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Focal representation of k-slant Helices in $\mathbb{E}^{m+1}$

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Abstract. The focal representation of a generic regular curve γ in $\mathbb{E}^{m+1}$ consists of the centers of the osculating hyperplanes. A k-slant helix γ in $\mathbb{E}^{m+1}$ is a (generic) regular curve whose unit normal vector V_k makes a constant angle with a fixed direction $\overrightarrow{U}$ in $\mathbb{E}^{m+1}$. In the present paper we proved that if γ is a k-slant helix in $\mathbb{E}^{m+1}$, then the focal representation C_γ of γ in $\mathbb{E}^{m+1}$ is an $(m-k+2)$-slant helix in $\mathbb{E}^{m+1}$.

1 Introduction

Curves with constant slope, or so-called general helices (inclined curves), are well-known curves in the classical differential geometry of space curves. They are defined by the property that the tangent makes a constant angle with a fixed line (the axis of the general helix) (see, [1], [4], [7] and [8]). In [10], the definition is more restrictive: the fixed direction makes constant angle with these all the vectors of the Frenet frame. It is easy to check that the definition

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only works in the odd dimensional case. Moreover, in the same reference, it is proven that the definition is equivalent to the fact the ratios $\frac{\kappa_2}{\kappa_1}, \frac{\kappa_4}{\kappa_3}, \dots$, κ_i being the curvatures, are constant. Further, J. Monterde has considered the Frenet curves in $\mathbb{E}^m$ which have constant curvature ratios (i.e., $\frac{\kappa_2}{\kappa_1}, \frac{\kappa_3}{\kappa_2}, \frac{\kappa_4}{\kappa_3}, \dots$ are constant) [14]. The Frenet curves with constant curvature ratios are called ccr-curves. Obviously, ccr-curves are a subset of generalized helices in the sense of [10]. It is well known that *curves with constant curvatures* (W-curves) are well-known ccr-curves [12], [15].

Recently, Izumiya and Takeuchi have introduced the concept of slant helix in Euclidean 3-space $\mathbb{E}^3$ by requiring that the normal lines make a constant angle with a fixed direction [11]. Further in [3] Ali and Turgut considered the generalization of the concept of slant helix to Euclidean n-space $\mathbb{E}^n$, and gave some characterizations for a non-degenerate slant helix. As a future work they remarked that it is possible to define a slant helix of type-k as a curve whose unit normal vector V_k makes a constant angle with a fixed direction $\overrightarrow{U}$ [9].

For a smooth curve (a source of light) γ in $\mathbb{E}^{m+1}$, the caustic of γ (defined as the envelope of the normal lines of γ) is a singular and stratified hypersurface. The focal curve of γ , C_γ , is defined as the singular stratum of dimension 1 of the caustic and it consists of the centers of the osculating hyperspheres of γ . Since the center of any hypersphere tangent to γ at a point lies on the normal plane to γ at that point, the focal curve of γ may be parametrized using the Frenet frame $(t, n_1, n_2, dots, n_m)$ of γ as follows:

$$C_\gamma(\theta) = (\gamma + c_1 n_1 + c_2 n_2 + \cdots + c_m n_m)(\theta),$$

where the coefficients $c_1, \dots, c_m$ are smooth functions that are called focal curvatures of γ [18].

This paper is organized as follows: Section 2 gives some basic concepts of the Frenet curves in $\mathbb{E}^{m+1}$. Section 3 tells about the focal representation of a generic curve given with a regular parametrization in $\mathbb{E}^{m+1}$. Further this section provides some basic properties of focal curves in $\mathbb{E}^{m+1}$ and the structure of their curvatures. In the final section we consider k-slant helices in $\mathbb{E}^{m+1}$. We prove that if γ is a k-slant helix in $\mathbb{E}^{m+1}$ then the focal representation C_γ of γ is an $(m-k+2)$-slant helix in $\mathbb{E}^{m+1}$.

