piecewise-constant inputs arising from sampled-data control, well-posedness of the open-/closed-loop control system does not involve difficult to check admissibility type assumptions. We need C to be bounded because the output $y(\cdot)$, which is sampled directly, needs to be continuous. If C was not bounded, then usually the free output $y(\cdot)$ would not be continuous so that sampling would require pre-filters.

(b) We emphasize that whilst our results are valid for a large class of infinite-dimensional systems, they are new even in the finite-dimensional case.

We assume that the steady-state gain matrix

$$\mathbf{G}(0) := -CA^{-1}B$$

is invertible. For stable systems given by (1) a non-adaptive, sampled-data low-gain integral controller with 'previous error integrator' takes the form:

$$u(t) = u_n \text{ for } t \in [t_n, t_{n+1}) \text{ with} \tag{2a}$$

$$u_{n+1} = u_n + K(r - y(t_n)). \tag{2b}$$

Analogue results for the current error integrator can be found in Özdemir [16].

Here $y(t_n)$ is the sampled output at the sampling time t_n. Usually, $t_n = n\tau$ where τ is the sampling period. One of our key ideas is to use the sampling time as a control parameter τ_n so that the sampling time is given instead by $t_{n+1} = t_n + \tau_n$, with $t_0 = 0$. This idea is not without precendent. Indeed variable and adaptive sampling has been used in a high-gain adaptive control context, see Owens [15] and Ilchmann and Townley [5]. Applying variation of constants to (1), (2) gives

$$x(t_{n+1}) = T(\tau_n)x_n + (T(\tau_n) - I)A^{-1}Bu_n.$$

Let $x_n := x(t_n)$. Then

$$x_{n+1} = T(\tau_n)x_n + (T(\tau_n) - I)A^{-1}Bu_n \tag{3a}$$

$$u_{n+1} = u_n + K(r - Cx_n). \tag{3b}$$

If we apply the change of coordinates

$$z_n = x_n + A^{-1}Bu_n \text{ and } v_n = u_n - u_r = u_n - \mathbf{G}(0)^{-1}r,$$

as in Logemann et al [6], then

$$z_{n+1} = (T(\tau_n) - A^{-1}BKC)z_n - A^{-1}BK\mathbf{G}(0)v_n \tag{4a}$$

$$v_{n+1} = -KCz_n + (I - K\mathbf{G}(0))v_n \tag{4b}$$

Here we clearly see how the gain K, the steady state gain $\mathbf{G}(0)$ and the variable sampling period τ_n influence the system. Our approach is to use τ_n as a tuning parameter, whilst choosing K (robustly) on the basis of steady-state information. The paper is organised as follows: In Section 2 we consider (4) with divergent τ_n. This allows us to study first the stability of a much simpler system with 'infinite sampling period'. Lemma 2.1 gives a simple criterion for choosing the matrix gain K based only on knowledge of the steady-state gain $\mathbf{G}(0)$. The main result is Theorem 2.2 which shows that (2) achieves tracking if the gain is chosen as in Lemma 2.1 and $\{\tau_n\}$ is divergent. In Section 3.1 we look at refinements to Lemma 2.1 by which the matrix gain is chosen robustly with respect to error in the measurement of $\mathbf{G}(0)$. In Section 3.2 we consider the possibility of input-nonlinearity. Finally in Section 3.3 we combine the criteria for choosing the gain, either via Lemma 2.1 or robustly as in Section 3.1, with convergent adaptation of the sampling period.

