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DOI: 10.7153/mia-18-100

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JACKSON–STECHKIN TYPE INEQUALITY IN WEIGHTED LORENTZ SPACES

RAMAZAN AKGÜN AND YUNUS EMRE YILDIRIR

(Communicated by J. Pečarić)

Abstract. In the present work we consider the modulus of smoothness, defined by means of the Steklov operator in weighted Lorentz spaces and prove the Jackson–Stechkin type direct theorem of trigonometric approximation. In the particular case we obtain a result on the constructive characterization of the generalized Lipschitz classes defined in these spaces. Simultaneous approximation of functions is also considered.

1. Introduction and the main results

Jackson–Stechkin type inequalities in the normed space under consideration estimate the order of decrease of the best approximation of a function by a finite dimensional subspace in terms of some characteristic of its smoothness.

The starting point here is the classical theorem of Jackson ([8]) on the best uniform approximation of a periodic function f by trigonometric polynomials of degree $\leq n$:

For any 2π -periodic continuous function f , the following inequality holds

$$E_n(f) \leq C\omega\left(f, \frac{1}{n+1}\right).$$

In this inequality, $E_n(f)$ denotes the best approximation of the function f by trigonometric polynomials of degree $\leq n$, i.e.,

$$E_n(f) := \inf_{T_n \in \mathcal{T}_n} \max_{x \in [0, 2\pi]} |f(x) - T_n(x)|,$$

where \mathcal{T}_n is the class of trigonometric polynomials of degree $\leq n$, and

$$\omega(f, \delta) := \sup_{|h| \leq \delta} \max_{x \in [0, 2\pi]} |f(x+h) - f(x)|$$

denotes the modulus of continuity of f .

In [15], Stechkin proved an analog of Jackson’s inequality for the Lebesgue spaces L^p , $1 \leq p \leq \infty$. The elegant representation of the corresponding results in the Lebesgue

Mathematics subject classification (2010): 41A25, 41A27, 42A10.

Keywords and phrases: Modulus of smoothness, trigonometric polynomials, weighted Lorentz spaces, Muckenhoupt weight, direct and inverse theorem.

spaces L^p , $1 \leq p < \infty$, can be found in [3, 16, 17]. In weighted Lebesgue spaces with weights satisfying the Muckenhoupt's condition A_p , $1 < p < \infty$, the direct theorem of trigonometric approximation in the following form was proved in [7].

$$E_n(f)_{L_w^p} \lesssim \omega_r\left(f, \frac{1}{n}\right)_{L_w^p} := \sup_{0 \leq h_i \leq 1/n} \left\| \prod_{i=1}^r (I - \sigma_{h_i}) f \right\|_{L_w^p}, \quad r \in \mathbb{N},$$

where I is the identity operator on $\mathbf{T} := [-\pi, \pi)$ and

$$\sigma_h f(x) := \frac{1}{2h} \int_{x-h}^{x+h} f(u) du, \quad x \in \mathbf{T}.$$

The shift operator σ_h and the modulus of smoothness $\omega_r(f, \cdot)_{L_w^p}$ are defined in this way, because the weighted Lebesgue space L_w^p is not, in general, invariant under the usual shift $f(\cdot) \rightarrow f(\cdot + h)$.

Some interesting results concerning to the best polynomial approximation in weighted Lebesgue spaces were also proved in [4, 5, 11, 19]. In weighted Lorentz spaces some converse theorems were obtained in [10, 18]. The detailed information on the weighted polynomial approximation can be found in the books [6, 12].