2 Basic concepts

Let $\gamma = \gamma(s) : I \to \mathbb{E}^{m+1}$ be a regular curve in $\mathbb{E}^{m+1}$, (i.e., $\|\gamma'(s)\|$ is nowhere zero) where I is an interval in $\mathbb{R}$. Then γ is called a *Frenet curve of osculating* *order* d, $(2 \leq d \leq m+1)$ if $\gamma'(s)$, $\gamma''(s)$,...,$\gamma^{(d)}(s)$ are linearly independent and $\gamma'(s)$, $\gamma''(s)$,...,$\gamma^{(d+1)}(s)$ are no longer linearly independent for all s in I [18]. In this case, $Im(\gamma)$ lies in a d-dimensional Euclidean subspace of $\mathbb{E}^{m+1}$. To each Frenet curve of rank d there can be associated the orthonormal d-frame $\{t, n_1, \ldots, n_{d-1}\}$ along γ, the Frenet r-frame, and $d-1$ functions $\kappa_1, \kappa_2, \ldots, \kappa_{d-1}$:$I \longrightarrow \mathbb{R}$, the Frenet curvatures, such that

$$\begin{bmatrix} t' \\ n_1' \\ n_2' \\ \ldots \\ n_{d-1}' \end{bmatrix} = v \begin{bmatrix} 0 & \kappa_1 & 0 & \ldots & 0 \\ -\kappa_1 & 0 & \kappa_2 & \ldots & 0 \\ 0 & -\kappa_2 & 0 & \ldots & 0 \\ \ldots & & & & \kappa_{d-1} \\ 0 & 0 & \ldots & -\kappa_{d-1} & 0 \end{bmatrix} \begin{bmatrix} t \\ n_1 \\ n_2 \\ \ldots \\ n_{d-1} \end{bmatrix} \quad (1)$$

where, v is the speed of γ. In fact, to obtain $t, n_1, \ldots, n_{d-1}$ it is sufficient to apply the Gram-Schmidt orthonormalization process to $\gamma'(s)$, $\gamma''(s)$,...,$\gamma^{(d)}(s)$. Moreover, the functions $\kappa_1, \kappa_2, \ldots, \kappa_{d-1}$ are easily obtained as by-product during this calculation. More precisely, $t, n_1, \ldots, n_{d-1}$ and $\kappa_1, \kappa_2, \ldots, \kappa_{d-1}$ are determined by the following formulas:

$$\begin{aligned} v_1(s) &:= \gamma'(s) \quad ; t := \frac{v_1(s)}{\|v_1(s)\|}, \\ v_\alpha(s) &:= \gamma^{(\alpha)}(s) - \sum_{i=1}^{\alpha-1} < \gamma^{(\alpha)}(s), v_i(s) > \frac{v_i(s)}{\|v_i(s)\|^2}, \quad (2) \\ \kappa_{\alpha-1}(s) &:= \frac{\|v_\alpha(s)\|}{\|v_{\alpha-1}(s)\| \, \|v_1(s)\|}, \\ n_{\alpha-1} &:= \frac{v_\alpha(s)}{\|v_\alpha(s)\|}, \end{aligned}$$

where $\alpha \in \{2, 3, \ldots, d\}$ (see, [8]).

A Frenet curve of rank d for which $\kappa_1, \kappa_2, \ldots, \kappa_{d-1}$ are constant is called (generalized) screw line or helix [6]. Since these curves are trajectories of the 1-parameter group of the Euclidean transformations, so, F. Klein and S. Lie called them W-*curves* [12]. For more details see also [5]. γ is said to have constant curvature ratios (that is to say, it is a ccr-curve) if all the quotients $\frac{\kappa_2}{\kappa_1}, \frac{\kappa_3}{\kappa_2}, \frac{\kappa_4}{\kappa_3}, \ldots, \frac{\kappa_i}{\kappa_{i-1}}$ $(1 \leq i \leq m-1)$ are constant [14], [15].

3 The focal representation of a curve in $\mathbb{E}^{m+1}$

The hyperplane normal to γ at a point is the union of all lines normal to γ at that point. The envelope of all hyperplanes normal to γ is thus a component of the focal set that we call the main component (the other component is the curve γ itself, but we will not consider it) [16].