2 Integral control with divergent sampling period and an infinite-sampling-period lemma

If, loosely speaking, we set the sampling period $\tau_n = \infty$ in (4), then we obtain the much simpler closed-loop system

$$z_{n+1} = -A^{-1}BKCz_n - A^{-1}BK\mathbf{G}(0)v_n \tag{5a}$$

$$v_{n+1} = -KCz_n + (I - K\mathbf{G}(0))v_n \tag{5b}$$

Lemma 2.1 *Suppose $\mathbf{G}(0)$ is invertible and $K \in \mathbb{R}^{m\times m}$ is such that*

$$det(\lambda(\lambda - 1)I + KG(0)) \tag{6}$$

has zeros inside the unit circle, equivalently so that the matrix

$$\begin{pmatrix} 0 & I \\ -K\mathbf{G}(0) & I \end{pmatrix}$$

is Schur. Then the system (5) is power stable, i.e. the operator

$$A_K = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} - \begin{pmatrix} A^{-1}B \\ I \end{pmatrix} K \begin{pmatrix} C & \mathbf{G}(0) \end{pmatrix} \quad (7)$$

has spectral radius less one.

Theorem 2.2 *Consider*

$$u(t) = u_n \ \text{ for } t \in [t_n, t_{n+1}), \ \text{ with} \quad (8a)$$

$$u_{n+1} = u_n + K(r - Cx(t_n)) \ \text{ and} \quad (8b)$$

$$t_{n+1} = \tau_n + t_n = f_n. \quad (8c)$$

Here $\{f_n\}$ is any divergent monotone sequence and K is chosen as in Lemma 2.1. If $u(t)$, given by (8a) and (8b), with sampling times t_n given by (8c), is applied to (1), then for each $x(0) \in X$ and $u_0 \in \mathbb{R}^m$ we have

$(i)\ \lim_{n\to\infty} \|r - Cx(t_n)\| = 0,\ (ii)\ \lim_{t\to\infty} u(t) = u_r := G(0)^{-1}r,$

$(iii)\ \lim_{t\to\infty} x(t) = x_r := -A^{-1}Bu_r, \qquad (iv)\ \lim_{t\to\infty} y(t) = r.$

Remark 2.3 *1. Note that there exists $M_P > 0$ so that*

$$\begin{pmatrix} z_n \\ v_n \end{pmatrix}^T P \begin{pmatrix} z_n \\ v_n \end{pmatrix} \le M_P \|\alpha_n\|^2.$$

So

$$0 \le V_{n+1} \le \left(1 - \frac{(1 - \tilde{M}e^{-w\tau_n})}{M_P}\right) V_n, \ \text{for all } n \ge N.$$

For finite-dimensional systems we could then use bounded invertibility of P to conclude that for $\epsilon \in (0, \frac{1}{M_P})$ we can find $M > 0$ so that

$$\left\| \begin{pmatrix} z_n \\ v_n \end{pmatrix} \right\| \le M\left(1 - \frac{1}{M_P} + \epsilon\right)^n \left\| \begin{pmatrix} z_0 \\ v_0 \end{pmatrix} \right\|$$

i.e

$$u_n \to u_r \quad \text{and} \quad x_n \to x_r$$

with exponential decay rate $\log_e(1 - \frac{1}{M_P} + \epsilon)$, which does not depend on $\{f_n\}$.

2. Each choice of $\{f_n\}_{n=0}^{\infty}$ gives a different N so that

$$1 - \tilde{M}e^{-w\tau_n} > \frac{1}{2} \ \text{for all } n \ge N. \quad (9)$$

holds. This in turn gives

$$\left\| \begin{pmatrix} z_n \\ v_n \end{pmatrix} \right\| \le L(1 - M_P^{-1} + \epsilon)^n \left\| \begin{pmatrix} z_0 \\ v_0 \end{pmatrix} \right\| \quad (10)$$

for all $n \ge 0$, where L depends on $\{f_n\}_{n=0}^{\infty}$. However, this exponential convergence is with respect to n and not t_n. In continuous time we have

$$x(t_{n+1}) = T(\tau_n)x_n + (T(\tau_n) - I)A^{-1}Bu_n,$$

with u_n given by (8) Hence the exponential convergence of α_n with respect to n leads via $\tau_n = f_n$, to slower continuous-time convergence of $x(t) \to x_r$ as $t \to \infty$. Note that a more rapidly diverging $f_n \nearrow \infty$ gives slower $t-$convergence, but a smaller N and so smaller L in (10). This leads to a trade-off between a reduced overshoot (smaller L) and slower continuous-time convergence that more rapidly diverging $\{f_n\}$ gives. An interesting question is how to find the best compromise choice for f_n.