A measurable function $w : \mathbf{T} \rightarrow [0, \infty]$ is called a weight function if the preimage $w^{-1}(\{0, \infty\})$ has Lebesgue measure zero. Let w be a weight function and $f_w^*(t)$ be a decreasing rearrangement of $f : \mathbf{T} \rightarrow \mathbb{R}$ with respect to the Borel measure

$$w(e) = \int_e w(x) dx,$$

i.e.,

$$f_w^*(t) = \inf\{\tau \geq 0 : w(x \in \mathbf{T} : |f(x)| > \tau) \leq t\}.$$

Let $1 < p, q < \infty$ and let $L_w^{pq}(\mathbf{T})$ be a weighted Lorentz space, i.e., the set of all measurable functions for which

$$\|f\|_{L_w^{pq}} = \left(\int_{\mathbf{T}} (f^{**}(t))^q t^{\frac{q}{p}} \frac{dt}{t} \right)^{1/q} < \infty,$$

where

$$f^{**}(t) = \frac{1}{t} \int_0^t f_w^*(u) du.$$

If $p = q$, $L_w^{pq}(\mathbf{T})$ is turn into the weighted Lebesgue space $L_w^p(\mathbf{T})$.

The weights w used in the paper are those which belong to the Muckenhoupt's ([13]) class $A_p(\mathbf{T})$, i.e., they satisfy the condition

$$\sup \frac{1}{|I|} \int_I w(x) dx \left(\frac{1}{|I|} \int_I w^{1-p'}(x) dx \right)^{p-1} = C_{A_p} < \infty, \quad p' := \frac{p}{p-1}$$

where the supremum is taken with respect to all the intervals I with length $\leq 2\pi$ and $|I|$ denotes the length of I . The constant C_{A_p} is called the Muckenhoupt constant of w .

The modulus of smoothness of a function $f \in L_w^{pq}(\mathbf{T})$ is given by

$$\Omega_r(f, \delta)_{L_w^{pq}} := \sup_{0 \leq h_i \leq \delta, i=1, \dots, r} \left\| \prod_{i=1}^r (I - \sigma_{h_i}) f \right\|_{L_w^{pq}}, \quad r \in \mathbb{N}.$$

Whenever $w \in A_p(\mathbf{T})$, $1 < p, q < \infty$, the Hardy-Littlewood maximal function of $f \in L_w^{pq}(\mathbf{T})$ belongs to $L_w^{pq}(\mathbf{T})$ ([2, Theorem 3]). Therefore the average $\sigma_n f$ belongs to $L_w^{pq}(\mathbf{T})$. Thus $\Omega_r(f, \delta)_{L_w^{pq}}$ makes sense for every $w \in A_p(\mathbf{T})$.

By $E_n(f)_{L_w^{pq}}$ we denote the best approximation of $f \in L_w^{pq}(\mathbf{T})$ by trigonometric polynomials of degree $\leq n$, i.e.,

$$E_n(f)_{L_w^{pq}} = \inf_{T_n \in \mathcal{T}_n} \|f - T_n\|_{L_w^{pq}}.$$

Since $L_w^{pq}(\mathbf{T}) \subset L^1(\mathbf{T})$ when $w \in A_p(\mathbf{T})$, $1 < p, q < \infty$ (see [10, the proof of Prop. 3.3]), we can define the Fourier series of $f \in L_w^{pq}(\mathbf{T})$

$$f(x) \sim \frac{a_0(f)}{2} + \sum_{k=1}^{\infty} (a_k(f) \cos kx + b_k(f) \sin kx) \tag{1.1}$$

and the conjugate Fourier series

$$\tilde{f}(x) \sim \sum_{k=1}^{\infty} (a_k(f) \sin kx - b_k(f) \cos kx).$$

Here $a_0(f), a_k(f), b_k(f), k = 1, \dots$, are Fourier coefficients of f .

The relation \lesssim is defined as “ $A \lesssim B \Leftrightarrow$ there exists a positive constant C , independent of essential parameters, such that $A \leq CB$.”

In this work we prove the direct and simultaneous theorems of approximation theory in the weighted Lorentz spaces using the modulus of smoothness $\Omega_r(f, \cdot)_{L_w^{pq}}$.

Our new results are the following.

