The refined direct and converse inequalities of trigonometric approximation in weighted variable exponent Lebesgue spaces

Article in Georgian Mathematical Journal · September 2011

DOI: 10.1515/GMJ.2011.0037

2 authors:

Ramazan Akgün
Balikesir University

Vakhtang Kokilashvili
A. Razmadze Mathematical Institute

Georgian Math. J. **18** (2011), 399–423
DOI 10.1515/GMJ.2011.0037

The refined direct and converse inequalities of trigonometric approximation in weighted variable exponent Lebesgue spaces

Ramazan Akgün and Vakhtang Kokilashvili

Abstract. Refined direct and converse theorems of trigonometric approximation are proved in the variable exponent Lebesgue spaces with weights satisfying some Muckenhoupt A_p-condition. As a consequence, the refined versions of Marchaud and its converse inequalities are obtained.

Keywords. Weighted fractional modulus of smoothness, direct theorem, converse theorem, fractional derivative, variable exponent Lebesgue space.

2010 Mathematics Subject Classification. 26A33, 41A10, 41A17, 41A25, 42A10.

1 Introduction and auxiliary results

It is well known that sharp Jackson [51] and converse [50] inequalities[1]

$$\frac{c_1(r,p)}{n^r}\left\{\sum_{\nu=1}^{n}\nu^{\beta r-1}E_{\nu-1}^{\beta}(f)_p\right\}^{\frac{1}{\beta}} \leq \omega_r\left(f,\frac{1}{n}\right)_p \leq \frac{c_2(r,p)}{n^r}\left\{\sum_{\nu=1}^{n}\nu^{\gamma r-1}E_{\nu-1}^{\gamma}(f)_p\right\}^{\frac{1}{\gamma}} \quad (1)$$

of trigonometric approximation for the classical Lebesgue space $L^p(\boldsymbol{T})$, with $1 < p < \infty$, hold with positive constants $c_1(r,p)$ and $c_2(r,p)$. We define

$$E_n(f)_p := \inf\{\|f-T\|_p : T \in \mathcal{T}_n\},$$

where $\mathcal{T}_n$ is the class of trigonometric polynomials of degree not greater than n, and $f \in L^p(\boldsymbol{T})$. Also we set $\gamma := \min\{2,p\}$, $r \in \mathbb{N} := \{1,2,3,\dots\}$ and $\beta := \max\{2,p\}$. Finally $T_h f(\cdot) := f(\cdot + h)$, $h \in \mathbb{R}$, is a translation operator,

[1] We will denote by $c_1(\cdots), c_2(\cdots), \dots, c_i(\cdots), \dots$ constants that are different on different occurrences and absolute or dependent only on the parameters given in brackets.

I is an identity operator,

$$\omega_r(f,\delta)_p := \sup\{\|(T_h - I)^r f\|_p : 0 < h \le \delta\}$$

is the *r-th modulus of smoothness* of f and $\boldsymbol{T} := [0, 2\pi)$. Inequalities like (1) have wide applications in embedding theorems [39, 47], in the study of absolute convergent Fourier series [24, 25, 48], investigation of the properties of conjugate functions [8] and characterizations of Lipschitz classes [31, 47, 50, 51]. Using weights satisfying the Muckenhoupt A_p-condition (see the definition below) inequalities (1) also hold, in a certain form, for Lebesgue spaces $L^p(\boldsymbol{T}, \omega)$ with A_p-weights [1]:

Theorem A. *Let $1 < p < \infty$, $\omega \in A_p$ and $f \in L^p(\boldsymbol{T}, \omega)$. If $n \in \mathbb{N}$ and $r \in \mathbb{R}^+ := (0, \infty)$, then there exist constants $c_3(r, p)$, $c_4(r, p) > 0$ such that*

$$\frac{c_3(r,p)}{n^{2r}} \left\{ \sum_{\nu=1}^{n} \nu^{2\beta r - 1} E_\nu^\beta(f)_{p,\omega} \right\}^{\frac{1}{\beta}} \le \Omega_r\left(f, \frac{1}{n}\right)_{p,\omega} \le \frac{c_4(r,p)}{n^{2r}} \left\{ \sum_{\nu=1}^{n} \nu^{2\gamma r - 1} E_{\nu-1}^\gamma(f)_{p,\omega} \right\}^{\frac{1}{\gamma}}$$

holds.