3. For systems with small time constants the use of the above sampled-data integral controllers with divergent sampling period is appealing. Indeed, in contrast to the sampled-data control with adaptive gain, considerably more use is made of available step-response data. The algorithm can be made more practical by allowing reset of the sampling time, in particular in response to set-point changes.

The main benefit of our approach is that we use available step-response data. In applications this data will be subject to experimental error. In Section 3 we consider refinements to the selection of K which take account of the uncertainty in $\mathbf{G}(0)$. We also consider the possibility of input nonlinearities and adaptation of the sampling period.

3 Robustness and Sampling Period Adaptation

3.1 Robustness to Experimental Error

The steady state gain $\mathbf{G}(0)$ is determined by step response experiments. In practice we will only know $\mathbf{G}(0)$ approximately and the true value of $\mathbf{G}(0)$ will be a perturbation of the value obtained experimentally. This uncertainty in the value of $\mathbf{G}(0)$ can be due to measurement noise or else to the use of finite-time, as opposed to steady-state, experiments when determining $\mathbf{G}(0)$.

Denote the measured $\mathbf{G}(0)$ by $\mathbf{G}_{expt}(0)$. Suppose

$$\mathbf{G}(0) = \mathbf{G}_{expt}(0) + D\Delta E,$$

where $D \in \mathbb{R}^{n\times q}$, $E \in \mathbb{R}^{r\times n}$ are fixed and $\Delta \in \mathbb{R}^{q\times r}$ is unknown but $\|\Delta\| < \delta$, some $\delta > 0$. This is the set-up of the so-called structured stability radius, see Hinrichsen and Pritchard [4]. For simplicity consider the unstructured case $E = D = I$. Then

$$\mathbf{G}(0) = \mathbf{G}_{expt}(0) + \Delta.$$

Of course we must have that $\mathbf{G}(0) = \mathbf{G}_{expt}(0) + \Delta$ is invertible for all $\|\Delta\| < \delta$. Now

$$(\mathbf{G}_{expt}(0) + \Delta)^{-1} = \left(\mathbf{G}_{expt}(0)(I + \mathbf{G}_{expt}(0)^{-1}\Delta)\right)^{-1}.$$

It follows that necessarily $\delta \leq \|\mathbf{G}_{expt}(0)^{-1}\|^{-1}$. Indeed for

$$\hat{\Delta} = -\frac{vu^T}{\|\mathbf{G}_{expt}(0)^{-1}\|^2},$$

where $\|v\| = 1, \mathbf{G}_{expt}(0)^{-1}v = u$ and $\|u\| = \|\mathbf{G}_{expt}(0)^{-1}\|$, $\|\hat{\Delta}\| = \|\mathbf{G}_{expt}(0)^{-1}\|^{-1}$ and $\mathbf{G}_{expt}(0) + \hat{\Delta}$ is singular. We need to choose K, on the basis of the experimental $\mathbf{G}_{expt}(0)$, such that

$$\begin{pmatrix} 0 & I \\ -K\mathbf{G}(0) & I \end{pmatrix} + \begin{pmatrix} 0 \\ K \end{pmatrix} \Delta \begin{pmatrix} I & 0 \end{pmatrix}$$

is Schur (as in Lemma 2.1). Using stability radius techniques [4] this is guaranteed if

$$\|\Delta\| \leq \inf_{|z|=1} \frac{1}{\|(\mathbf{G}_{expt}(0) + z(z-1)K^{-1})^{-1}\|}. \quad (11)$$

In order to allow for the maximum experimental error (i.e maximum $\delta > 0$) we should choose K to maximise the right-hand side of (11). Now clearly for any choice of K, the right-hand side of (11) is not greater than $\|\mathbf{G}_{expt}(0)^{-1}\|^{-1}$ (just choose $z = 1$). Hence the maximum possible $\delta > 0$ is

$$\max_K \inf_{|z|=1} \frac{1}{\|(\mathbf{G}_{expt}(0) + z(z-1)K^{-1})^{-1}\|}.$$