THEOREM 1. *Let $w \in A_p(\mathbf{T})$, $1 < p, q < \infty$, $r \in \mathbb{N}$. Then for every $f \in W_{pq,w}^r$, the inequality*

$$\|f - S_n(f)\|_{L_w^{pq}} \lesssim \frac{1}{(n+1)^r} \left\| f^{(r)} - S_n(f^{(r)}) \right\|_{L_w^{pq}}, \quad n \in \mathbb{N}$$

holds with a positive constant depending only on r, p, q and the Muckenhoupt constant C_{A_p} of w , where $S_n(f)$ denotes the n .th partial sum of the Fourier series (1.1) of f .

We define $W_{pq,w}^r := \left\{ g \in L_w^{pq} : g^{(r)} \in L_w^{pq} \right\}$. Theorem 1 gives the following corollary.

COROLLARY 1. Let $w \in A_p(\mathbf{T})$, $1 < p, q < \infty$, $r, n \in \mathbb{N}$. Then for every $f \in W_{pq,w}^r$, the inequalities

$$E_n(f)_{L_w^{pq}} \lesssim \frac{1}{(n+1)^r} E_n(f^{(r)})_{L_w^{pq}},$$

and

$$E_n(f)_{L_w^{pq}} \lesssim \frac{1}{(n+1)^r} \|f^{(r)}\|_{L_w^{pq}},$$

hold with some positive constants depending only on r, p, q and the Muckenhoupt constant C_{A_p} of w .

THEOREM 2. Let $f \in L_w^{pq}(\mathbf{T})$, $w \in A_p(\mathbf{T})$, $1 < p, q < \infty$. Then we have the following estimate

$$E_n(f)_{L_w^{pq}} \lesssim \Omega_r \left(f, \frac{1}{n} \right)_{L_w^{pq}}, \quad r \in \mathbb{N}, \tag{1.2}$$

and

$$\|f - S_n(f)\|_{L_w^{pq}} \lesssim \Omega_r \left(f, \frac{1}{n} \right)_{L_w^{pq}} \tag{1.3}$$

for $n \in \mathbb{N}$, with some constants depending only on r, p, q and the Muckenhoupt constant C_{A_p} of w .

We note that the sharp inverse inequality to the Jackson-Stechkin type inequality was proved in [10]. In the sequel we use a weak version of inverse estimate:

Let $f \in L_w^{pq}(\mathbf{T})$, $w \in A_p(\mathbf{T})$, $1 < p, q < \infty$. Then

$$\Omega_r \left(f, \frac{1}{n} \right)_{L_w^{pq}} \lesssim \frac{1}{n^{2r}} \sum_{k=0}^n (k+1)^{2r-1} E_k(f)_{L_w^{pq}}, \quad r \in \mathbb{N} \tag{1.4}$$

holds for $n \in \mathbb{N}$, with some constant depending only on r, p, q and the Muckenhoupt constant C_{A_p} of w .

From Theorem 2 and (1.4), we obtain the following Marchaud type inequality.

COROLLARY 2. Let $f \in L_w^{pq}(\mathbf{T})$, $w \in A_p(\mathbf{T})$, $1 < p, q < \infty$. Then we have

$$\Omega_r(f, \delta)_{L_w^{pq}} \lesssim \delta^{2r} \int_{\delta}^1 \frac{\Omega_{r+1}(f, u)_{L_w^{pq}} du}{u^{2r}}, \quad 0 < \delta < 1,$$

for $r \in \mathbb{N}$.

From Theorem 2 and (1.4), we also obtain the following estimate.

THEOREM 3. Let $f \in L_w^{pq}(\mathbf{T})$, $w \in A_p(\mathbf{T})$, $1 < p, q < \infty$. If

$$E_n(f)_{L_w^{pq}} \lesssim n^{-\alpha}, \quad n \in \mathbb{N}$$

for some $\alpha > 0$, then, for a given $r \in \mathbb{N}$, we have the estimations

$$\Omega_r(f, \delta)_{L_w^{pq}} = \begin{cases} \delta^\alpha & , r > \alpha/2; \\ \delta^{2r} \log \frac{1}{\delta} & , r = \alpha/2; \\ \delta^{2r} & , r < \alpha/2. \end{cases}$$

