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Toric ideals of simple surface singularities

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Abstract

In this paper, we study a class of toric ideals obtained by using some geometric data of ADE trees which are the minimal resolution graphs of rational surface singularities. We compute explicit Gröbner bases for these toric ideals that are also minimal generating sets consisting of large number of binomials of degree ≤ 4. In particular, they give rise to squarefree initial ideals as well.

Keywords: toric ideals, simple surface singularities, semigroup of Lipman

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1. Introduction

Algebraic varieties having squarefree initial ideals are of special interest. Many authors have presented squarefree initial ideals arising from different contexts, see for instance [5, 11, 13, 14, 16]. Normal toric ideals are known to have at least one squarefree term in each minimal binomial generator by [19, Proposition 4.1] and [17, Lemma 6.1]. They have Cohen-Macaulay initial ideals when their configurations are Δ-normal, see [18]. These suggest that they have (at least simplicial ones) squarefree initial ideals with respect to a term order. The challenge lies in the choice of a correct term order. Motivated by fundamental questions in combinatorial commutative algebra and its applications to statistics and optimization, recently, with the aid of Gale diagrams, Dueck et al. [8] have succeeded to show the existence of a term order with respect to which normal toric ideals of codimension 2 have squarefree initial ideals. They have also proven that the Gröbner bases giving rise to these initial ideals constitute minimal generating sets for the toric ideals.

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The aim of the present paper is to extend the discussion to certain examples of normal toric ideals of higher codimension. As a case study, we concentrate on certain toric ideals of higher codimension arising from singularity theory that are promising because of the speciality of the corresponding singularities. These are the simplicial normal toric ideals corresponding to the simple or ADE surface singularities. In section 3, we prove that toric ideals of DE type singularities have squarefree initial ideals. Our methods are computational and use the configurations given in [1]. The reduced Gröbner bases we obtain are also shown to be minimal generating sets containing a large number of binomials of degree at most 4, see section 4. In the last section, we speculate on initial ideals of A_n-type trees whose configurations seem impossible to give a closed form.

2. Preliminaries

2.1. Gröbner basis. Let $\mathcal{A} = \{\mathbf{a_1}, \ldots, \mathbf{a_N}\}$ be a configuration in $\mathbb{Z}^n$ and $K[\mathcal{A}] := K[\{x_\mathbf{a} \,|\, \mathbf{a} \in \mathcal{A}\}]$ denote the polynomial ring in variables $x_\mathbf{a}$ with $\mathbf{a} \in \mathcal{A}$ over the field K. Consider the affine semigroup $\mathbb{N}\mathcal{A} = \{\lambda_1\mathbf{a_1} + \cdots + \lambda_N\mathbf{a_N} \; : \; \lambda_i \in \mathbb{N}\}$ and let $K[\mathbb{N}\mathcal{A}] := K[\{\mathbf{u^a} \,|\, \mathbf{a} \in \mathcal{A}\}]$ be the associated semigroup ring. The *toric ideal* $I_\mathcal{A}$ of $\mathcal{A}$ is the kernel of the following K-algebra epimorphism:

$$\pi : K[\mathcal{A}] \to K[\mathbb{N}\mathcal{A}], \quad \pi(x_\mathbf{a}) := \mathbf{u^a} = u_1^{a_1} \cdots u_n^{a_n}.$$

It is known that $I_\mathcal{A}$ is a prime ideal generated by binomials $x_\mathbf{a} - x_\mathbf{b}$ with $\pi(x_\mathbf{a}) = \pi(x_\mathbf{b})$ [20]. The zero set of $I_\mathcal{A}$ is called the toric variety $V_\mathcal{A}$ of $\mathcal{A}$.

The *initial monomial*, $in(f)$, of a polynomial $f \in I_\mathcal{A} \setminus \{0\}$ is the greatest monomial of f with respect to a term order on the monomials of $K[\mathcal{A}]$. The *initial ideal*, $in(I_\mathcal{A})$, of $I_\mathcal{A}$ is a *monomial* ideal generated by all initial monomials of polynomials in $I_\mathcal{A}$. A finite subset $\mathcal{G} \subset I_\mathcal{A}$ is called a Gröbner basis of $I_\mathcal{A}$ if $in(I_\mathcal{A}) = in(\mathcal{G})$, where $in(\mathcal{G})$ is the monomial ideal generated by initial monomials of polynomials in $\mathcal{G}$. The following is the key in proving our main results.

2.1. Lemma. [2, Lemma 1.1] *With the preceding notation, let M and M' be monomials in $K[\mathcal{A}]$. The finite set $\mathcal{G}$ is a Gröbner basis of $I_\mathcal{A}$ if and only if $\pi(M) \neq \pi(M')$ for all $M \notin in(\mathcal{G})$ and $M' \notin in(\mathcal{G})$ with $M \neq M'$.*

2.2. ADE-trees. Here, we briefly review basics of ADE-trees, see [4, 3, 23, 9, 10] for more details. Let Γ be a weighted graph without loops, with vertices $C_1, \ldots, C_n$ and with weight $w_i \geq 2$ at each vertex C_i. The incidence matrix $\mathcal{M}(\Gamma) = [c_{ij}]$, associated with Γ is a symmetric matrix and defined in the following way: $c_{ii} = -w_i$ and c_{ij} is the number of edges linking the vertices C_i and C_j whenever $i \neq j$. On the free abelian group $\mathcal{L}$ generated by the vertices C_i of Γ, $\mathcal{M}(\Gamma)$ defines a symmetric bilinear form $(Y \cdot Z)$ for a pair (Y, Z) of elements in $\mathcal{L}$ via $(C_i \cdot C_j) := c_{ij}$. The elements $C = \sum_{i=1}^n m_i C_i$ of $\mathcal{L}$ will be called *cycles* of the graph Γ where $m_i \in \mathbb{Z}$. A *positive cycle* is a non-zero cycle with non-negative coefficients.

If $w_i = 2$ for all i and $C \cdot C \leq -2$ for any cycle then Γ is of type A_n, D_n, E_6, E_7 and E_8. It is well known that these are the Dynkin diagrams obtained as the minimal resolution graphs of the rational singularities of complex surfaces. The semigroup of Lipman is the set

$$\mathcal{E}^+(\Gamma) := \{C \in \mathcal{L} \mid (C \cdot C_i) \leq 0 \quad \text{for} \quad 1 \leq i \leq n\}.$$

which is not empty since $M(\Gamma)$ is negative definite in this case. By [15], each element of this set corresponds to a function in the maximal ideal of the local ring of the singularity on the surface having Γ as the minimal resolution graph.

In [22] and [1], the authors have studied the structure of this semigroup and provided an algorithm to find a generating set over $\mathbb{Z}$ by associating an affine toric variety $V_{\mathcal{A}}$, c.f. also [21]. This toric variety corresponds to the configuration $\mathcal{A}$ of the smallest n-tuples $(d_1,\ldots,d_n)\in\mathbb{N}^n$ such that $(C\cdot C_i)=-d_i$ for $C\in\mathcal{E}^+(\Gamma)$. The interested reader can see [1] for the details.

3. Squarefree initial ideals

In this section, we obtain reduced Gröbner bases for toric ideals of affine toric varieties corresponding to DE-type singularities. Throughout the section, we assume that the first term of a binomial is its initial monomial for a fixed term order. In order to find the set $\mathcal{A}$ which determines the parametrization of the toric variety $V_{\mathcal{A}}$, we use Proposition 3.9 and 3.12 in [1].

3.1. D_n-type singularities. We have $n\geq 4$. Since toric ideals behave in a different manner when n is even and odd, we discuss two cases separately.

When $n=2m$: Let $J=\{3,5,\ldots,n-1\}$ and $J^c=\{2,4,\ldots,n-2\}$. Consider the subset

$$D_{2m}:=\{2\mathbf{e}_i,\mathbf{e}_j,2\mathbf{e}_1,2\mathbf{e}_n,\mathbf{e}_k+\mathbf{e}_\ell,\mathbf{e}_i+\mathbf{e}_1+\mathbf{e}_n\,|\,i,k,\ell\in J,\ j\in J^c\text{ and }k<\ell\},$$

where $\{\mathbf{e}_1,\ldots,\mathbf{e}_n\}$ is the canonical basis of $\mathbb{Z}^n$. Then we introduce one variable for each element in the set D_{2m} and define the polynomial ring $K[D_{2m}]$ to be the K-algebra generated by the set of these variebles

$$\{x_1,\ldots,x_n,x_{j,k},y_i\mid\text{where }i,j,k\in J\text{ and }j<k\}.$$

Similarly we define the semigroup ring $K[\mathbb{N}D_{2m}]$ to be the K-algebra generated by

$$\{u_i^2,u_j,u_1^2,u_n^2,u_ku_\ell,u_iu_1u_n\,|\,i,k,\ell\in J,\ j\in J^c\text{ and }k<\ell\}.$$

The toric ideal $I_{D_{2m}}$ is thus the kernel of $\pi:K[D_{2m}]\to K[\mathbb{N}D_{2m}]$ which is defined as:

$$\pi(x_i)=u_i^2,\ \pi(x_j)=u_j,\ \pi(x_1)=u_1^2,\ \pi(x_n)=u_n^2,\ \pi(x_{k,\ell})=u_ku_\ell,$$

$$\pi(y_i)=u_iu_1u_n$$

for all $i,k,\ell\in J$, $j\in J^c$ with $k<\ell$.