Theorem 3.1

$$\max_K \min_{|z|=1} \frac{1}{\|(\mathbf{G}_{expt}(0) + z(z-1)K^{-1})^{-1}\|} = \frac{1}{\|\mathbf{G}_{expt}(0)^{-1}\|}$$

and K achieves the maximum if $K^{-1} = \mathbf{G}_{expt}(0)H$ where $H = H^T > 0$, and $\lambda_{min}(H) \geq 3$.

Example 3.2 *Consider system (1) with $X = \mathbb{R}^3$ and*

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -10 & -17 & -8 \end{pmatrix}, B = \begin{pmatrix} 2 & 3 \\ -1 & -2 \\ -15 & -6 \end{pmatrix},$$

$$C = \begin{pmatrix} 2 & 1 & 4 \\ 1 & 2 & 3 \end{pmatrix}.$$ *In this case $\mathbf{G}(s)$ equals*

$$\begin{pmatrix} \frac{-57s^2-5s+42}{s^3+8s^2+17s+10} & \frac{-20s^2+38s+108}{s^3+8s^2+17s+10} \\ \frac{-45s^2-40s+1}{s^3+8s^2+17s+10} & \frac{-19s^2-10s+29}{s^3+8s^2+17s+10} \end{pmatrix}.$$

We assume that knowledge of $\mathbf{G}(0)$ can only be obtained from steady-state experiments. To simulate steady-state experimental conditions we truncate the step response of the system at $t = 3.5$. This gives

$$\mathbf{G}_{exp}(0) = \begin{pmatrix} 4.5 & 10.5 \\ 0.1 & 2.5 \end{pmatrix}, \textit{whilst } \mathbf{G}(0) = \begin{pmatrix} 4.2 & 10.8 \\ 0.1 & 2.9 \end{pmatrix}.$$

In this case $\|\mathbf{G}(0) - \mathbf{G}_{exp}(0)\| = 0.5389$ and $\|\mathbf{G}_{exp}^{-1}(0)\|^{-1} = 0.8747$, Theorem 3.1 applies and we can choose $K = H^{-1}\mathbf{G}_{exp}^{-1}(0)$ with $\lambda_{min}(H) \geq 3$. Note that $\mathbf{G}_{exp}(0)$ is poorly conditioned. In the simulations we use

$$H = \begin{pmatrix} 4.5 & 0 \\ 0 & 4.5 \end{pmatrix}, \textit{ so that } K = \begin{pmatrix} 0.0545 & -0.2288 \\ -0.0022 & 0.0980 \end{pmatrix}$$

assume steady-state initial conditions $x(0) = (0,0,0)^T, u(0) = (0,0)^T$, with stepped-reference $r(t) = (1,1)^T$ $t < 130$, $r(t) = (2,2)^T$ $t \geq 130$, and choose $\tau_n = \log(n+2)$.