If we define the generalized Lipschitz class $Lip(\alpha, L_w^{pq})$ for $\alpha > 0$ and $k := [\alpha/2] + 1$, $[x] := \max\{n \in \mathbb{Z} : n \leq x\}$ as

$$Lip(\alpha, L_w^{pq}) := \left\{ f \in L_w^{pq} : \Omega_k(f, \delta)_{L_w^{pq}} \lesssim \delta^\alpha, \quad \delta > 0 \right\},$$

then by virtue of Theorems 3 and 2 we obtain the following result which gives a constructive characterization of the Lipschitz classes $Lip(\alpha, L_w^{pq})$.

COROLLARY 3. *Let $f \in L_w^{pq}(\mathbf{T})$, $w \in A_p(\mathbf{T})$, $1 < p, q < \infty$ and $\alpha > 0$. The following assertions are equivalent.*

$$(i) f \in Lip(\alpha, L_w^{pq}) \quad (ii) E_n(f)_{L_w^{pq}} \lesssim n^{-\alpha}, \quad n \in \mathbb{N}.$$

Jackson’s second type inequality is given in the following theorem.

THEOREM 4. *Let $w \in A_p(\mathbf{T})$, $1 < p, q < \infty$ and $r, k \in \mathbb{N}$. Then for every $f \in W_{pq,w}^r$, the inequality*

$$E_n(f)_{L_w^{pq}} \lesssim \frac{1}{(n+1)^r} \Omega_k\left(f^{(r)}, \frac{1}{n}\right)_{L_w^{pq}}, \quad n \in \mathbb{N}$$

holds with a positive constant depending only on r, p, q and the Muckenhoupt constant C_{A_p} of w .

Simultaneous approximation estimates are given in the next two theorems.

THEOREM 5. *Let $w \in A_p(\mathbf{T})$, $1 < p, q < \infty$ and $r, k, l \in \mathbb{N}$. Then for every $f \in W_{pq,w}^r$ and $0 \leq k \leq r$ the inequality*

$$\left\| f^{(k)} - S_n^{(k)}(f) \right\|_{L_w^{pq}} \lesssim \frac{1}{n^{r-k}} \Omega_l\left(f^{(r)}, \frac{1}{n}\right)_{L_w^{pq}}, \quad n \in \mathbb{N}$$

holds with a positive constant depending only on r, p, q and the Muckenhoupt constant C_{A_p} of w .

THEOREM 6. *Let $w \in A_p(\mathbf{T})$, $1 < p, q < \infty$ and $r, k, n \in \mathbb{N}$. Then for every $f \in W_{pq,w}^r$ and $0 \leq k \leq r$ the inequality*

$$\left\| f^{(k)} - S_n^{(k)}(f) \right\|_{L_w^{pq}} \lesssim \frac{1}{n^{r-k}} E_n\left(f^{(r)}\right)_{L_w^{pq}}$$

holds with a positive constant depending only on r, p, q and the Muckenhoupt constant C_{A_p} of w .

2. Proofs of the main results

To prove Theorem 2 we need the following lemma. If $A \lesssim B$ and $B \lesssim A$, simultaneously, we will write $A \approx B$.

For an $f \in L_w^{pq}(\mathbf{T})$ and $r \in \mathbb{N}$ the Peetre’s K -functional is defined as

$$K(f, t; L_w^{pq}, W_{pq,w}^r) := \inf_{g \in W_{pq,w}^r} \left\{ \|f - g\|_{L_w^{pq}} + t^r \|g^{(r)}(x)\|_{L_w^{pq}} \right\}$$

for $t > 0$.

LEMMA 1. Let $f \in L_w^{pq}(T)$, $r \in \mathbb{N}$, $w \in A_p(\mathbf{T})$, $1 < p, q < \infty$ and $t, k > 0$. Then we have

$$\Omega_r(f, t)_{L_w^{pq}} \approx K(f, t; L_w^{pq}, W_{pq,w}^{2r}) \tag{2.1}$$

and

$$\Omega_r(f, kt)_{L_w^{pq}} \lesssim (1 + [k])^{2r} \Omega_r(f, t)_{L_w^{pq}},$$

with some constant depending only on p, q and the Muckenhoupt constant C_{A_p} of w .