Here we used fractional weighted moduli of smoothness $\Omega_r(f, \cdot)_{p,\omega}$ (cf. [3, 7]) other than $\omega_r(f, \cdot)_p$ because the translation operator T_h is, in general, not continuous in weighted spaces, for example in weighted Lebesgue spaces $L^p(\boldsymbol{T}, \omega)$, in (weighted) variable exponent Lebesgue spaces. Variable exponent Lebesgue spaces $L^{p(x)}$ and the corresponding Sobolev type spaces $W^{p(x)}$ have wide applications in elasticity theory [52], fluid mechanics [40, 41], differential operators [13, 41], non-linear Dirichlet boundary value problems [33], non-standard growth [34, 52] and variational calculus [43]. The first article on $L^{p(x)}$ was [37] and later the research was carried out for rather general modular spaces [36]. $L^{p(x)}$ is an example of modular spaces [18, 35] and Sharapudinov [45] obtained the topological properties of $L^{p(x)}$. Furthermore, if $p^* := \operatorname{ess\,sup}_{x \in \boldsymbol{T}} p(x) < \infty$, then $L^{p(x)}$ is a particular case of Musielak–Orlicz spaces [35]. In subsequent years various mathematicians investigated the main properties of spaces $L^{p(x)}$, e.g. [15, 33, 42, 45]. In $L^{p(x)}$ there is a rich theory of boundedness of integral transforms of various type; see [12, 26, 43, 46]. For $p(x) := p$, $1 < p < \infty$, $L^{p(x)}$ coincides with the Lebesgue space $L^p(\boldsymbol{T})$ and basic problems of trigonometric approximation in $L^p(\boldsymbol{T})$ are well known. For a complete treatise of polynomial approximation we refer to the books [9, 11, 32, 38, 44, 49]. Approximation by algebraic polynomials and rational functions in Lebesgue spaces, Orlicz spaces, symmetric spaces and their weighted versions in sufficiently smooth complex domains and curves was investigated in [4–6, 19, 20, 22]. In harmonic and Fourier analysis some of the operators (for example a partial sum operator of Fourier series, a conjugate operator, differentiation operator, translation operator T_h, $h \in \mathbb{R}$) have been extensively used to prove direct and converse type approximation inequalities. Since $L^{p(x)}$ is not translation invariant [33], using Butzer–Wehrens type moduli of smoothness (see [10, 14, 16, 30]), Israfilov, Kokilashvili and Samko [21] obtained direct and converse trigonometric approximation theorems in weighted variable exponent Lebesgue spaces $L_{\omega}^{p(\cdot)}$.

In the present paper we investigate the approximation properties of a trigonometric system in $L_{\omega}^{p(\cdot)}$. We consider the fractional order moduli of smoothness and obtain the improved direct and converse theorems of trigonometric polynomial approximation in $L_{\omega}^{p(\cdot)}$.

A function $\omega : \boldsymbol{T} \to [0, \infty]$ will be called a weight if ω is measurable and almost everywhere (a.e.) positive. For a weight ω we denote by $L^p(\boldsymbol{T}, \omega)$ the weighted Lebesgue space of 2π periodic measurable functions $f : \boldsymbol{T} \to \mathbb{C}$ such that $f\omega^{1/p} \in L^p(\boldsymbol{T})$, where $\mathbb{C}$ is a complex plane. We set $\|f\|_{p,\omega} := \|f\omega^{1/p}\|_p$ for $f \in L^p(\boldsymbol{T}, \omega)$.

Let $\mathcal{P}$ be the class of Lebesgue measurable functions $p(x) : \boldsymbol{T} \to (1, \infty)$ such that $1 < p_* := \operatorname{ess\,inf}_{x \in \boldsymbol{T}} p(x) \leq p^* < \infty$. We define the class $L_{2\pi}^{p(\cdot)}$ of 2π periodic measurable functions $f : \boldsymbol{T} \to \mathbb{C}$ satisfying

$$\int_{-\pi+c}^{\pi+c} |f(x)|^{p(x)} dx < \infty$$

for any real number c and $p \in \mathcal{P}$.