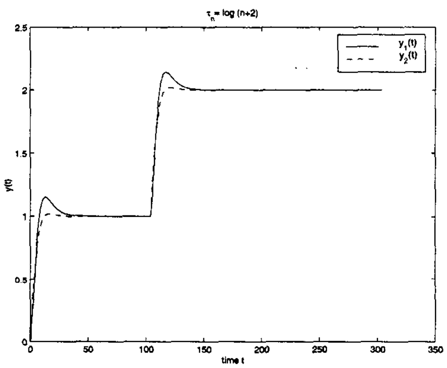

Figure 1: Output $y(t)$

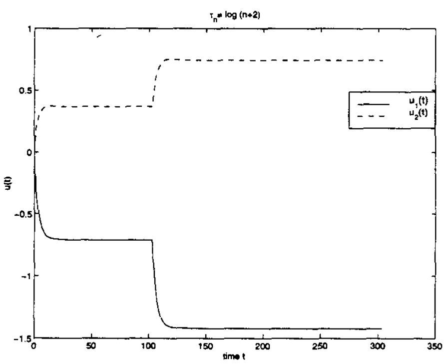

Figure 2: Input $u(t)$

The open-loop step responses produce quite significant over-shoot (typically 100%) and the rise-time is of the order of 5. In the closed loop simulations the overshoot is approximately 25% whilst the rise time is of the order 15-30. We emphasize that the only information used in the controller design was quite poor measurement of the steady-state gain (recall $\|\mathbf{G}(0) - \mathbf{G}_{exp}(0)\| = 0.5389$ and $\|\mathbf{G}_{exp}(0)^{-1}\| = 0.8747$.)

3.2 Robustness to Input Nonlinearity

In the previous subsection we considered robustness in the choice of K with respect to uncertainty in experimental measurement of the steady-state gain. Another common source of uncertainty in low-gain integral control is that due to input saturation or more generally input nonlinearity. Low-gain integral control for infinite dimensional systems in the presence of input nonlinearity has been studied by Logemann, Ryan and Townley [6](continuous time), Logemann and Ryan [7](continuous time, adaptive), Logemann and Mawby [18](continuous time, hysteresis nonlinearity). We consider sampled-data low-gain I-control with input nonlinearity and in particular the robustness of the design of K with respect to such input nonlinearity. We restrict attention to the single input-single output (SISO) case and suppose that the input to the system u is replaced by $\Phi(u)$ so that

$$\dot{x}(t) = Ax(t) + B\Phi(u_{n+1}), \qquad x(0) \in X, \qquad (12a)$$
$$y(t) = Cx(t) \qquad (12b)$$

with $u(t)$ given by (8). Then after sampling the closed-loop system becomes

$$z_{n+1} = T(\tau_n)z_n + (T(\tau_n) - I)A^{-1}B\Phi(v_n) \quad (13a)$$
$$v_{n+1} = v_n + k(r - Cz_n). \quad (13b)$$

where $k > 0$ is the scalar integrator gain. We assume throughout this section that there exists v_r such that $\Phi(v_r) = \Phi_r$ where $\mathbf{G}(0)\Phi_r = r$. Introducing variables $z_n = x_n - x_r$, $v_n = u_n - v_r$ and $\Psi(v) = \Phi(v + v_r) - \Phi_r$, then (13) becomes

$$z_{n+1} = T(\tau_n)z_n + (T(\tau_n) - I)A^{-1}B\Psi(v_n) \quad (14a)$$
$$v_{n+1} = v_n - kCz_n. \quad (14b)$$

As in subsection 2.1 we first consider (14) with "$\tau_n = \infty$." Then (14) becomes

$$z_{n+1} = -A^{-1}B\Psi(v_n) \quad (15a)$$
$$v_{n+1} = v_n - kCz_n. \quad (15b)$$

Lemma 3.3 (∞ − **Sampling Period Lemma**) *Define*

$$V_n = k^2(Cz_n)^2 + (v_n - kCz_n)^2$$

Then $V_{n+1} - V_n$, computed along solutions of (15a) and (15b), satisfies

$$V_{n+1} - V_n \leq 3k^2\mathbf{G}(0)^2\Psi^2(v_n) - 2k\mathbf{G}(0)v_n\Psi(v_n) \quad (16a)$$

If $\Psi^2(v) \leq v\Psi(v)$ and $k\mathbf{G}(0) \in (0, \frac{2}{3})$, then there exists $\epsilon > 0$ such that

$$V_{n+1} \leq V_n - \epsilon\Psi^2(v_n). \qquad (17a)$$

Theorem 3.4 *Consider sampled-data low-gain I-control of a continuous-time exponentially stable infinite dimensional system defined by equations (12). Define the control input by (8). If*

$$k\mathbf{G}(0) \in (0, \frac{2}{3}) \text{ and } \tau_n \geq \alpha\log(n+2), \text{ with } \alpha w > 1,$$

then

$$(i) \lim_{n\to\infty} \|x_n - x_r\| = 0, \ (ii) \lim_{t\to\infty} \Phi(u(t)) = u_r := \mathbf{G}(0)^{-1}r$$
$$(iii) \lim_{t\to\infty} x(t) = x_r := -A^{-1}Bu_r, \ (iv) \lim_{t\to\infty} y(t) = r.$$