Proof. If $h \in W_{pq,w}^{2r}(\mathbf{T})$, then from subadditivity of $\Omega_r(f, \cdot)_{L_w^{pq}}$ and $\Omega_r(h, t)_{L_w^{pq}} \lesssim t^{2r} \|h^{(2r)}\|_{L_w^{pq}}$ (see Lemma 4.1 of [10])

$$\Omega_r(f, t)_{L_w^{pq}} \lesssim \|f - h\|_{L_w^{pq}} + t^{2r} \|h^{(2r)}\|_{L_w^{pq}}.$$

Taking infimum on h we get $\Omega_r(f, t)_{L_w^{pq}} \lesssim K(f, t; L_w^{pq}, W_{pq,w}^{2r})$.

We define

$$(L_\delta f)(x) := 3\delta^{-3} \int_0^\delta \int_0^u \int_{-t}^t f(x+s) ds dt du, \quad x \in \mathbf{T}.$$

From [1, p.15]

$$\frac{d^{2r}}{dx^{2r}} L_\delta^r f = \frac{c}{\delta^{2r}} (I - \sigma_\delta)^r, \quad r \in \mathbb{N}.$$

Because of estimates

$$\|L_\delta f\|_{L_w^{pq}} \lesssim \frac{\delta^{-3}}{3} \int_0^\delta \int_0^u 2t \|\sigma_t f\|_{L_w^{pq}} dt du \lesssim \|f\|_{L_w^{pq}}$$

the operator L_δ is bounded in L_w^{pq} .

Defining $A_\delta^r := I - (I - L_\delta^r)^r$ we obtain

$$\begin{aligned} \left\| \frac{d^{2r}}{dx^{2r}} A_\delta^r f \right\|_{L_w^{pq}} &\lesssim \left\| \frac{d^{2r}}{dx^{2r}} L_\delta^r f \right\|_{L_w^{pq}} = \frac{1}{\delta^{2r}} \|(I - \sigma_\delta)^r\|_{L_w^{pq}} \\ &\lesssim \frac{1}{\delta^{2r}} \Omega_r(f, \delta)_{L_w^{pq}} \end{aligned}$$

and hence $A_{\delta}^r f \in W_{pq,w}^{2r}(\mathbf{T})$. Since L_{δ} is bounded in L_w^{pq} and $I - L_{\delta}^r = (I - L_{\delta}) \sum_{j=0}^{r-1} L_{\delta}^j$ we have

$$\begin{aligned} \|(I - L_{\delta}^r)g\|_{L_w^{pq}} &\lesssim \|(I - L_{\delta})g\|_{L_w^{pq}} \lesssim \delta^{-3} \int_0^{\delta} \int_0^u 2t \|(I - \sigma_t)g\|_{L_w^{pq}} dt du \\ &\lesssim \sup_{0 < t \leq \delta} \|(I - \sigma_t)g\|_{L_w^{pq}} \end{aligned}$$

for any $g \in L_w^{pq}$.

Applying this inequality r times in $\|f - A_{\delta}^r f\|_{L_w^{pq}} = \|(I - L_{\delta}^r)^r f\|_{L_w^{pq}}$ we obtain

$$\begin{aligned} \|f - A_{\delta}^r f\|_{L_w^{pq}} &\lesssim \sup_{0 < t_1 \leq \delta} \|(I - \sigma_{t_1})(I - L_{\delta}^r)^{r-1} f\|_{L_w^{pq}} \\ &\lesssim \sup_{0 < t_1, t_2 \leq \delta} \|(I - \sigma_{t_1})(I - \sigma_{t_2})(I - L_{\delta}^r)^{r-2} f\|_{L_w^{pq}} \\ &\leq \dots \leq \sup_{\substack{0 < t_i \leq \delta \\ i=1,2,\dots,r}} \left\| \prod_{i=1}^r (I - \sigma_{t_i}) f(x) \right\|_{L_w^{pq}} = \Omega_r(f, \delta)_{L_w^{pq}}. \end{aligned}$$