The class $L_{2\pi}^{p(\cdot)}$ is a Banach space [33] with the norm

$$\|f\|_{\boldsymbol{T},p(\cdot)} := \inf_{\alpha > 0} \left\{ \int_{\boldsymbol{T}} \left| \frac{f(x)}{\alpha} \right|^{p(x)} dx \leq 1 \right\}.$$

Let $\omega : \boldsymbol{T} \to [0, \infty]$ be a 2π periodic weight. We will denote by $L_{\omega}^{p(\cdot)}$ the class of Lebesgue measurable functions $f : \boldsymbol{T} \to \mathbb{C}$ satisfying $\omega f \in L_{2\pi}^{p(\cdot)}$. The weighted Lebesgue space with variable exponent $L_{\omega}^{p(\cdot)}$ is a Banach space with the norm $\|f\|_{p(\cdot),\omega} := \|\omega f\|_{\boldsymbol{T},p(\cdot)}$.

For given $p \in \mathcal{P}$ the class of weights ω satisfying the condition [17]

$$\|\omega^{p(x)}\|_{A_{p(\cdot)}} := \sup_{B \in \mathcal{B}} \frac{1}{|B|^{p_B}} \|\omega^{p(x)}\|_{L^1(B)} \left\| \frac{1}{\omega^{p(x)}} \right\|_{B,(p'(\cdot)/p(\cdot))} < \infty$$

will be denoted by $A_{p(\cdot)}$. Here

$$p_B := \left(\frac{1}{|B|}\int_B \frac{1}{p(x)}dx\right)^{-1}$$

and $\mathcal{B}$ is the class of all balls in $\boldsymbol{T}$.

The variable exponent $p(x)$ is said to satisfy *the local log-Hölder continuity condition* if there exists a positive constant c_5 such that

$$|p(x_1) - p(x_2)| \le \frac{c_5}{\log(e + 1/|x_1 - x_2|)} \quad \text{for all } x_1, x_2 \in \boldsymbol{T}. \tag{2}$$

We will denote by $\mathcal{P}_{\pm}^{\log}$ the class of all $p \in \mathcal{P}$ satisfying (2).

Let $f \in L_\omega^{p(\cdot)}$ and

$$\mathcal{A}_h f(x) := \frac{1}{h}\int_{x-h/2}^{x+h/2} f(t)dt, \quad x \in \boldsymbol{T},$$

be Steklov's mean operator. If $p \in \mathcal{P}_{\pm}^{\log}$, then it was proved in [17] that the Hardy–Littlewood maximal operator $\mathcal{M}$ is bounded in $L_\omega^{p(\cdot)}$ if and only if $\omega \in A_{p(\cdot)}$.

Therefore if $p \in \mathcal{P}_{\pm}^{\log}$ and $\omega \in A_{p(\cdot)}$, then $\mathcal{A}_h$ is bounded in $L_\omega^{p(\cdot)}$. Using these facts and setting $x, h \in \boldsymbol{T}$, $0 \le r$ we define, via binomial expansion, that

$$\begin{aligned}\sigma_h^r f(x) &:= (I - \mathcal{A}_h)^r f(x) \\ &= \sum_{k=0}^{\infty}(-1)^k \binom{r}{k}\frac{1}{h^k}\int_{-h/2}^{h/2}\cdots\int_{-h/2}^{h/2} f(x + u_1 + \cdots + u_k)du_1\cdots du_k,\end{aligned}$$

where $f \in L_\omega^{p(\cdot)}$, $\binom{r}{k} := \frac{r(r-1)\cdots(r-k+1)}{k!}$ for $k > 1$, $\binom{r}{1} := r$ and $\binom{r}{0} := 1$.