Remark 3.5 *Let us compare our estimates on the gain k for $\tau_n \nearrow \infty$ with existing Positive Real (PR) estimates on the gain for fixed τ (see [8]). First, denote $G_d(z)$ the transfer function of the discrete-time system obtained by applying sampled data control:*

$$G_d(z) = C(zI - T(\tau))^{-1}(T(\tau) - I)A^{-1}B.$$

Now for a discrete-time system with transfer function $G(z)$, subject to input nonlinearity Φ with $\Phi^2 \leq u\Phi$ a (PR) estimate [1] *for the gain k so that the I-controller*

$$u_{n+1} = u_n + k(r - y(n\tau))$$

achieves tracking of r is given by

$$1 + kRe\frac{G_d(z)}{z-1} \geq 0, \ for \ \forall\, |z| \geq 1.$$

Applying this result to the sampled system, i.e. with $G(z) = G_d(z)$ we have

$$Re\left(\frac{G_d(z)}{z-1}\right) = Re\left(\frac{G_d(z) - G_d(1)}{z-1}\right) + G_d(1)Re\frac{1}{z-1}.$$

After some manipulation this becomes

$$1 - \frac{kG(0)}{2} + kReE(1) \geq 0. \qquad (18)$$

Where $E(z)$ is the z-transform of the step-response error. Now

$$E(1) = \sum_{j=0}^{\infty} C(T(j\tau))A^{-1}B = C(I - T(\tau))^{-1}A^{-1}B$$

and $\lim_{\tau\to\infty} C(I - T(\tau))^{-1}A^{-1}B = CA^{-1}B = -\mathbf{G}(0)$. It follows that if $k\mathbf{G}(0) < \frac{2}{3}$ i.e. the condition imposed in Theorem 3.4, then (18) holds for all large enough τ. When τ is not large, we can estimate the discrete time condition (18) i.e.

$$1 - \frac{kG(0)}{2} + k\sum_{j=0}^{\infty} e(j) \geq 0$$

by

$$1 - \frac{3k\mathbf{G}(0)}{2} - k\frac{J}{\tau} \geq 0.$$

Here J is the area between the steady state $\mathbf{G}(0)$ and step-response.

[1] Townley, Logemann and Ryan, Personal Communication

3.3 Integral Control with Fixed-Gain and Adaptive Sampling

In this subsection we develop an algorithm for on-line adaptation of the sampling period. From the analysis of Sections 2.1 and 3 it is reasonable that τ_n should be increasing when e_n is large. This gives us the idea to choose $\tau_n = \alpha \log \gamma_n$ where γ_n increases if e_n is not converging to zero.

Theorem 3.6 *Let $r \in \mathbb{R}^m$ be an arbitrary constant reference signal. Define*

$$u(t) = u_n \text{ for } t \in [t_n, t_{n+1}) \text{ where} \quad (19a)$$
$$u_{n+1} = u_n + K(r - Cx(t_n)), \quad (19b)$$
$$\tau_n = t_{n+1} - t_n = \alpha \log \gamma_n \quad (19c)$$
$$\gamma_{n+1} = \gamma_n + \|r - y(t_n)\|^2 . \quad (19d)$$

Choose any $\alpha > 0$ and $K > 0$ so that the zeros of $\det(\lambda(\lambda - I) + K\mathbf{G}(0))$ have modulus less than one. If $u(t)$ given by (19a) and (19b), with sampling times t_n given by (19c) where γ_n is given by (19d), is applied to (1), then for all $x(0) \in X, u_0 \in \mathbb{R}^m$ and $\gamma_0 > 1$

$$(a) \lim_{n\to\infty} \gamma_n = \gamma_\infty < \infty, \ (b) \lim_{n\to\infty} \tau_n = \tau_\infty < \infty$$

and (i)-(iv) of Theorem 2.2 hold.