Definition 1 *Given a generic curve (i.e., a Frenet curve of osculating order* $m+1$*)* $\gamma : \mathbb{R} \to \mathbb{E}^{m+1}$*, let* $F : \mathbb{E}^{m+1} \times \mathbb{R} \to \mathbb{R}$ *be the* $(m+1)$*-parameter family of real functions given by*

$$F(q, \theta) = \frac{1}{2} \|q - \gamma(\theta)\|^2 . \tag{3}$$

The caustic of the family F *is given by the set*

$$\left\{ q \in \mathbb{E}^{m+1} : \exists \theta \in \mathbb{R} : F'_q(\theta) = 0 \text{ and } F''_q(\theta) = 0 \right\} \tag{4}$$

[16].

Proposition 1 *[17] The caustic of the family* $F(q, \theta) = \frac{1}{2} \|q - \gamma(\theta)\|^2$ *coincides with the focal set of the curve* $\gamma : \mathbb{R} \to \mathbb{E}^{m+1}$.

Definition 2 *The center of the osculating hypersphere of* γ *at a point lies in the hyperplane normal to the* γ *at that point. So we can write*

$$C_\gamma = \gamma + c_1 n_1 + c_2 n_2 + \cdots + c_m n_m, \tag{5}$$

which is called the focal curve of γ*, where* $c_1, c_2, \ldots, c_m$ *are smooth functions of the parameter of the curve* γ*. We call the function* c_i *the* i^{th} *focal curvature of* γ*. Moreover, the function* c_1 *never vanishes and* $c_1 = \frac{1}{\kappa_1}$ *[18].*

The focal curvatures of γ, parametrized by arc length s, satisfy the following "scalar Frenet equations" for $c_m \neq 0$:

$$\begin{aligned} 1 &= \kappa_1 c_1 \\ c_1 &= \kappa_2 c_2 \\ c_2 &= -\kappa_2 c_1 + \kappa_3 c_3 \\ &\cdots \\ c_{m-1} &= -\kappa_{m-1} c_{m-2} + \kappa_m c_m \\ c_m - \frac{(R_m^2)'}{2c_m} &= -\kappa_m c_{m-1} \end{aligned} \tag{6}$$

where R_m is the radius of the osculating m-sphere. In particular $R_m^2 = \|C_\gamma - \gamma\|^2$ [18].

Theorem 1 [16] *Let* $\gamma : s \to \gamma(s) \in \mathbb{E}^{m+1}$ *be a regular generic curve. Write* $\kappa_1, \kappa_2, \ldots, \kappa_m$ *for its Euclidean curvatures and* $\{t, n_1, n_2, \ldots, n_m\}$ *for its Frenet Frame. For each non-vertex* $\gamma(s)$ *of* γ, *write* $\varepsilon(s)$ *for the sign of* $(c_m' + c_{m-1}\kappa_m)(s)$ *and* $\delta_\alpha(s)$ *for the sign of* $(-1)^\alpha \varepsilon(s)\kappa_m(s)$, $\alpha = 1, \ldots, m$. *Then the following holds:*

a) *The Frenet frame* $\{T, N_1, N_2, \ldots, N_m\}$ *of* C_γ *at* $C_\gamma(s)$ *is well-defined and its vectors are given by* $T = \varepsilon n_m$, $N_\alpha = \delta_\alpha n_{m-l}$, *for* $l = 1, \ldots, m-1$, *and* $N_m = \pm t$. *The sign in* $\pm t$ *is chosen in order to obtain a positive basis.*

b) *The Euclidean curvatures* $K_1, K_2, \ldots, K_m$ *of the parametrized focal curve of* γ, $C_\gamma : s \to C_\gamma(s)$, *are related to those of* γ *by:*

$$\frac{K_1}{|\kappa_m|} = \frac{K_2}{\kappa_{m-1}} = \cdots = \frac{|K_m|}{\kappa_1} = \frac{1}{|c_m' + c_{m-1}\kappa_m|}, \tag{7}$$

the sign of K_m *is equal to* δ_m *times the sign chosen in* $\pm t$.