Since the binomial coefficients satisfy

$$\left|\binom{r}{k}\right| \le \frac{c_6(r)}{k^{r+1}}, \quad k \in \mathbb{N},$$

we get

$$\sum_{k=0}^{\infty}\left|\binom{r}{k}\right| < \infty$$

and therefore if $p \in \mathcal{P}_{\pm}^{\log}$, $\omega \in A_{p(\cdot)}$ and $f \in L_\omega^{p(\cdot)}$, then there is a positive constant $c_7(r, p)$ such that

$$\|\sigma_h^r f\|_{p(\cdot),\omega} \le c_7(r, p)\|f\|_{p(\cdot),\omega} < \infty \tag{3}$$

holds. For $0 \le r$, we can now define the *fractional moduli of smoothness of index* r for $p \in \mathcal{P}_{\pm}^{\log}$, $\omega \in A_{p(\cdot)}$ and $f \in L_{\omega}^{p(\cdot)}$ as

$$\Omega_r(f,\delta)_{p(\cdot),\omega} := \sup_{0<h_i,t\le\delta} \left\| \prod_{i=1}^{[r]} (I - \mathcal{A}_{h_i}) \sigma_t^{r-[r]} f \right\|_{p(\cdot),\omega}, \quad \delta \ge 0,$$

where $\Omega_0(f,\delta)_{p(\cdot),\omega} := \|f\|_{p(\cdot),\omega}$;

$$\prod_{i=1}^{0} (I - \mathcal{A}_{h_i}) \sigma_t^r f := \sigma_t^r f \quad \text{for } 0 < r < 1;$$

and $[r]$ denotes the integer part of the real number r.

We have by (3) that if $p \in \mathcal{P}_{\pm}^{\log}$, $\omega \in A_{p(\cdot)}$ and $f \in L_{\omega}^{p(\cdot)}$, then there exists a positive constant $c_8(r,p)$ such that

$$\Omega_r(f,\delta)_{p(\cdot),\omega} \le c_8(r,p) \|f\|_{p(\cdot),\omega}.$$

Remark 1. The modulus of smoothness $\Omega_r(f,\delta)_{p(\cdot),\omega}$, $r \in \mathbb{R}^+$, has the following properties for $p \in \mathcal{P}_{\pm}^{\log}$, $\omega \in A_{p(\cdot)}$ and $f \in L_{\omega}^{p(\cdot)}$:

(i) $\Omega_r(f,\delta)_{p(\cdot),\omega}$ is a non-negative and non-decreasing function of $\delta \ge 0$,

(ii) $\Omega_r(f_1+f_2,\cdot)_{p(\cdot),\omega} \le \Omega_r(f_1,\cdot)_{p(\cdot),\omega} + \Omega_r(f_2,\cdot)_{p(\cdot),\omega}$,

(iii) $\lim_{\delta\to 0^+} \Omega_r(f,\delta)_{p(\cdot),\omega} = 0$.

If $p \in \mathcal{P}_{\pm}^{\log}$ and $\omega \in A_{p(\cdot)}$, then $\omega^{p(x)} \in L^1(\boldsymbol{T})$. This implies that the set of trigonometric polynomials is dense in $L_{\omega}^{p(\cdot)}$; cf. [27]. Therefore approximation problems make sense in $L_{\omega}^{p(\cdot)}$. On the other hand, if $p \in \mathcal{P}_{\pm}^{\log}$ and $\omega \in A_{p(\cdot)}$, then $L_{\omega}^{p(\cdot)} \subset L^1(\boldsymbol{T})$, where $p'(x) := p(x)/(p(x)-1)$ is the conjugate exponent of $p(x)$.

For a given $f \in L^1(\boldsymbol{T})$, let

$$f(x) \sim \frac{a_0(f)}{2} + \sum_{k=1}^{\infty} (a_k(f)\cos kx + b_k(f)\sin kx) = \sum_{k=-\infty}^{\infty} c_k(f) e^{ikx} \quad (4)$$

be the *Fourier series* of f with $c_k(f) = \frac{1}{2}(a_k(f) - i b_k(f))$. We set

$$L_0^1(\boldsymbol{T}) := \{f \in L^1(\boldsymbol{T}) : c_0(f) = 0 \text{ for the series in (4)}\}.$$

Let $\alpha \in \mathbb{R}^+$ be given. We define the *fractional derivative* of a function $f \in L_0^1(\boldsymbol{T})$ as

$$f^{(\alpha)}(x) := \sum_{k=-\infty}^{\infty} c_k(f)(ik)^\alpha e^{ikx}$$

provided the right-hand side exists, where $(ik)^\alpha := |k|^\alpha e^{(1/2)\pi i\alpha \operatorname{sign} k}$ as the principal value. We will say that a function $f \in L_\omega^{p(\cdot)}$ has fractional derivative of degree $\alpha \in \mathbb{R}^+$ if there exists a function $g \in L_\omega^{p(\cdot)}$ such that its Fourier coefficients satisfy $c_k(g) = c_k(f)(ik)^\alpha$. In that case we will write $f^{(\alpha)} = g$.