Note: $\alpha > 0$ plays a similar role as in Theorem 3.4. It helps to improve speed of response/convergence.

Remark 3.7 *We can clearly choose the gain K. in Theorem 3.6 simply as in Theorem 2.2 or robustly as in Theorem 3.1.*

References

[1] K.J. ÅSTRÖM, A robust sampled regulator for stable systems with monotone step responses, *Automatica* **16** (1980), pp.313–315.

[2] P.A. COOK, Controllers with universal tracking properties, *Proc. of Int. IMA Conf. on Control: Modelling, Computation, Information*, Manchester, 1992.

[3] E.J. DAVISON, Multivariable tuning regulators: the feedforward and robust control of a general servomechanism problem, *IEEE Trans. Auto. Control* **21** (1976), pp. 35–47

[4] D. HINRICHSEN AND A. J. PRITCHARD, Stability radius for structured perturbations and the algebraic Riccati equation, *Systems and Control Letters*, **8**, (1986), pp. 105–113.

[5] A. ILCHMANN AND S. TOWNLEY, Adaptive sampling control of high-gain stabilizable systems, *IEEE Trans. Auto. Control* **vol.44 no. 10** (1999)

[6] H. LOGEMANN, E.P. RYAN AND S. TOWNLEY, Integral control of infinite-dimensional linear systems subject to input saturation, *SIAM J. Control & Optim.*, **vol.36**, **no. 6**, (1998), pp. 1940–1961.

[7] H. LOGEMANN AND E.P. RYAN, Time-varying and adaptive integral control of infinite-dimensional regular systems with input nonlinearities, *Mathematics Preprint, 98/12, University of Bath*, (1998).

[8] H. LOGEMANN, E.P. RYAN AND S. TOWNLEY, Integral Control of Linear Systems with Actuator Nonlinearities: Lover Bounds for the Maximal Regulating Gain, *IEEE Trans. Auto. Control*, **vol. 44, no. 6**, (1999).

[9] H. LOGEMANN AND S. TOWNLEY, Low-gain control of uncertain regular linear systems, *SIAM J. Control & Optim.* **35** (1997), pp. 78–116

[10] H. LOGEMANN AND S. TOWNLEY, Discrete-time low-gain control of uncertain infinite-dimensional systems, *IEEE Trans. Auto. Control* **42** (1997), pp. 22–37.

[11] H. LOGEMANN AND S. TOWNLEY, Adaptive integral control for multi-variable systems, in preparation.

[12] J. LUNZE, Determination of robust multivariable I-controllers by means of experiments and simulation, *Syst. Anal. Model. Simul.***2**(1985), pp. 227–249.

[13] B. MÅRTENSSON, Adaptive Stabilization, Ph.D. thesis, *Lund Institute of Technology, Dept. of Automatic Control, 1986.*

[14] D.E. MILLER AND E.J. DAVISON, The self-tuning robust servomechanism problem, *IEEE Trans. Auto. Control* **34** (1989), pp. 511–523.

[15] D. H. OWENS, Adaptive stabilization using a variable sampling rate, *Int. J. Control.*, **vol. 63**, **no. 1**, (1996) pp. 107–119.

[16] N. ÖZDEMIR, Robust and Adaptive Sampled data I-control, Ph.D Thesis, University of Exeter, School of Mathematical Sciences, UK, **March** (2000).

[17] N. ÖZDEMIR AND S. TOWNLEY, Adaptive low-gain control of infinite dimensional systems by means of sampling time adaptation, *Methods and Models in Automation and Robotics* **25-29 August**(1998), Miedzyzdroje, Poland, pp. 63–68.

[18] H. LOGEMANN AND A.D. MAWBY, Integral Control of Distributed parameter systems with input relay hysteresis, *UKACC International Conference on CONTROL'98,* **1-4 September 1998**, Swansea, UK pp. 1236–1241.