Using the equivalence (2.1) we have

$$\begin{aligned} \Omega_r(f, kt)_{L_w^{pq}} &\lesssim \inf_{g \in W_{pq,w}^{2r}} \left\{ \|f - g\|_{L_w^{pq}} + (kt)^{2r} \|g^{(2r)}(x)\|_{L_w^{pq}} \right\} \\ &\lesssim (1 + [k])^{2r} \inf_{g \in W_{pq,w}^{2r}} \left\{ \|f - g\|_{L_w^{pq}} + t^{2r} \|g^{(2r)}(x)\|_{L_w^{pq}} \right\} \\ &\lesssim (1 + [k])^{2r} \Omega_r(f, t)_{L_w^{pq}} \end{aligned}$$

and the lemma is proved. \square

Proof of Theorem 1. We know that (see [9, Theorem 6.6.2], [10])

$$\|S_n(f)\|_{L_w^{pq}} \lesssim \|f\|_{L_w^{pq}}, \quad \|\tilde{f}\|_{L_w^{pq}} \lesssim \|f\|_{L_w^{pq}},$$

$$\|f - S_n(f)\|_{L_w^{pq}} \lesssim E_n(f)_{L_w^{pq}} \text{ and } S_n(t, f^{(r)}) = S_n^{(r)}(t, f).$$

Then (see inequality 6.15 of [14])

$$f(x) - S_n(x, f) = \sum_{k=n+1}^{\infty} \frac{1}{k^r \pi} \int_{\mathbf{T}} \left(f^{(r)}(t) - S_n(t, f^{(r)}) \right) \cos\left(k(x-t) - \frac{r\pi}{2}\right) dt.$$

When $r = 2l$

$$\begin{aligned} f(x) - S_n(x, f) &= (-1)^l \sum_{k=n+1}^{\infty} \frac{1}{k^r} \left[a_k \left(f^{(r)} - S_n \left(f^{(r)} \right) \right) \cos(kx) + \right. \\ &\quad \left. + b_k \left(f^{(r)} - S_n \left(f^{(r)} \right) \right) \sin(kx) \right] \\ &= (-1)^l \sum_{k=n+1}^{\infty} \frac{A_k \left(f^{(r)} - S_n \left(f^{(r)} \right) \right)}{k^r} \\ &= (-1)^l \sum_{k=n+1}^{\infty} \left(k^{-r} - (k+1)^{-r} \right) S_k \left(x, f^{(r)} - S_n \left(f^{(r)} \right) \right). \end{aligned}$$

Hence

$$\begin{aligned} \|f - S_n(f)\|_{L_w^{pq}} &\leq \sum_{k=n+1}^{\infty} \left(k^{-r} - (k+1)^{-r} \right) \left\| S_k \left(f^{(r)} - S_n \left(f^{(r)} \right) \right) \right\|_{L_w^{pq}} \\ &\lesssim \sum_{k=n+1}^{\infty} \left(k^{-r} - (k+1)^{-r} \right) \left\| f^{(r)} - S_n \left(f^{(r)} \right) \right\|_{L_w^{pq}} \\ &\lesssim \frac{1}{(n+1)^r} \left\| f^{(r)} - S_n \left(f^{(r)} \right) \right\|_{L_w^{pq}}. \end{aligned}$$