Let $W_{p(\cdot),\omega}^\alpha$, $p \in \mathcal{P}$, $\alpha > 0$, be the class of functions $f \in L_\omega^{p(\cdot)}$ such that $f^{(\alpha)}$ is an element of $L_\omega^{p(\cdot)}$. Then $W_{p(\cdot),\omega}^\alpha$ becomes a Banach space with the norm

$$\|f\|_{W_{p(\cdot),\omega}^\alpha} := \|f\|_{p(\cdot),\omega} + \|f^{(\alpha)}\|_{p(\cdot),\omega}.$$

For $f \in L_\omega^{p(\cdot)}$ we set

$$E_n(f)_{p(\cdot),\omega} := \inf\{\|f - T\|_{p(\cdot),\omega} : T \in \mathcal{T}_n\}$$

The following approximation theorems were proved in [2]:

Theorem B. *If $p \in \mathcal{P}_\pm^{\log}$, $\omega^{-p_0} \in A_{(\frac{p(\cdot)}{p_0})'}$ for some $p_0 \in (1, p_*)$ and $f \in L_\omega^{p(\cdot)}$, then there is a positive constant $c_9(r,p)$ such that*

$$E_n(f)_{p(\cdot),\omega} \le c_9(r,p)\Omega_r\Big(f, \frac{1}{n+1}\Big)_{p(\cdot),\omega}$$

holds for $r \in \mathbb{R}^+$ and $n = 0, 1, 2, 3, \ldots$

Theorem C. *Under the conditions of Theorem* B *there exists a positive constant $c_{10}(r,p)$ such that the inequality*

$$\Omega_r\Big(f, \frac{1}{n+1}\Big)_{p(\cdot),\omega} \le \frac{c_{10}(r,p)}{(n+1)^r}\sum_{\nu=0}^{n}(\nu+1)^{r-1}E_\nu(f)_{p(\cdot),\omega}$$

holds for $r \in \mathbb{R}^+$ and $n = 0, 1, 2, 3, \ldots$

Theorem D. *Under the conditions of Theorem* B *if*

$$\sum_{\nu=1}^{\infty} \nu^{\alpha-1}E_\nu(f)_{p(\cdot),\omega} < \infty$$

for some $\alpha \in (0, \infty)$*, then* $f \in W^{\alpha}_{p(\cdot),\omega}$ *and there is a positive constant* $c_{11}(\alpha, p)$ *such that*

$$E_n(f^{(\alpha)})_{p(\cdot),\omega} \leq c_{11}(\alpha, p)\left((n+1)^{\alpha} E_n(f)_{p(\cdot),\omega} + \sum_{\nu=n+1}^{\infty} \nu^{\alpha-1} E_\nu(f)_{p(\cdot),\omega}\right)$$

holds for $r \in \mathbb{R}^+$ *and* $n = 0, 1, 2, 3, \ldots$

Theorem E. *Under the conditions of Theorem* B *if* $r \in \mathbb{R}^+$ *and*

$$\sum_{\nu=1}^{\infty} \nu^{\alpha-1} E_\nu(f)_{p(\cdot),\omega} < \infty$$

for some $\alpha > 0$*, then there exists a positive constant* $c_{12}(\alpha, r, p)$ *such that*

$$\Omega_r\left(f^{(\alpha)}, \frac{1}{n+1}\right)_{p(\cdot),\omega} \leq c_{12}(\alpha, r, p)\left(\frac{1}{(n+1)^r} \sum_{\nu=0}^{n} (\nu+1)^{\alpha+r-1} E_\nu(f)_{p(\cdot),\omega} + \sum_{\nu=n+1}^{\infty} \nu^{\alpha-1} E_\nu(f)_{p(\cdot),\omega}\right)$$

holds, where $n = 0, 1, 2, 3, \ldots$

These inequalities are not the best possible ones, and in the present paper we investigate the improvements of Theorems B–E.