We next define the ordering $\succ^{even}$ to be the reverse lexicographic ordering imposed by:

$$x_1\succ\cdots\succ x_{n-1}\succ x_n\succ x_{j_1,j_2}\succ x_{j_3,j_4}\succ y_{k_1}\succ y_{k_2}$$

where $j_1,j_2,j_3,j_4,k_1,k_2\in J$ with $j_2<j_4$ or $j_2=j_4,\ j_1<j_3$; and $k_1<k_2$.

Then a squarefree initial ideal for $I_{D_{2m}}$ is given by the following theorem, since the first monomial of a binomial is its initial term.

3.1. Theorem. *The following set $\mathcal{G}_{D_{2m}}$*

$$\begin{array}{lll}
x_{i,k}x_{j,\ell}-x_{i,j}x_{k,\ell} & x_{i,\ell}x_{j,k}-x_{i,j}x_{k,\ell} & i<j<k<\ell\\
x_{i,j}x_{i,k}-x_ix_{j,k} & x_jx_{i,k}-x_{i,j}x_{j,k} & i<j<k\\
x_kx_{i,j}-x_{i,k}x_{j,k} & x_{j,k}y_i-x_{i,j}y_k & i<j<k\\
x_{i,k}y_j-x_{i,j}y_k & & i<j<k\\
x_ix_j-x_{i,j}^2 & x_jy_i-x_{i,j}y_j & i<j\\
x_{i,j}y_i-x_iy_j & x_{i,j}x_1x_n-y_iy_j & i<j\\
x_ix_1x_n-y_i^2 & & i\in J
\end{array}$$

is a Gröbner basis of $I_{D_{2m}}$ with respect to the ordering $\succ^{even}$ defined above.

Proof. Let M and M' be two monomials in $K[D_{2m}]$ with $M \notin in(\mathcal{G}_{D_{2m}})$ and $M' \notin in(\mathcal{G}_{D_{2m}})$, where $in(\mathcal{G}_{D_{2m}})$ is the monomial ideal generated by initial terms of binomials in $\mathcal{G}_{D_{2m}}$. Since $x_i x_j \in in(\mathcal{G}_{D_{2m}})$, we may assume that

$$M = x_a^p x_1^{\alpha_1} x_n^{\alpha_n} x_{b_1,c_1} \cdots x_{b_q,c_q} y_{d_1} \cdots y_{d_r} \quad \text{and}$$
$$M' = x_{a'}^{p'} x_1^{\alpha_1'} x_n^{\alpha_n'} x_{b_1',c_1'} \cdots x_{b_{q'}',c_{q'}'} y_{d_1'} \cdots y_{d_{r'}'}, \quad \text{where}$$
$$x_a \succ x_{b_1,c_1} \succ \cdots \succ x_{b_q,c_q} \succ y_{d_1} \succ \cdots \succ y_{d_r},$$
$$x_{a'} \succ x_{b_1',c_1'} \succ \cdots \succ x_{b_{q'}',c_{q'}'} \succ y_{d_1'} \succ \cdots \succ y_{d_{r'}'}.$$

First, we observe that the ordering above implies that $c_1 \le \cdots \le c_q$, $c_1' \le \cdots \le c_{q'}'$ and $d_1 \le \cdots \le d_r$, $d_1' \le \cdots \le d_{r'}'$. Moreover, we have $b_1 < c_1 \le b_2 < c_2 \le \cdots \le b_q < c_q$ and $b_1' < c_1' \le b_2' < c_2' \le \cdots \le b_{q'}' < c_{q'}'$, since $x_{i,k}x_{j,\ell}, x_{i,\ell}x_{j,k}, x_{i,j}x_{i,k} \in in(\mathcal{G}_{D_{2m}})$.

The images of M and M' are found easily as

$$\pi(M) = u_a^{2p} u_1^{2\alpha_1+r} u_n^{2\alpha_n+r} u_{b_1} u_{c_1} \cdots u_{b_q} u_{c_q} u_{d_1} \cdots u_{d_r}$$
$$\pi(M') = u_{a'}^{2p'} u_1^{2\alpha_1'+r'} u_n^{2\alpha_n'+r'} u_{b_1'} u_{c_1'} \cdots u_{b_{q'}'} u_{c_{q'}'} u_{d_1'} \cdots u_{d_{r'}'}.$$

In what follows we will prove that $\pi(M) = \pi(M') \Rightarrow M = M'$, by the virtue of Lemma 2.1. It follows from $\pi(M) = \pi(M')$ that we have the following identities

$$2\alpha_1 + r = 2\alpha_1' + r' \tag{3.1}$$
$$2\alpha_n + r = 2\alpha_n' + r' \tag{3.2}$$
$$2p + 2q + r = 2p' + 2q' + r' \tag{3.3}$$
$$\alpha_1 - \alpha_n = \alpha_1' - \alpha_n' \qquad \text{(follows directly from (3.1) and (3.2)).} \tag{3.4}$$

To accomplish our goal $M = M'$, we will assume now that $M \ne M'$ to obtain a contradiction in all possible cases considered below. Since $M \ne M'$, we may suppose further that they have no variable in common without loss of generality. This is because $in(\mathcal{G}_{D_{2m}})$ is an ideal and $M, M' \notin in(\mathcal{G}_{D_{2m}})$ implies that the new monomials obtained by dividing M and M' by their greatest common divisor will also lie outside of $in(\mathcal{G}_{D_{2m}})$.

If $\alpha_1 > 0$ and $\alpha_n > 0$ then $\alpha_1' = \alpha_n' = 0$, as M and M' have no common variable. Since $x_{i,j}x_1x_n$, $x_kx_1x_n \in in(\mathcal{G}_{D_{2m}})$, we have $p = q = 0$. This implies that $r = 2p' + 2q' + r'$ by (3.3) and thus $2p' + 2q' + 2\alpha_1 = 0$ by (3.1), a contradiction.

If $\alpha_1 > 0$ and $\alpha_n = 0$ then $\alpha_1' = 0$ which implies together with (3.4) that $\alpha_1 = -\alpha_n' \le 0$, contradiction. The case $\alpha_1 = 0$ and $\alpha_n > 0$ is done similarly. So, we have only the case where $\alpha_1 = 0$ and $\alpha_n = 0$. A similar argument shows that $\alpha_1' = \alpha_n' = 0$. In this case $r = r'$ by (3.1).

Case I: Assume $r = r' > 0$. Since $x_j y_i \in in(\mathcal{G}_{D_{2m}})$, for all $i < j$, it follows that $a \le d_r$. Again by $x_{i,j}y_i, x_{j,k}y_i, x_{i,k}y_j \in in(\mathcal{G}_{D_{2m}})$, for all $i < j < k$, we have $(b_q <)c_q \le d_r$ and $(b_{q'}' <)c_{q'}' \le d_{r'}'$. Hence, d_r (resp. d_r') is the biggest index appearing in $\pi(M)$ (resp. $\pi(M')$). Since $\pi(M) = \pi(M')$, it follows that $d_r = d_r'$. But this implies that y_{d_r} is a variable appearing in both M and M', contradiction.

Case II: Assume $r = r' = 0$. If $q = 0$ then $\pi(M) = \pi(M')$ implies that $u_a^{2p} = u_{a'}^{2p'} u_{b_1'} u_{c_1'} \cdots u_{b_{q'}'} u_{c_{q'}'}$, which is possible only if $q' = 0$ as $b_{q'}' < c_{q'}'$. But in this case $a = a'$ and x_a is a common variable of M and M', a contradiction. Thus $q > 0$ and $q' > 0$.