When $r = 2l + 1$ we have $\cos \left(k \left(x - t \right) - \frac{\pi}{r} \right) = \sin \left(k \left(x - t \right) \left(-1 \right)^l \right)$ and

$$\begin{aligned} f(x) - S_n(x, f) &= (-1)^l \sum_{k=n+1}^{\infty} \frac{1}{k^r} \left[a_k \left(f^{(r)} - S_n \left(f^{(r)} \right) \right) \sin(kx) - \right. \\ &\quad \left. - b_k \left(f^{(r)} - S_n \left(f^{(r)} \right) \right) \cos(kx) \right] \\ &= (-1)^l \sum_{k=n+1}^{\infty} \frac{A_k \left(\widetilde{f^{(r)}} - S_n \left(\widetilde{f^{(r)}} \right) \right)}{k^r} \\ &= (-1)^l \sum_{k=n+1}^{\infty} \left(k^{-r} - (k+1)^{-r} \right) S_k \left(x, \widetilde{f^{(r)}} - S_n \left(\widetilde{f^{(r)}} \right) \right). \end{aligned}$$

Hence

$$\begin{aligned} \|f - S_n(f)\|_{L_w^{pq}} &\leq \sum_{k=n+1}^{\infty} \left(k^{-r} - (k+1)^{-r} \right) \left\| S_k \left(\widetilde{f^{(r)}} - S_n \left(\widetilde{f^{(r)}} \right) \right) \right\|_{L_w^{pq}} \\ &\lesssim \sum_{k=n+1}^{\infty} \left(k^{-r} - (k+1)^{-r} \right) \left\| \widetilde{f^{(r)}} - S_n \left(\widetilde{f^{(r)}} \right) \right\|_{L_w^{pq}} \\ &\lesssim \frac{1}{(n+1)^r} \left\| \widetilde{f^{(r)}} - S_n \left(\widetilde{f^{(r)}} \right) \right\|_{L_w^{pq}} \\ &\lesssim \frac{1}{(n+1)^r} \left\| f^{(r)} - S_n \left(f^{(r)} \right) \right\|_{L_w^{pq}}. \end{aligned}$$

and Theorem 1 is proved. \square

Proof of Theorem 2. Let $n \in \mathbb{N}$ and $f \in L_w^{pq}$ be fixed. We will use the operator $A_{1/n}^r f$. From Corollary 1 and Lemma 1

$$\begin{aligned} E_n(f)_{L_w^{pq}} &= E_n(f - A_{1/n}^r f + A_{1/n}^r f)_{L_w^{pq}} \leq E_n(f - A_{1/n}^r f)_{L_w^{pq}} + E_n(A_{1/n}^r f)_{L_w^{pq}} \\ &\lesssim \left\| f - A_{1/n}^r f \right\|_{L_w^{pq}} + n^{-2r} \left\| \frac{d^{2r}}{dx^{2r}} A_{1/n}^r f(x) \right\|_{L_w^{pq}} \\ &\lesssim \Omega_r \left(f, \frac{1}{n} \right)_{L_w^{pq}}. \end{aligned}$$

Hence $\|f - S_n(f)\|_{L_w^{pq}} \lesssim E_n(f)_{L_w^{pq}} \lesssim \Omega_r \left(f, \frac{1}{n} \right)_{L_w^{pq}}$. \square

Proof of Theorem 3. Let $f \in L_w^{pq}$ and

$$E_n(f)_{L_w^{pq}} \lesssim n^{-\alpha}, \quad n \in \mathbb{N}$$

for some $\alpha > 0$. We suppose that $\delta > 0$ and $n := [1/\delta]$. From (1.4) we get

$$\begin{aligned} \Omega_r(f, \delta)_{L_w^{pq}} &\leq \Omega_r \left(f, \frac{1}{n} \right)_{L_w^{pq}} \lesssim \frac{1}{n^{2r}} \sum_{j=0}^n (j+1)^{2r-1} E_j(f)_{L_w^{pq}} \\ &\lesssim \delta^{2r} \left(E_0(f)_{L_w^{pq}} + \sum_{j=1}^n j^{2r-1} E_j(f)_{L_w^{pq}} \right) \\ &\lesssim \delta^{2r} \left(E_0(f)_{L_w^{pq}} + \sum_{j=1}^n j^{2r-1-\alpha} \right). \end{aligned}$$