We need the following Marcinkiewicz multiplier and Littlewood–Paley type theorems:

Theorem F ([29]). *Let a sequence* $\{\lambda_\mu\}$ *of real numbers satisfy*

$$|\lambda_\mu| \leq A, \qquad \sum_{\mu=2^{m-1}}^{2^m-1} |\lambda_\mu - \lambda_{\mu+1}| \leq A \tag{5}$$

for all $\mu, m \in \mathbb{N}$*, where* A *does not depend on* μ *and* m*. Under the conditions of Theorem* B *there is a function* $F \in L^{p(\cdot)}_{\omega}$ *such that the series* $\sum_{k=-\infty}^{\infty} \lambda_k c_k e^{ikx}$ *is a Fourier series for* F *and*

$$\|F\|_{p(\cdot),\omega} \leq c_{13} A \|f\|_{p(\cdot),\omega}$$

holds with a positive constant c_{13} *not depending on* f*.*

Theorem G ([29]). *Under the conditions of Theorem* B *there are constants* $c_{14}(r,p)$, $c_{15}(r,p) > 0$ *such that*

$$c_{14}(p)\left\|\left(\sum_{\mu=\nu}^{\infty}|\Delta_\mu|^2\right)^{\frac{1}{2}}\right\|_{p(\cdot),\omega} \le \left\|\sum_{|\mu|=2^{\nu-1}}^{\infty} c_\nu e^{i\nu x}\right\|_{p(\cdot),\omega}$$
$$\le c_{15}(p)\left\|\left(\sum_{\mu=\nu}^{\infty}|\Delta_\mu|^2\right)^{\frac{1}{2}}\right\|_{p(\cdot),\omega}, \qquad (6)$$

where

$$\Delta_\mu := \Delta_\mu(x,f) := \sum_{|\nu|=2^{\mu-1}}^{2^\mu-1} c_\nu e^{i\nu x}.$$

Theorem H ([23]). *The space* $L_\omega^{p(\cdot)}$ *is q-concave, i.e., for* $0 \le f_i \in L_\omega^{p(\cdot)}$, $i = 1,2,3,\dots,n \in \mathbb{N}$ *the (generalized Minkowski) inequality*

$$\left\{\sum_{i=1}^{n}\|f_i\|_{p(\cdot),\omega}^q\right\}^{\frac{1}{q}} \le c_{16}\left\|\left(\sum_{i=1}^{n} f_i^q\right)^{\frac{1}{q}}\right\|_{p(\cdot),\omega}$$

holds if and only if $p(x) \le q$ *a.e.*

Proposition 1. *If* $p \in \mathcal{P}_{\pm}^{\log}$ *and* $\omega^{-p_0} \in A_{\left(\frac{p(\cdot)}{p_0}\right)'}$ *for some* $p_0 \in (1,p_*)$, *then* $\omega \in A_{p(\cdot)}$.

Proof. Using the Extrapolation Theorem 3.2 of [29] we obtain that the Hardy–Littlewood maximal operator $\mathcal{M}$ is bounded in $L_\omega^{p(\cdot)}$. This implies that $\omega \in A_{p(\cdot)}$; cf. [17]. □

The following weighted fractional Bernstein inequality holds.

Lemma A ([2]). *If* $p \in \mathcal{P}_{\pm}^{\log}$, $\omega^{-p_0} \in A_{\left(\frac{p(\cdot)}{p_0}\right)'}$ *for some* $p_0 \in (1,p_*)$ *and* $n \in \mathbb{N}$, *then there exists a constant* $c_{17}(\alpha,p) > 0$ *such that the inequality*

$$\|T_n^{(\alpha)}\|_{p(\cdot),\omega} \le c_{17}(\alpha,p)n^\alpha\|T_n\|_{p(\cdot),\omega}$$

holds for $\alpha \in \mathbb{R}^+$.