Since $x_j x_{i,k}, x_k x_{i,j} \in in(\mathcal{G}_{D_{2m}})$, we have $a \le c_q$ and $a' \le c_{q'}'$. Since $b_q < c_q$ and $b_{q'}' < c_{q'}'$, we observe that c_q (resp. $c_{q'}'$) is the biggest index appearing in $\pi(M)$ (resp. $\pi(M')$) which yields together with $\pi(M) = \pi(M')$ that $c_q = c_{q'}'$. In this case u_{b_q} and $u_{b_{q'}'}$ appear in $\pi(M) = \pi(M')$. Clearly $b_q > b_{q'}'$ or $b_q < b_{q'}'$, as otherwise M and M' would have a common variable x_{b_q,c_q}. If $b_q > b_{q'}' (> \cdots > b_1')$ then $b_q = a'$ as u_{b_q}

appears in $\pi(M')$. This forces that $b'_{q'} < b_q = a' < c_q = c'_{q'}$ which is impossible, since $x_j x_{i,k} \in in(\mathcal{G}_{D_{2m}})$. The other case $b_q < b'_{q'}$ is impossible by a similar argument. $\square$

3.2. Remark. Note that we have

$$|\mathcal{G}_{D_{2m}}| = 2\binom{m-1}{4} + 5\binom{m-1}{3} + 4\binom{m-1}{2} + \binom{m-1}{1}.$$

$\dim V_{D_{2m}} = 2m$, $\operatorname{codim} V_{D_{2m}} = m - 1 + \binom{m-1}{2}$.

When $n = 2m+1$ **:** Let $J = \{2, 4, \dots, n-1\}$ and $J^c = \{3, 5, \dots, n-2\}$. Consider the subset D_{2m+1} defined by

$$\{2\mathbf{e}_i, \mathbf{e}_j, 4\mathbf{e}_1, 4\mathbf{e}_n, \mathbf{e}_k + \mathbf{e}_\ell, \mathbf{e}_1 + \mathbf{e}_n, \mathbf{e}_i + 2\mathbf{e}_1, \mathbf{e}_i + 2\mathbf{e}_n, \mathbf{e}_i + 3\mathbf{e}_1 + \mathbf{e}_n, \mathbf{e}_i + \mathbf{e}_1 + 3\mathbf{e}_n \\ \mid i, k, \ell \in J,\ j \in J^c \text{ and } k < \ell\},$$

where $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ is the canonical basis of $\mathbb{Z}^n$. As before we introduce one variable for each member of D_{2m+1} and define the polynomial ring $K[D_{2m+1}]$ to be the K-algebra generated by the set

$$\{x_1, \dots, x_n, x_{j,k}, x_{1,n}, x_{i,1}, x_{i,n}, y_{i,1}, y_{i,n} \mid \text{where } i, j, k \in J \text{ and } j < k\}$$

and the semigroup ring $K[\mathbb{N}D_{2m+1}]$ to be the K-algebra generated by

$$\{u_i^2, u_j, u_1^4, u_n^4, u_k u_\ell, u_1 u_n, u_i u_1^2, u_i u_n^2, u_i u_1^3 u_n, u_i u_1 u_n^3 \mid i, k, \ell \in J,\ j \in J^c \text{ and } k < \ell\}.$$

The toric ideal $I_{D_{2m+1}}$ is thus the kernel of $\pi : K[D_{2m+1}] \to K[\mathbb{N}D_{2m+1}]$ which is defined as follows:

$$\pi(x_i) = u_i^2,\ \pi(x_j) = u_j,\ \pi(x_1) = u_1^4,\ \pi(x_n) = u_n^4,\ \pi(x_{k,\ell}) = u_k u_\ell,\ \pi(x_{1,n}) = u_1 u_n \\ \pi(x_{i,1}) = u_i u_1^2,\ \pi(x_{i,n}) = u_i u_n^2,\ \pi(y_{i,1}) = u_i u_1^3 u_n,\ \pi(y_{i,n}) = u_i u_1 u_n^3$$

for all $i, k, \ell \in J$, $j \in J^c$ with $k < \ell$.

Finally, we define the ordering $\succ^{odd}$ to be the reverse lexicographic ordering imposed by:

$$y_{i_1,1} \succ y_{i_2,1} \succ y_{i_1,n} \succ y_{i_2,n} \succ x_1 \succ \cdots \succ x_n \succ \\ \succ x_{j_1,j_2} \succ x_{j_3,j_4} \succ x_{k_1,1} \succ x_{k_2,1} \succ x_{\ell_1,n} \succ x_{\ell_2,n} \succ x_{1,n}$$

where $j_1, j_2, j_3, j_4, k_1, k_2, \ell_1, \ell_2 \in J$ with $j_2 < j_4$ or $j_2 = j_4$, $j_1 < j_3$ and $k_1 < k_2$ and $\ell_1 < \ell_2$.

Then a squarefree initial ideal for $I_{D_{2m+1}}$ is given by the following theorem as the first monomials are the initial terms with respect to the ordering $\succ^{odd}$.

3.3. Theorem. *The following set* $\mathcal{G}_{D_{2m+1}}$

$x_{i,k}x_{j,\ell} - x_{i,j}x_{k,\ell}$	$x_{i,\ell}x_{j,k} - x_{i,j}x_{k,\ell}$	$i < j < k < \ell \in J$
$x_{j,k}x_{i,n-1} - x_{i,j}x_{k,n-1}$	$x_{i,k}x_{j,n-1} - x_{i,j}x_{k,n-1}$	$i < j < k \in J$
$x_{j,k}x_{i,n} - x_{i,j}x_{k,n}$	$x_{i,k}x_{j,n} - x_{i,j}x_{k,n}$	$i < j < k \in J$
$x_j x_{i,k} - x_{i,j}x_{j,k}$	$x_{i,j}x_{i,k} - x_i x_{j,k}$	$i < j < k \in J$
$x_k x_{i,j} - x_{i,k}x_{j,k}$		$i < j < k \in J$
$x_i x_j - x_{i,j}^2$	$x_{i,j}x_1 - x_{i,n-1}x_{j,n-1}$	$i < j \in J$
$x_{i,j}x_n - x_{i,n}x_{j,n}$	$x_{i,j}x_{i,n-1} - x_i x_{j,n-1}$	$i < j \in J$
$x_{i,j}x_{i,n} - x_i x_{j,n}$	$x_j x_{i,n-1} - x_{i,j}x_{j,n-1}$	$i < j \in J$
$x_j x_{i,n} - x_{i,j}x_{j,n}$	$x_{j,n-1}x_{i,n} - x_{i,n-1}x_{j,n}$	$i < j \in J$
$x_{i,n-1}x_{j,n} - x_{1,n}^2 x_{i,j}$		$i < j \in J$
$x_i x_1 - x_{i,n-1}^2$	$x_i x_n - x_{i,n}^2$	$i \in J$
$x_{i,n-1}x_{i,n} - x_{1,n}^2 x_i$	$x_{i,n}x_1 - x_{1,n}^2 x_{i,n-1}$	$i \in J$
$y_{i,1} - x_{1,n}x_{i,1}$	$y_{i,n} - x_{1,n}x_{i,n}$	$i \in J$
$x_{i,n-1}x_n - x_{1,n}^2 x_{i,n}$	$x_1 x_n - x_{1,n}^4$	$i \in J$

is a Gröbner basis of $I_{\mathcal{D}_{2m+1}}$ with respect to the ordering $\succ^{odd}$.

Proof. Let M and M' be two monomials in $K[D_{2m+1}]$ with $M \notin in(\mathcal{G}_{D_{2m+1}})$ and $M' \notin in(\mathcal{G}_{D_{2m+1}})$, where $in(\mathcal{G}_{D_{2m+1}})$ is the monomial ideal generated by initial terms of binomials in $\mathcal{G}_{D_{2m+1}}$. Since $y_{i,1}, y_{i,n}, x_i x_j \in in(\mathcal{G}_{D_{2m+1}})$, we may assume that

$$\begin{aligned} M &= x_a^p x_1^{\alpha_1} x_n^{\alpha_n} x_{1,n}^{\beta} x_{b_1,c_1} \cdots x_{b_q,c_q} x_{d_1,n-1} \cdots x_{d_r,n-1} x_{e_1,n} \cdots x_{e_s,n} \quad \text{and} \\ M' &= x_{a'}^{p'} x_1^{\alpha'_1} x_n^{\alpha'_n} x_{1,n}^{\beta'} x_{b'_1,c'_1} \cdots x_{b'_{q'},c'_{q'}} x_{d'_1,n-1} \cdots x_{d'_{r'},n-1} x_{e'_1,n} \cdots x_{e'_{s'},n}, \end{aligned}$$

where the variables are ordered with respect to

$$x_a \succ x_{b_1,c_1} \succ \cdots \succ x_{b_q,c_q} \succ x_{d_1,n-1} \succ \cdots \succ x_{d_r,n-1} \succ x_{e_1,n} \succ \cdots \succ x_{e_s,n},$$

$$x_{a'} \succ x_{b'_1,c'_1} \succ \cdots \succ x_{b'_{q'},c'_{q'}} \succ x_{d'_1,n-1} \succ \cdots \succ x_{d'_{r'},n-1} \succ x_{e'_1,n} \succ \cdots \succ x_{e'_{s'},n}.$$

First, we observe that the ordering above implies that $c_1 \le \cdots \le c_q$, $c'_1 \le \cdots \le c'_{q'}$, $d_1 \le \cdots \le d_r$, $d'_1 \le \cdots \le d'_{r'}$ and $e_1 \le \cdots \le e_r$, $e'_1 \le \cdots \le e'_{r'}$. Moreover, we have $b_1 < c_1 \le b_2 < c_2 \le \cdots \le b_q < c_q$ and $b'_1 < c'_1 \le b'_2 < c'_2 \le \cdots \le b'_{q'} < c'_{q'}$, since $x_{i,k}x_{j,\ell}, x_{i,\ell}x_{j,k}, x_{i,j}x_{i,k} \in in(\mathcal{G}_{D_{2m+1}})$.