That is the Frenet formulas of C_γ at $C_\gamma(s)$ are

$$\begin{aligned} T' &= \frac{1}{A}|\kappa_m| N_1 \\ N_1' &= \frac{1}{A}(-|\kappa_m| T + \kappa_{m-1} N_2) \\ N_2' &= \frac{1}{A}(-|\kappa_{m-1}| N_1 + \kappa_{m-2} N_3) \\ &\cdots \\ N_{m-1}' &= \frac{1}{A}(-\kappa_2 N_{m-2} \mp \delta_m \kappa_1 N_m) \\ N_m' &= \frac{1}{A} \mp \delta_m \kappa_1 N_{m-1} \end{aligned} \tag{8}$$

where $A = |c_m' + c_{m-1}\kappa_m|$.

Corollary 1 *Let* $\gamma = \gamma(s)$ *be a regular generic curve in* $\mathbb{E}^{m+1}$ *and* $C_\gamma : s \to C_\gamma(s)$ *be the focal representation of* γ. *Then the Frenet frame of* C_γ *becomes as follows;*

i) *If* m *is even, then*

$$\begin{aligned} T &= n_m \\ N_1 &= -n_{m-1} \\ N_2 &= n_{m-2} \\ &\cdots \\ N_{m-1} &= -n_1 \\ N_m &= t \end{aligned} \tag{9}$$

ii) *If* m *is odd, then*

$$\begin{aligned} T &= n_m \\ N_1 &= -n_{m-1} \\ N_2 &= n_{m-2} \\ &\dots \\ N_{m-1} &= n_1 \\ N_m &= -t. \end{aligned} \tag{10}$$

Proof. By the use of (7) with (8) we get the result. □

4 k-Slant helices

Let $\gamma = \gamma(s) : I \to \mathbb{E}^{m+1}$ be a regular generic curve given with arclength parameter. Further, let $\overrightarrow{U}$ be a unit vector field in $\mathbb{E}^{m+1}$ such that for each $s \in I$ the vector $\overrightarrow{U}$ is expressed as the linear combinations of the orthogonal basis $\{V_1(s), V_2(s), \dots, V_{m+1}(s)\}$ with

$$\overrightarrow{U} = \sum_{j=1}^{m+1} a_j(s) V_j(s). \tag{11}$$

where $a_j(s)$ are differentiable functions, $1 \leq j \leq m+1$.

Differentiating $\overrightarrow{U}$ and using the Frenet equations (1), one can get

$$\frac{d\overrightarrow{U}}{ds} = \sum_{i=1}^{m+1} P_i(s) V_i(s), \tag{12}$$

where

$$\begin{aligned} P_1(s) &= a_1' - \kappa_1 a_2, \qquad (13) \\ P_i(s) &= a_i' + \kappa_{i-1} a_{i-1} - \kappa_i a_{i+1}, \ 2 \leq i \leq m, \\ P_{m+1}(s) &= a_{m+1}' + \kappa_m a_m. \end{aligned}$$

If the vector field $\overrightarrow{U}$ is constant then the following system of ordinary differential equations are obtained

$$\begin{aligned} 0 &= a_1' - \kappa_1 a_2, \\ 0 &= a_2' + \kappa_1 a_1 - \kappa_2 a_3, \\ 0 &= a_i' + \kappa_{i-1} a_{i-1} - \kappa_i a_{i+1}, \ 3 \leq i \leq m, \\ 0 &= a_{m+1}' + \kappa_m a_m. \end{aligned} \tag{14}$$

Definition 3 *Recall that a unit speed generic curve* $\gamma = \gamma(s) : I \to \mathbb{E}^{m+1}$ *is called a* k*-type slant helix if the vector field* V_k $(1 \leq k \leq m+1)$ *makes a constant angle* θ_k *with the fixed direction* $\overrightarrow{U}$ *in* $\mathbb{E}^{m+1}$*, that is*

$$< \overrightarrow{U}, V_k >= \cos\theta_k, \ \ \theta_k \neq \frac{\pi}{2}. \tag{15}$$

A 1-type slant helix is known as cylindrical helix [2] or generalized helix [13], [4]. For the characterization of generalized helices in $(n+2)$-dimensional Lorentzian space $\mathbb{L}^{n+2}$ see [19].