If $2r > \alpha$, then we get $\Omega_r(f, \delta)_{L_w^{pq}} \lesssim \delta^\alpha$. If $2r = \alpha$, then

$$\sum_{j=1}^n j^{2r-1-\alpha} = \sum_{j=1}^n j^{-1} \leq 1 + \log(1/\delta)$$

and hence $\Omega_r(f, \delta)_{L_w^{pq}} \lesssim \delta^{2r} \log(1/\delta)$. If $2r < \alpha$, then the series $\sum_{j=0}^n j^{2r-1-\alpha}$ is convergent and

$$\Omega_r(f, \delta)_{L_w^{pq}} \lesssim \delta^{2r} \left(E_0(f)_{L_w^{pq}} + \sum_{j=1}^n j^{2r-1-\alpha} \right) \lesssim \delta^{2r}$$

holds. \square

Proof of Theorem 4. Using Corollary 1 and (1.2) we find

$$\begin{aligned} E_n(f)_{L_w^{pq}} &\lesssim \frac{1}{(n+1)^r} E_n \left(f^{(r)}, \frac{1}{n} \right)_{L_w^{pq}} \\ &\lesssim \frac{1}{(n+1)^r} \Omega_k \left(f^{(r)}, \frac{1}{n} \right)_{L_w^{pq}}. \quad \square \end{aligned}$$

Proof of Theorem 5. For $f \in W_{pq,w}^r$ we have $f^{(k)} \in W_{pq,w}^{r-k}$. Using Corollary 1, $S_n^{(k)}(f) = S_n(f^{(k)})$ and (1.3) we find

$$\begin{aligned} \left\| f^{(k)} - S_n^{(k)}(f) \right\|_{L_w^{pq}} &= \left\| f^{(k)} - S_n(f^{(k)}) \right\|_{L_w^{pq}} \\ &\lesssim \frac{1}{n^{r-k}} \Omega_l \left(f^{(r)}, \frac{1}{n} \right)_{L_w^{pq}}. \quad \square \end{aligned}$$

Proof of Theorem 6. Let $q, t_n^* \in \mathcal{T}_n$ and $E_n(f^{(k)})_{L_w^{pq}} = \left\| f^{(k)} - q \right\|_{L_w^{pq}}, E_n(f)_{L_w^{pq}} = \left\| f - t_n^* \right\|_{L_w^{pq}}$. Then using $(S_n(f, \cdot))^{(k)} = S_n(f^{(k)}, \cdot)$

$$\begin{aligned} \left\| f^{(k)} - S_n^{(k)}(f) \right\|_{L_w^{pq}} &\leq \left\| f^{(k)} - S_n(f^{(k)}, \cdot) \right\|_{L_w^{pq}} + \left\| S_n^{(k)}(f) - (t_n^*)^{(k)} \right\|_{L_w^{pq}} \\ &\leq \left\| f^{(k)} - q \right\|_{L_w^{pq}} + \left\| q - S_n(f^{(k)}, \cdot) \right\|_{L_w^{pq}} + \left\| (S_n(f, \cdot) - t_n^*)^{(k)} \right\|_{L_w^{pq}} \\ &\lesssim E_n(f^{(k)})_{L_w^{pq}} + \left\| S_n(q - f^{(k)}, \cdot) \right\|_{L_w^{pq}} + n^k \left\| S_n(f, \cdot) - t_n^* \right\|_{L_w^{pq}} \\ &\lesssim E_n(f^{(k)})_{L_w^{pq}} + n^k \left\| S_n(f, \cdot) - S_n(t_n^*, \cdot) \right\|_{L_w^{pq}} \\ &\lesssim n^{k-r} E_n(f^{(r)})_{L_w^{pq}} + n^k E_n(f)_{L_w^{pq}} \lesssim n^{k-r} E_n(f^{(r)})_{L_w^{pq}} \end{aligned}$$

and the proof of Theorem is completed. \square

Acknowledgements. Authors are indebted to referee for valuable and constructive suggestions which improve the presentation of the paper. The first author was supported by Balikesir University Scientific Research Unit with contract 2014.058.

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(Received May 16, 2014)

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