The images of M and M' are found as follows

$$\begin{aligned} \pi(M) &= u_a^{2p} u_1^{4\alpha_1+\beta+2r} u_n^{4\alpha_n+\beta+2s} u_{b_1} u_{c_1} \cdots u_{b_q} u_{c_q} u_{d_1} \cdots u_{d_r} u_{e_1} \cdots u_{e_s} \\ \pi(M') &= u_{a'}^{2p'} u_1^{4\alpha'_1+\beta'+2r'} u_n^{4\alpha'_n+\beta'+2s'} u_{b'_1} u_{c'_1} \cdots u_{b'_{q'}} u_{c'_{q'}} u_{d'_1} \cdots u_{d'_{r'}} u_{e'_1} \cdots u_{e'_{s'}}. \end{aligned}$$

It follows from $\pi(M) = \pi(M')$ that we have the following identities

$$2p + 2q + r + s = 2p' + 2q' + r' + s' \tag{3.5}$$

$$4\alpha_1 + \beta + 2r = 4\alpha'_1 + \beta' + 2r' \tag{3.6}$$

$$4\alpha_n + \beta + 2s = 4\alpha'_n + \beta' + 2s' \tag{3.7}$$

$$2\alpha_1 - 2\alpha_n + r - s = 2\alpha'_1 - 2\alpha'_n + r' - s' \text{ (follows from (3.6) and (3.7)).} \tag{3.8}$$

To accomplish our goal $M = M'$, we will assume contrarily that $M \neq M'$ and obtain a contradiction in all possible cases considered below. Since $M \neq M'$, we may suppose further that they have no variable in common without loss of generality.

Since $x_1 x_n \in in(\mathcal{G}_{D_{2m+1}})$, it follows that α_1 and α_n can not be positive simultaneously. If $\alpha_1 > 0$ then $\alpha_n = 0$ and $\alpha'_1 = 0$ immediately. That $p = q = s = 0$ follows respectively from $x_i x_1, x_{i,j} x_1, x_{i,n} x_1 \in in(\mathcal{G}_{D_{2m+1}})$. Thus equations 3.5 and 3.8 become

$$\begin{aligned} r &= 2p' + 2q' + r' + s' \\ 2\alpha_1 + r &= -2\alpha'_n + r' - s' \end{aligned}$$

and we have $2\alpha_1 = -2(p' + q' + s' + \alpha'_n) \le 0$, contradiction. If $\alpha_n > 0$ then clearly $\alpha'_n = 0$ and $\alpha_1 = 0$. That $p = q = r = 0$ follows respectively from $x_i x_n, x_{i,j} x_n, x_{i,n-1} x_n \in in(\mathcal{G}_{D_{2m+1}})$. Thus equations 3.5 and 3.8 become

$$\begin{aligned} s &= 2p' + 2q' + r' + s' \\ -2\alpha_n - s &= 2\alpha'_1 + r' - s' \end{aligned}$$

and we have $2\alpha_n = -2(p' + q' + r' + \alpha'_1) \le 0$, contradiction. So, both $\alpha_1 = \alpha_n = 0$. One can show that $\alpha'_1 = 0$ and $\alpha'_n = 0$ by a similar argument.

Now, $x_{j,n-1} x_{i,n}, x_{i,n-1} x_{j,n}, x_{i,n-1} x_{i,n} \in in(\mathcal{G}_{D_{2m+1}})$ implies that r and s (resp. r' and s') can not be positive at the same time.

If $r > 0$, then $s = 0$ in which case equation 3.8 becomes $r = r' - s'$. If $r' > 0$, then $s' = 0$ and we have $r = r' > 0$, which is impossible as in this case, d_r would be equal to $d'_{r'}$ since these are the biggest indices of variables in M and M', x_{d_r} would be a common variable. If $s' > 0$, then $r' = 0$ and we have $r = -s'$, contradiction as $r > 0$ and $s' > 0$.

If $s > 0$, then $r = 0$ in which case equation 3.8 becomes $-s = r' - s'$. If $r' > 0$, then $s' = 0$ and we have $-s = r'$, which contradicts the assumption that $s > 0$ and $r' > 0$. If $s' > 0$, then $r' = 0$ and we have $s = s' > 0$, which is impossible as in this case e_s would be $e'_{s'}$ and since these are the biggest indices of variables in M and M', x_{e_s} would be a common variable.

Hence, $r = s = 0$ and this implies together with equation 3.8 that $r' = s'$. Since they can not be positive simultaneously, $r' = s' = 0$ as well. After all these observations, equation 3.6 reveals that $\beta = \beta'$. Since M and M' have no common variable, it follows that $\beta = \beta' = 0$.

If $q = 0$ then $\pi(M) = \pi(M')$ implies that $u_a^{2p} = u_{a'}^{2p'} u_{b'_1} u_{c'_1} \cdots u_{b'_{q'}} u_{c'_{q'}}$, which is possible only if $q' = 0$ as $b'_{q'} < c'_{q'}$. But in this case $a = a'$ and x_a is a common variable of M and M', a contradiction. Similarly, $q' = 0$ gives rise to a contradiction. Thus $q > 0$ and $q' > 0$.

Since $x_j x_{i,k}, x_k x_{i,j} \in in(\mathcal{G}_{D_{2m+1}})$, we have $a \leq c_q$ and $a' \leq c'_{q'}$. Since $b_q < c_q$ and $b'_{q'} < c'_{q'}$, we observe that c_q (resp. $c'_{q'}$) is the biggest index appearing in $\pi(M)$ (resp. $\pi(M')$) which yields together with $\pi(M) = \pi(M')$ that $c_q = c'_{q'}$. In this case u_{b_q} and $u_{b'_{q'}}$ appear in $\pi(M) = \pi(M')$. Clearly $b_q > b'_{q'}$ or $b_q < b'_{q'}$, as otherwise M and M' would have a common variable x_{b_q,c_q}. If $b_q > b'_{q'} (> \cdots > b'_1)$ then $b_q = a'$ as u_{b_q} appears in $\pi(M')$. This forces that $b'_{q'} < b_q = a' < c_q = c'_{q'}$ which is impossible, since $x_j x_{i,k} \in in(\mathcal{G}_{D_{2m+1}})$. The other case $b_q < b'_{q'}$ is impossible by a similar argument. $\square$

3.4. Remark. Note that if $n = 2m + 1$ we have,

$$|\mathcal{G}_{D_{2m+1}}| = 2\binom{m}{4} + 7\binom{m}{3} + 9\binom{m}{2} + 7\binom{m}{1} + \binom{m}{0}.$$

$\dim V_{D_{2m+1}} = 2m + 1$, $\operatorname{codim} V_{D_{2m+1}} = 2m + 1 + \binom{m}{2}$.