We give the following result;

Theorem 2 *Let* $\gamma = \gamma(s)$ *be a regular generic curve in* $\mathbb{E}^{m+1}$*. If* $C_\gamma : s \to C_\gamma(s)$ *is the focal representation of* γ *then the following statements are valid;*

i) If γ *is a* 1*-slant helix then the focal representation* C_γ *of* γ *is an* $(m+1)$*-slant helix in* $\mathbb{E}^{m+1}$.

ii) If γ *is an* $(m+1)$*-slant helix then the focal representation* C_γ *of* γ *is a* 1*-slant helix in* $\mathbb{E}^{m+1}$.

iii) If γ *is a k-slant helix* $(2 < k < m)$ *then the focal representation* C_γ *of* γ *is an* $(m-k+2)$*-slant helix in* $\mathbb{E}^{m+1}$.

Proof. i) Suppose γ is a 1-slant helix in $\mathbb{E}^{m+1}$. Then by Definition 3 the vector field V_1 makes a constant angle θ_1 with the fixed direction $\overrightarrow{U}$ defined in (11), that is

$$< \overrightarrow{U}, V_1 >= \cos\theta_1, \ \ \theta_1 \neq \frac{\pi}{2}. \tag{16}$$

For the focal representation $C_\gamma(s)$ of γ, we can choose the orthogonal basis

$$\{V_1(s) = t, V_2(s) = n_1, \ldots, V_{m+1}(s) = n_m\}$$

such that the equalities (9) or (10) is hold. Hence, we get,

$$< \overrightarrow{U}, V_1 >=< \overrightarrow{U}, t >=< \overrightarrow{U}, \pm N_m >= cons. \tag{17}$$

where $\{T, N_1, N_2, \ldots, N_m\}$ is the Frenet frame of C_γ at point $C_\gamma(s)$. From the equality (17) it is easy to see that C_γ is an (m+1)-slant helix of $\mathbb{E}^{m+1}$.

ii) Suppose γ is an $(m+1)$-slant helix in $\mathbb{E}^{m+1}$. Then by Definition 3 the vector field V_{m+1} makes a constant angle θ_{m+1} with the fixed direction $\overrightarrow{U}$ defined in (11), that is

$$< \overrightarrow{U}, V_{m+1} >= \cos\theta_{m+1}, \ \ \theta_{m+1} \neq \frac{\pi}{2}. \tag{18}$$

For the focal representation $C_\gamma(s)$ of γ, one can get

$$< \overrightarrow{U}, V_{m+1} >=< \overrightarrow{U}, n_m >=< \overrightarrow{U}, T >= cons. \tag{19}$$

where $\{V_1 = t, V_2 = n_1, \ldots, V_{m+1} = n_m\}$ and $\{T, N_1, N_2, \ldots, N_m\}$ are the Frenet frame of γ and C_γ, respectively. From the equality (19) it is easy to see that C_γ is a 1-slant helix of $\mathbb{E}^{m+1}$.

iii) Suppose γ is a k-slant helix in $\mathbb{E}^{m+1}$ $(2 \le k \le m)$. Then by Definition 3 the vector field V_k makes a constant angle θ_k with the fixed direction $\overrightarrow{U}$ defined in (11), that is

$$< \overrightarrow{U}, V_k >= \cos\theta_k,\ \theta_k \neq \frac{\pi}{2}, 2 \le k \le m. \tag{20}$$

Let $C_\gamma(s)$ be the focal representation of γ. Then using the equalities (9) or (10) we get

$$< \overrightarrow{U}, V_k >=< \overrightarrow{U}, n_{k-1} >=< \overrightarrow{U}, N_{m-k+1} >= cons.,\ 2 \le k \le m \tag{21}$$

where

$$\{V_1 = t, V_2 = n_1, \ldots, V_{m+1} = n_m\}$$

and

$$\left\{\widetilde{V}_1 = T, \widetilde{V}_2 = N_1, \ldots, \widetilde{V}_{m-k+2} = N_{m-k+1}, \ldots, \widetilde{V}_{m+1} = N_m\right\}$$

are the Frenet frame of γ and C_γ, respectively. From the equality (21) it is easy to see that C_γ is an $(m - k + 2)$-slant helix of $\mathbb{E}^{m+1}$. □

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