3.2. E_n-type Singularities. We will give Gröbner bases of toric ideals $I_{\mathcal{E}_n}$, where $n = 6, 7, 8$, without proofs, as they can easily be checked by a computation in Cocoa [7]. To begin with, let us define the set $\mathcal{E}_6 \subset \mathbb{Z}^6$:

$$\{3\mathbf{e}_1, 3\mathbf{e}_2, \mathbf{e}_3, 3\mathbf{e}_4, 3\mathbf{e}_5, \mathbf{e}_6, \mathbf{e}_1{+}\mathbf{e}_2, \mathbf{e}_1{+}\mathbf{e}_5, \mathbf{e}_2{+}\mathbf{e}_4, \mathbf{e}_4{+}\mathbf{e}_5, 2\mathbf{e}_2{+}\mathbf{e}_5, \mathbf{e}_2{+}2\mathbf{e}_5, 2\mathbf{e}_1{+}\mathbf{e}_4, \mathbf{e}_1{+}2\mathbf{e}_4\}.$$

Let $K[\mathcal{E}_6]$ be the polynomial ring $K[x_1, \ldots, x_{14}]$ with 14 variables and $K[\mathbb{N}\mathcal{E}_6]$ be the semigroup ring generated over K by monomials $u^{\mathbf{a}}$ with $\mathbf{a} \in \mathcal{E}_6$. Then, as before, the toric ideal $I_{\mathcal{E}_6}$ is the kernel of the epimorphism defined by sending the i-th variable x_i to $u^{\mathbf{a}_i}$, where $\mathbf{a}_i$ denotes the i-th element in $\mathcal{E}_6$, for all $i = 1, \ldots, 14$. Similarly, we define the set $\mathcal{E}_7 \subset \mathbb{Z}^7$:

$$\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, 2\mathbf{e}_4, \mathbf{e}_5, 2\mathbf{e}_6, 2\mathbf{e}_7, \mathbf{e}_4 + \mathbf{e}_6, \mathbf{e}_4 + \mathbf{e}_7, \mathbf{e}_6 + \mathbf{e}_7\}.$$

Again, $K[\mathcal{E}_7]$ denotes the polynomial ring $K[x_1, \ldots, x_{10}]$ with 10 variables and $K[\mathbb{N}\mathcal{E}_7]$ be the semigroup ring generated over K by monomials $u^{\mathbf{a}}$ with $\mathbf{a} \in \mathcal{E}_7$. Thus, the toric ideal $I_{\mathcal{E}_7}$ is the kernel of the epimorphism defined by sending the i-th variable x_i to $u^{\mathbf{a}_i}$, where $\mathbf{a}_i$ denotes the i-th element in $\mathcal{E}_7$, for all $i = 1, \ldots, 10$. Finally, the set $\mathcal{E}_8 \subset \mathbb{Z}^8$ is defined as $\{\mathbf{e}_1, \ldots, \mathbf{e}_8\}$.

3.5. Theorem. *With the notations above we have the following:*

(1) *A Gröbner basis for $I_{\mathcal{E}_6}$ with respect to lexicographic ordering with $x_1 > x_2 > x_3 > x_4 > x_5 > x_6 > x_{11} > x_{12} > x_{13} > x_{14} > x_7 > x_8 > x_9 > x_{10}$ is given by*

$$\begin{array}{llll}
x_7x_{10} - x_8x_9, & x_{13}x_{10} - x_{14}x_8, & x_{13}x_9 - x_{14}x_7, & x_{12}x_{14} - x_8x_9x_{10},\\
x_{12}x_{13} - x_8^2x_9, & x_{11}x_{10} - x_{12}x_9, & x_{11}x_8 - x_{12}x_7, & x_{11}x_{14} - x_8x_9^2,\\
x_{11}x_{13} - x_7x_8x_9, & x_5x_9 - x_{12}x_{10}, & x_5x_7 - x_{12}x_8, & x_5x_{14} - x_8x_{10}^2,\\
x_5x_{13} - x_8^2x_{10}, & x_5x_{11} - x_{12}^2, & x_4x_8 - x_{14}x_{10}, & x_4x_7 - x_{14}x_9,\\
x_4x_{13} - x_{14}^2, & x_4x_{12} - x_9x_{10}^2, & x_4x_{11} - x_9^2x_{10}, & x_4x_5 - x_{10}^3,\\
x_2x_{10} - x_{11}x_9, & x_2x_8 - x_{11}x_7, & x_2x_{14} - x_7x_9^2, & x_2x_{13} - x_7^2x_9,\\
x_2x_{12} - x_{11}^2, & x_2x_5 - x_{11}x_{12}, & x_2x_4 - x_9^3, & x_1x_{10} - x_{13}x_8,\\
x_1x_9 - x_{13}x_7, & x_1x_{14} - x_{13}^3, & x_1x_{11} - x_7x_8^2, & x_1x_{11} - x_7^2x_8,\\
x_1x_5 - x_8^3, & x_1x_4 - x_{13}x_{14}, & x_1x_2 - x_7^3. &
\end{array}$$

(2) *A Gröbner basis for $I_{\mathcal{E}_7}$ with respect to lexicographic ordering with $x_1 > x_2 > x_3 > x_4 > x_5 > x_6 > x_7 > x_8 > x_9 > x_{10}$ is given by the following binomials*

$$x_7x_8 - x_9x_{10}, \quad x_6x_9 - x_8x_{10}, \quad x_6x_7 - x_{10}^2, \quad x_4x_{10} - x_8x_9, \quad x_4x_7 - x_9^2, \quad x_4x_6 - x_8^2.$$

(3) *The toric ideal $I_{\mathcal{E}_8} = (0)$.*

4. Minimal generating sets

In this part, using [6] we show that the Gröbner bases obtained in the previous section are in fact minimal generating sets for each toric ideal. This will be achieved as follows.

Since our semigroups $\mathbb{N}\mathcal{A}$ are pointed, there is a partial order on them given by

$$\mathbf{c} \le \mathbf{d} \Leftrightarrow \text{there is a } \mathbf{c}' \in \mathbb{N}\mathcal{A} \text{ such that } \mathbf{c} + \mathbf{c}' = \mathbf{d}.$$

As $I_{\mathcal{A}}$ is generated by binomials $x_{\mathbf{a}} - x_{\mathbf{b}}$ with $\pi(x_{\mathbf{a}}) = \pi(x_{\mathbf{b}})$, $x_{\mathbf{a}}$ and $x_{\mathbf{b}}$ will have the same $\mathcal{A}$-*degree*. Recall that for $\mathbf{p} = (p_1, \dots, p_N) \in \mathbb{N}^N$, the $\mathcal{A}$-*degree* of the monomial $x^{\mathbf{p}} := x_1^{p_1} \dots x_N^{p_N}$ is $\deg_{\mathcal{A}}(x^{\mathbf{p}}) = p_1\mathbf{a}_1 + \dots + p_N\mathbf{a}_N \in \mathbb{N}\mathcal{A}$. A vector $\mathbf{b} \in \mathbb{N}\mathcal{A}$ is called a *Betti* $\mathcal{A}$-degree, if $I_{\mathcal{A}}$ has a minimal generating set containing an element of $\mathcal{A}$-degree $\mathbf{b}$. Since *Betti* $\mathcal{A}$-degrees are independent of the minimal generating sets our Gröbner bases will determine all the candidate vectors $\mathbf{b} \in \mathbb{N}\mathcal{A}$.

For a vector $\mathbf{b} \in \mathbb{N}\mathcal{A}$, $G(\mathbf{b})$ is the graph with vertices the elements of the fiber

$$deg_{\mathcal{A}}^{-1}(\mathbf{b}) = \{x^{\mathbf{p}} \mid deg_{\mathcal{A}}(x^{\mathbf{p}}) = \mathbf{b}\}$$

and edges all the sets $\{x^{\mathbf{p}}, x^{\mathbf{q}}\}$, whenever $x^{\mathbf{p}} - x^{\mathbf{q}} \in I_{\mathcal{A},\mathbf{b}}$, where the ideal $I_{\mathcal{A},\mathbf{b}}$ is defined by $I_{\mathcal{A},\mathbf{b}} = \langle x^{\mathbf{p}} - x^{\mathbf{p}} \mid deg_{\mathcal{A}}(x^{\mathbf{p}}) = deg_{\mathcal{A}}(x^{\mathbf{q}}) \lneqq \mathbf{b}\rangle$.

For each possible *Betti* $\mathcal{A}$-degree $\mathbf{b}$, we consider the complete graph $S_{\mathbf{b}}$ with vertices $G(\mathbf{b})_i$, the connected components of $G(\mathbf{b})$. Let $T_{\mathbf{b}}$ be a spanning tree of $S_{\mathbf{b}}$. Then $\mathcal{F}_{T_{\mathbf{b}}}$ is the collection of binomials $x^{\mathbf{p}} - x^{\mathbf{q}}$ corresponding to edges $\{x^{\mathbf{p}}, x^{\mathbf{q}}\}$ of $T_{\mathbf{b}}$ with $x^{\mathbf{p}} \in G(\mathbf{b})_i$ and $x^{\mathbf{q}} \in G(\mathbf{b})_j$. We will use the following to show the minimality of the generating sets given by the Gröbner bases presented in section 3.

4.1. Theorem. [6, Theorem 2.6]. *$\mathcal{F} = \bigcup_{\mathbf{b}\in\mathbb{N}\mathcal{A}} \mathcal{F}_{T_{\mathbf{b}}}$ is a minimal generating set of $I_{\mathcal{A}}$.*

Notice that if $\mathbf{b}$ is not a *Betti*$\mathcal{A}$-degree, then $\mathcal{F}_{T_{\mathbf{b}}} = \emptyset$ and that the number of possible spanning trees determine the number of different minimal generating sets.

4.1. Even Case $\mathcal{D}_{2m}$. We consider the subset $\mathcal{D}_{2m}$ defined by,

$$\mathcal{D}_{2m} := \{2\mathbf{e}_i, \mathbf{e}_j, 2\mathbf{e}_1, 2\mathbf{e}_n, \mathbf{e}_k + \mathbf{e}_\ell, \mathbf{e}_i + \mathbf{e}_1 + \mathbf{e}_n \,|\, i, k, \ell \in J,\, j \in J^c \text{ and } k < \ell\},$$

where $J = \{3, 5, \dots, n-1\}$, $J^c = \{2, 4, \dots, n-2\}$ and $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ is the canonical basis of $\mathbb{Z}^n$. Recall that the elements of $\mathcal{D}_{2m}$ are the $\mathcal{D}_{2m}$-degrees of the variables x_i, x_j, x_1, x_n, $x_{k,l}$ and y_i respectively.

By Theorem 3.1, we see that $I_{\mathcal{D}_{2m}}$ is generated by the set $\mathcal{G}_{\mathcal{D}_{2m}}$

$$\begin{array}{lll}
x_{i,k}x_{j,\ell}-x_{i,j}x_{k,\ell} & x_{i,\ell}x_{j,k}-x_{i,j}x_{k,\ell} & i<j<k<\ell\in J\\
x_{i,j}x_{i,k}-x_ix_{j,k} & x_jx_{i,k}-x_{i,j}x_{j,k} & i<j<k\in J\\
x_kx_{i,j}-x_{i,k}x_{j,k} & x_{j,k}y_i-x_{i,j}y_k & i<j<k\in J\\
x_{i,k}y_j-x_{i,j}y_k & & i<j<k\in J\\
x_ix_j-x_{i,j}^2 & x_jy_i-x_{i,j}y_j & i<j\in J\\
x_{i,j}y_i-x_iy_j & x_{i,j}x_1x_n-y_iy_j & i<j\in J\\
x_ix_1x_n-y_i^2 & & i\in J.
\end{array}$$

Therefore, possible *Betti* $\mathcal{D}_{2m}$-degrees are

$$\begin{array}{ll}
\mathbf{b_1}=2\mathbf{e}_i+2\mathbf{e}_j, & \mathbf{b_2}=\mathbf{e}_i+\mathbf{e}_j+2\mathbf{e}_1+2\mathbf{e}_n,\\
\mathbf{b_3}=2\mathbf{e}_i+\mathbf{e}_j+\mathbf{e}_1+\mathbf{e}_n, & \mathbf{b_4}=\mathbf{e}_i+2\mathbf{e}_j+\mathbf{e}_1+\mathbf{e}_n,\\
\mathbf{b_5}=2\mathbf{e}_i+\mathbf{e}_j+\mathbf{e}_k, & \mathbf{b_6}=\mathbf{e}_i+2\mathbf{e}_j+\mathbf{e}_k,\\
\mathbf{b_7}=\mathbf{e}_i+\mathbf{e}_j+2\mathbf{e}_k, & \mathbf{b_8}=\mathbf{e}_i+\mathbf{e}_j+\mathbf{e}_k+\mathbf{e}_1+\mathbf{e}_n,\\
\mathbf{b_9}=2\mathbf{e}_i+2\mathbf{e}_1+2\mathbf{e}_n, & \mathbf{b_{10}}=\mathbf{e}_i+\mathbf{e}_j+\mathbf{e}_k+\mathbf{e}_\ell
\end{array}$$

Next we prove that these binomials constitute a minimal generating set for $I_{\mathcal{D}_{2m}}$.

4.2. Proposition. *The set $\mathcal{G}_{\mathcal{D}_{2m}}$ is a minimal generating set of $I_{\mathcal{D}_{2m}}$.*

Proof. Since there is no binomial in $I_{\mathcal{D}_{2m},\mathbf{b_1}}$, $G(\mathbf{b_1})$ consists of two connected components $\{x_ix_j\}$ and $\{x_{i,j}^2\}$. Similarly, $G(\mathbf{b_2})$ has $\{x_{i,j}x_1x_n\}$ and $\{y_iy_j\}$, $G(\mathbf{b_3})$ has $\{x_{i,j}y_i\}$ and $\{x_iy_j\}$, $G(\mathbf{b_4})$ has $\{x_jy_i\}$ and $\{x_{i,j}y_j\}$, $G(\mathbf{b_5})$ has $\{x_{i,j}x_{i,k}\}$ and $\{x_ix_{j,k}\}$, $G(\mathbf{b_6})$ has $\{x_jx_{i,k}\}$ and $\{x_{i,j}x_{j,k}\}$, $G(\mathbf{b_7})$ has $\{x_kx_{i,j}\}$ and $\{x_{i,k}x_{j,k}\}$, $G(\mathbf{b_9})$ has $\{x_ix_1x_n\}$ and $\{y_i^2\}$ as its connected components.

By Corollary 2.10 in [6], these graphs determine all indispensable binomials of $I_{\mathcal{D}_{2m}}$. Since these binomials are indispensable, they must belong to any minimal generating set. Let us find the other binomials needed to obtain a minimal generating set for $I_{\mathcal{D}_{2m}}$.

$G(\mathbf{b_8})$ and $G(\mathbf{b_{10}})$ have three connected components: $\{x_{i,j}y_k\}\cup\{x_{j,k}y_i\}\cup\{x_{i,k}y_j\}$ and $\{x_{i,j}x_{k,\ell}\}\cup\{x_{i,k}x_{j,\ell}\}\cup\{x_{i,\ell}x_{j,k}\}$, respectively. Since each connected component of these graphs is a singleton, the complete graphs $S_{\mathbf{b_8}}$ and $S_{\mathbf{b_{10}}}$ are triangles obtained by joining connected components of $G(\mathbf{b_8})$ and $G(\mathbf{b_{10}})$, respectively. Thus, spanning trees of these complete graphs can be obtained by deleting one edge from the triangle.

Therefore, in a minimal generating set only one of the following three binomial couples may appear corresponding to $G(\mathbf{b_8})$;

$$\begin{gathered}
x_{i,j}y_k-x_{j,k}y_i \text{ and } x_{i,j}y_k-x_{i,k}y_j, \text{ or}\\
x_{j,k}y_i-x_{i,j}y_k \text{ and } x_{j,k}y_i-x_{i,k}y_j, \text{ or}\\
x_{i,k}y_j-x_{i,j}y_k \text{ and } x_{i,k}y_j-x_{j,k}y_i
\end{gathered}$$

and similarly for $G(\mathbf{b_{10}})$;

$$\begin{gathered}
x_{i,j}x_{k,\ell}-x_{i,k}x_{j,\ell} \text{ and } x_{i,j}x_{k,\ell}-x_{i,\ell}x_{j,k}, \text{ or}\\
x_{i,k}x_{j,\ell}-x_{i,j}x_{k,\ell} \text{ and } x_{i,k}x_{j,\ell}-x_{i,\ell}x_{j,k}, \text{ or}\\
x_{i,\ell}x_{j,k}-x_{i,j}x_{k,\ell} \text{ and } x_{i,\ell}x_{j,k}-x_{i,k}x_{j,\ell}.
\end{gathered}$$

Hence, there are many different minimal generating sets for the toric ideal $I_{\mathcal{D}_{2m}}$, and in particular the set $\mathcal{G}_{\mathcal{D}_{2m}}$ is a minimal generating set of $I_{\mathcal{D}_{2m}}$. □

4.2. Odd Case $\mathcal{D}_{2m+1}$. In this case, we consider the set $\mathcal{D}_{2m+1}\subset\mathbb{Z}^n$ given by

$$\{2\mathbf{e}_i,\mathbf{e}_j,4\mathbf{e}_1,4\mathbf{e}_n,\mathbf{e}_k+\mathbf{e}_\ell,\mathbf{e}_1+\mathbf{e}_n,\mathbf{e}_i+2\mathbf{e}_1,\mathbf{e}_i+2\mathbf{e}_n,\mathbf{e}_i+3\mathbf{e}_1+\mathbf{e}_n,\mathbf{e}_i+\mathbf{e}_1+3\mathbf{e}_n$$
$$\mid i,k,\ell\in J,\ j\in J^c \text{ and } k<\ell\},$$

where $J = \{2, 4, \dots, n-1\}$, $J^c = \{3, 5, \dots, n-2\}$ and $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ is the canonical basis of $\mathbb{Z}^n$. Again, the $\mathcal{D}_{2m+1}$-degrees of the variables are exactly the elements of $\mathcal{D}_{2m+1}$ as before.

By Theorem 3.3, we see that $I_{\mathcal{D}_{2m+1}}$ is generated by the set $\mathcal{G}_{\mathcal{D}_{2m+1}}$ of binomials

$x_{i,k}x_{j,\ell} - x_{i,j}x_{k,\ell}$	$x_{i,\ell}x_{j,k} - x_{i,j}x_{k,\ell}$	$i < j < k < \ell \in J$
$x_{j,k}x_{i,n-1} - x_{i,j}x_{k,n-1}$	$x_{i,k}x_{j,n-1} - x_{i,j}x_{k,n-1}$	$i < j < k \in J$
$x_{j,k}x_{i,n} - x_{i,j}x_{k,n}$	$x_{i,k}x_{j,n} - x_{i,j}x_{k,n}$	$i < j < k \in J$
$x_jx_{i,k} - x_{i,j}x_{j,k}$	$x_{i,j}x_{i,k} - x_ix_{j,k}$	$i < j < k \in J$
$x_kx_{i,j} - x_{i,k}x_{j,k}$		$i < j < k \in J$
$x_ix_j - x_{i,j}^2$	$x_{i,j}x_1 - x_{i,n-1}x_{j,n-1}$	$i < j \in J$
$x_{i,j}x_n - x_{i,n}x_{j,n}$	$x_{i,j}x_{i,n-1} - x_ix_{j,n-1}$	$i < j \in J$
$x_{i,j}x_{i,n} - x_ix_{j,n}$	$x_jx_{i,n-1} - x_{i,j}x_{j,n-1}$	$i < j \in J$
$x_jx_{i,n} - x_{i,j}x_{j,n}$	$x_{j,n-1}x_{i,n} - x_{i,n-1}x_{j,n}$	$i < j \in J$
$x_{i,n-1}x_{j,n} - x_{1,n}^2x_{i,j}$		$i < j \in J$
$x_ix_1 - x_{i,n-1}^2$	$x_ix_n - x_{i,n}^2$	$i \in J$
$x_{i,n-1}x_{i,n} - x_{1,n}^2x_i$	$x_{i,n}x_1 - x_{1,n}^2x_{i,n-1}$	$i \in J$
$y_{i,1} - x_{1,n}x_{i,1}$	$y_{i,n} - x_{1,n}x_{i,n}$	$i \in J$
$x_{i,n-1}x_n - x_{1,n}^2x_{i,n}$	$x_1x_n - x_{1,n}^4$	$i \in J$

Therefore, possible Betti $\mathcal{D}_{2m+1}$-degrees are

$$\begin{array}{ll}
\mathbf{b_1} = 2\mathbf{e}_i + 2\mathbf{e}_j, & \mathbf{b_2} = 4\mathbf{e}_1 + \mathbf{e}_i + \mathbf{e}_j,\\
\mathbf{b_3} = \mathbf{e}_i + \mathbf{e}_j + 4\mathbf{e}_n, & \mathbf{b_4} = 2\mathbf{e}_1 + 2\mathbf{e}_i + \mathbf{e}_j,\\
\mathbf{b_5} = 2\mathbf{e}_i + \mathbf{e}_j + 2\mathbf{e}_n, & \mathbf{b_6} = 2\mathbf{e}_1 + \mathbf{e}_i + 2\mathbf{e}_j,\\
\mathbf{b_7} = \mathbf{e}_i + 2\mathbf{e}_j + 2\mathbf{e}_n, & \mathbf{b_8} = 2\mathbf{e}_1 + \mathbf{e}_i + \mathbf{e}_j + 2\mathbf{e}_n,\\
\mathbf{b_9} = 2\mathbf{e}_1 + \mathbf{e}_i + \mathbf{e}_j + \mathbf{e}_k, & \mathbf{b_{10}} = \mathbf{e}_i + \mathbf{e}_j + \mathbf{e}_k + 2\mathbf{e}_n,\\
\mathbf{b_{11}} = \mathbf{e}_i + 2\mathbf{e}_j + \mathbf{e}_k, & \mathbf{b_{12}} = 2\mathbf{e}_i + \mathbf{e}_j + \mathbf{e}_k,\\
\mathbf{b_{13}} = \mathbf{e}_i + \mathbf{e}_j + 2\mathbf{e}_k, & \mathbf{b_{14}} = \mathbf{e}_i + \mathbf{e}_j + \mathbf{e}_k + \mathbf{e}_\ell,\\
\mathbf{b_{15}} = 4\mathbf{e}_1 + 2\mathbf{e}_i, & \mathbf{b_{16}} = 2\mathbf{e}_i + 4\mathbf{e}_n,\\
\mathbf{b_{17}} = 2\mathbf{e}_1 + 2\mathbf{e}_i + 2\mathbf{e}_n, & \mathbf{b_{18}} = 4\mathbf{e}_1 + \mathbf{e}_i + 2\mathbf{e}_n,\\
\mathbf{b_{19}} = 3\mathbf{e}_1 + \mathbf{e}_i + \mathbf{e}_n, & \mathbf{b_{20}} = \mathbf{e}_1 + \mathbf{e}_i + 3\mathbf{e}_n\\
\mathbf{b_{21}} = 2\mathbf{e}_1 + \mathbf{e}_i + 4\mathbf{e}_n, & \mathbf{b_{22}} = 4\mathbf{e}_1 + 4\mathbf{e}_n.
\end{array}$$

Next we prove that these binomials constitute a minimal generating set for $I_{\mathcal{D}_{2m+1}}$.

4.3. Proposition. *The set $\mathcal{G}_{\mathcal{D}_{2m+1}}$ is a minimal generating set of $I_{\mathcal{D}_{2m+1}}$.*

Proof. There is no binomial in $I_{\mathcal{D}_{2m+1},\mathbf{b_1}}$. Thus, the graph $G(\mathbf{b_1})$ consists of two connected components $\{x_ix_j\}$ and $\{x_{i,j}^2\}$. Similarly, $G(\mathbf{b_2})$ has $\{x_{i,j}x_1\}$ and $\{x_{i,n-1}x_{j,n-1}\}$, $G(\mathbf{b_3})$ has $\{x_{i,j}x_n\}$ and $\{x_{i,n}x_{j,n}\}$, $G(\mathbf{b_4})$ has $\{x_{i,j}x_{i,n-1}\}$ and $\{x_ix_{j,n-1}\}$, $G(\mathbf{b_5})$ has $\{x_{i,j}x_{i,n}\}$ and $\{x_ix_{j,n}\}$, $G(\mathbf{b_6})$ has $\{x_jx_{i,n-1}\}$ and $\{x_{i,j}x_{j,n-1}\}$, $G(\mathbf{b_7})$ has $\{x_jx_{i,n}\}$ and $\{x_{i,j}x_{j,n}\}$, $G(\mathbf{b_{11}})$ has $\{x_jx_{i,k}\}$ and $\{x_{i,j}x_{j,k}\}$, $G(\mathbf{b_{12}})$ has $\{x_{i,j}x_{i,k}\}$ and $\{x_ix_{j,k}\}$, $G(\mathbf{b_{13}})$ has $\{x_kx_{i,j}\}$ and $\{x_{i,k}x_{j,k}\}$, $G(\mathbf{b_{15}})$ has $\{x_i, x_1\}$ and $\{x_{i,n-1}^2\}$, $G(\mathbf{b_{16}})$ has $\{x_i, x_n\}$ and $\{x_{i,n}^2\}$, $G(\mathbf{b_{17}})$ has $\{x_{i,n-1}x_{i,n}\}$ and $\{x_{1,n}^2x_i\}$, $G(\mathbf{b_{18}})$ has $\{x_{i,n}x_1\}$ and $\{x_{1,n}^2x_{i,n-1}\}$, $G(\mathbf{b_{19}})$ has $\{y_{i,1}\}$ and $\{x_{1,n}x_{i,1}\}$, $G(\mathbf{b_{20}})$ has $\{y_{i,n}\}$ and $\{x_{1,n}x_{i,n}\}$, $G(\mathbf{b_{21}})$ has $\{x_{i,n-1}x_n\}$ and $\{x_{1,n}^2x_{i,n}\}$, and finally $G(\mathbf{b_{22}})$ has $\{x_1x_n\}$ and $\{x_{1,n}^4\}$ as its connected components.

Indispensable binomials of $I_{\mathcal{D}_{2m+1}}$ are all determined by these graphs by Corollary 2.10 in [6] and hence, corresponding binomials belong to any minimal generating set.

The other graphs $G(\mathbf{b_8})$, $G(\mathbf{b_9})$, $G(\mathbf{b_{10}})$ and $G(\mathbf{b_{14}})$ have three connected components:

$$\{x_{i,n-1}x_{j,n}\} \cup \{x_{j,n-1}x_{i,n}\} \cup \{x_{1,n}^2x_{i,j}\}, \{x_{j,k}x_{i,n-1}\} \cup \{x_{i,j}x_{k,n-1}\} \cup \{x_{i,k}x_{j,n-1}\},$$

$$\{x_{j,k}x_{i,n}\} \cup \{x_{i,j}x_{k,n}\} \cup \{x_{i,k}x_{j,n}\} \text{ and } \{x_{i,k}x_{j,\ell}\} \cup \{x_{i,j}x_{k,\ell}\} \cup \{x_{i,\ell}x_{j,k}\}$$

respectively. Each connected component of these graphs is a singleton. Therefore, the complete graphs are triangles obtained by joining the connected components of the graphs $G(\mathbf{b_8})$, $G(\mathbf{b_9})$, $G(\mathbf{b_{10}})$ and $G(\mathbf{b_{14}})$, respectively. Thus, we obtain the spanning trees by deleting one edge from each triangle.

Therefore, in a minimal generating set only one of the following three binomial couples may appear corresponding to $G(\mathbf{b_8})$;

$$\begin{gathered} x_{i,n-1}x_{j,n} - x_{1,n}^2x_{i,j} \text{ and } x_{i,n-1}x_{j,n} - x_{j,n-1}x_{i,n}, \\ x_{1,n}^2x_{i,j} - x_{i,n-1}x_{j,n} \text{ and } x_{1,n}^2x_{i,j} - x_{j,n-1}x_{i,n}, \\ x_{j,n-1}x_{i,n} - x_{i,n-1}x_{j,n} \text{ and } x_{j,n-1}x_{i,n} - x_{1,n}^2x_{i,j} \end{gathered}$$

and the same is true for the following couples corresponding to $G(\mathbf{b_9})$;

$$\begin{gathered} x_{j,k}x_{i,n-1} - x_{i,j}x_{k,n-1} \text{ and } x_{j,k}x_{i,n-1} - x_{i,k}x_{j,n-1}, \\ x_{i,j}x_{k,n-1} - x_{j,k}x_{i,n-1} \text{ and } x_{i,j}x_{k,n-1} - x_{i,k}x_{j,n-1}, \\ x_{i,k}x_{j,n-1} - x_{j,k}x_{i,n-1} \text{ and } x_{i,k}x_{j,n-1} - x_{i,j}x_{k,n-1} \end{gathered}$$

and similarly for $G(\mathbf{b_{10}})$;

$$\begin{gathered} x_{j,k}x_{i,n} - x_{i,j}x_{k,n} \text{ and } x_{j,k}x_{i,n} - x_{i,k}x_{j,n}, \\ x_{i,j}x_{k,n} - x_{j,k}x_{i,n} \text{ and } x_{i,j}x_{k,n} - x_{i,k}x_{j,n}, \\ x_{i,k}x_{j,n} - x_{j,k}x_{i,n} \text{ and } x_{i,k}x_{j,n} - x_{i,j}x_{k,n} \end{gathered}$$

and for $G(\mathbf{b_{14}})$;

$$\begin{gathered} x_{i,k}x_{j,\ell} - x_{i,j}x_{k,\ell} \text{ and } x_{i,k}x_{j,\ell} - x_{i,\ell}x_{j,k}, \\ x_{i,j}x_{k,\ell} - x_{i,k}x_{j,\ell} \text{ and } x_{i,j}x_{k,\ell} - x_{i,\ell}x_{j,k}, \\ x_{i,\ell}x_{j,k} - x_{i,j}x_{k,\ell} \text{ and } x_{i,\ell}x_{j,k} - x_{i,k}x_{j,\ell}. \end{gathered}$$

These discussions show that there are many minimal generating sets for $I_{\mathcal{D}_{2m+1}}$ and in particular, the set $\mathcal{G}_{\mathcal{D}_{2m+1}}$ is a minimal generating set of $I_{\mathcal{D}_{2m+1}}$. $\square$

4.3. E_n**-type.** In this case, it is easy to check that the Gröbner basis given in Theorem 3.5 constitutes a minimal generating set for each $n = 6$, 7, 8. Indeed, there is nothing to prove for the case of $n = 8$, as the corresponding toric ideal is trivial. In the case of $n = 7$, the corresponding toric ideal is generated minimally by the 6 binomials given in Theorem 3.5 (2) as we explain now. Let $\mathbf{b}$ be the $\mathcal{E}_7$-degree of a binomial given in Theorem 3.5 (2). Since the graph $G(\mathbf{b})$ has two connected components, the complete graph $S_{\mathbf{b}}$ (and its spanning tree $T_{\mathbf{b}}$) is a line segment and thus $\mathcal{F}_{T_b}$ is a singleton. As the connected components of $G(\mathbf{b})$ are singletons, $\mathcal{F}_{T_b}$ *must* consist of the binomial we have started with. This means that the binomial is *indispensable*, i.e. appears in any minimal generating set. Therefore the toric ideal has a unique minimal generating set provided by Theorem 3.5 (2).

As for the case of $n = 6$, we have a generating set given in Theorem 3.5 (1) consisting of 35 binomials. Let $\mathbf{b} = 2\mathbf{e}_1 + 2\mathbf{e}_2 + \mathbf{e}_4 + \mathbf{e}_5$ which is the $\mathcal{E}_6$-degree of the binomial $x_{11}x_{13} - x_7x_8x_9$. The graph $G(\mathbf{b})$ has two connected components $\{x_{11}x_{13}\}$ and $\{x_7x_8x_9, x_7^2x_{10}\}$. As before the complete graph $S_{\mathbf{b}}$ (and its spanning tree $T_{\mathbf{b}}$) is a line segment and thus $\mathcal{F}_{T_b}$ is a singleton but it changes according to which monomial we choose from the second component of $G(\mathbf{b})$. So, $\mathcal{F}_{T_b}$ is either $\{x_{11}x_{13} - x_7x_8x_9\}$ or $\{x_{11}x_{13} - x_7^2x_{10}\}$. We have the same situation for the following degrees:

$$\begin{aligned} \mathbf{b} &= \mathbf{e}_1 + 2\mathbf{e}_2 + 2\mathbf{e}_4 + \mathbf{e}_5, \\ \mathbf{b} &= 2\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_4 + 2\mathbf{e}_5, \\ \mathbf{b} &= \mathbf{e}_1 + \mathbf{e}_2 + 2\mathbf{e}_4 + 2\mathbf{e}_5. \end{aligned}$$

It is a standard procedure to check that the other 31 binomials given in Theorem 3.5 (1) are indispensable, so there are 8 different minimal generating sets for the toric ideal including the one provided by Theorem 3.5 (1).

5. What about A_n-type?

There are two ways to study the question of whether or not toric ideals of these configurations have squarefree initial ideals. The first one is to produce an example with no squarefree initial ideal using computer programs. In order to achieve this goal one has to find all possible initial ideals for a fixed configuration. The toric ideal corresponding to A_2 is generated by a binomial with a squarefree monomial. One can compute 29 different initial ideals for the toric ideal of A_3 and obtain the unique squarefree one generated by 6 monomials by using e.g. Gfan [12]. As long as n gets larger values listing all the possible initial ideals (or regular triangulations of the corresponding convex polytope) using computer programs becomes problematic. In the second way, one has to determine the correct term order with respect to which the initial ideal is generated by squarefree monomials by heuristic/experimental methods. For the toric ideal of A_4 the lexicographic ordering with $x_{14} > x_{12} > x_{10} > x_9 > x_7 > x_4 > x_8 > x_6 > x_5 > x_3 > x_{11} > x_1 > x_2$ gives a Gröbner basis consisting of 54 binomials with a squarefree initial ideal. Similarly, the toric ideal of A_5 has a squarefree initial ideal generated by 105 monomials which are obtained as the initial terms with respect to the lexicographic ordering with $x_{19} > x_{18} > x_{17} > x_{11} > x_{10} > x_3 > x_{16} > x_{13} > x_7 > x_{15} > x_{14} > x_{12} > x_8 > x_5 > x_9 > x_4 > x_6 > x_2 > x_1$. However, for larger values of n, proving the existence of squarefree initial ideals is difficult as well. This is due to the fact that there is no general formula for the vector configuration as in the case of D-type, although one can compute them one by one with e.g. CoCoA using the algorithm described in [21].

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