Lemma 1. *Let $1 < p_* \le 2$. Then for an arbitrary system of functions $\{\varphi_j(x)\}_{j=1}^m$, $\varphi_j \in L_\omega^{p(\cdot)}$ we have*

$$\left\| \left(\sum_{j=1}^m \varphi_j^2 \right)^{\frac{1}{2}} \right\|_{p(\cdot),\omega} \le \left(\sum_{j=1}^m \|\varphi_j\|_{p(\cdot),\omega}^{p_*} \right)^{\frac{1}{p_*}}.$$

Proof. The result follows from

$$\left\| \left(\sum_{j=1}^m \varphi_j^2 \right)^{\frac{1}{2}} \right\|_{p(\cdot),\omega} = \left\| \left(\sum_{j=1}^m \varphi_j^2 \right)^{\frac{p_*}{2}\frac{1}{p_*}} \right\|_{p(\cdot),\omega} \le \left\| \left(\sum_{j=1}^m |\varphi_j|^{p_*} \right)^{\frac{1}{p_*}} \right\|_{p(\cdot),\omega}$$
$$= \left\| \sum_{j=1}^m |\varphi_j|^{p_*} \right\|_{\frac{p(\cdot)}{p_*},\omega}^{\frac{1}{p_*}} \le \left(\sum_{j=1}^m \|\varphi_j^{p_*}\|_{\frac{p(\cdot)}{p_*},\omega} \right)^{\frac{1}{p_*}} = \left(\sum_{j=1}^m \|\varphi_j\|_{p(\cdot),\omega}^{p_*} \right)^{\frac{1}{p_*}}. \quad \square$$

Lemma 2. *Let $p_* > 2$. Then for an arbitrary system of functions $\{\varphi_j(x)\}_{j=1}^m$, $\varphi_j \in L_\omega^{p(\cdot)}$ we have*

$$\left\| \left(\sum_{j=1}^m \varphi_j^2 \right)^{\frac{1}{2}} \right\|_{p(\cdot),\omega} \le \left(\sum_{j=1}^m \|\varphi_j\|_{p(\cdot),\omega}^2 \right)^{\frac{1}{2}}.$$

Proof. We have

$$\left\| \left(\sum_{j=1}^m \varphi_j^2 \right)^{\frac{1}{2}} \right\|_{p(\cdot),\omega} = \left\| \sum_{j=1}^m \varphi_j^2 \right\|_{\frac{p(\cdot)}{2},\omega}^{\frac{1}{2}}$$
$$\le \left(\sum_{j=1}^m \|\varphi_j^2\|_{\frac{p(\cdot)}{2},\omega} \right)^{\frac{1}{2}} = \left(\sum_{j=1}^m \|\varphi_j\|_{p(\cdot),\omega}^2 \right)^{\frac{1}{2}}. \qquad \square$$

2 Main results

The following theorem is an improvement of Theorem B.

Theorem 1. *If $p \in \mathcal{P}_\pm^{\log}$, $\omega^{-p_0} \in A_{\left(\frac{p(\cdot)}{p_0}\right)'}$ for some $p_0 \in (1, p_*)$, $n \in \mathbb{N}$, $r \in \mathbb{R}^+$, $\beta_1 := \max(2, p^*)$ and $f \in L_\omega^{p(\cdot)}$, then there is a positive constant $c_{18}(r,p)$ such that*

$$\frac{1}{n^{2r}} \left\{ \sum_{\nu=1}^n \nu^{2\beta_1 r - 1} E_\nu^{\beta_1}(f)_{p(\cdot),\omega} \right\}^{\frac{1}{\beta_1}} \le c_{18}(r,p)\,\Omega_r\left(f, \frac{1}{n}\right)_{p(\cdot),\omega}$$

holds.

Remark 2. Since $E_n(f)_{p(\cdot),\omega} \downarrow 0$ we have

$$E_n(f)_{p(\cdot),\omega} \leq \frac{c(r,p)}{n^{2r}} \left\{ \sum_{\nu=1}^{n} \nu^{2\beta_1 r-1} E_\nu^{\beta_1}(f)_{p(\cdot),\omega} \right\}^{\frac{1}{\beta_1}}$$

and therefore the inequality in Theorem 1 is an improvement of the inequality in Theorem B.

By $A \overset{a,b}{\preccurlyeq} B$ we mean that there exists a constant $c > 0$ depending only on the parameters a, b such that $A \leq cB$.

Proof of Theorem 1. Let $r \in \mathbb{R}^+$, $\beta_1 = \max(2, p^*)$, $n \in \mathbb{N}$, and suppose that the number $m \in \mathbb{N}$ satisfies $2^m \leq n \leq 2^{m+1}$. Using $E_n(f)_{p(\cdot),\omega} \downarrow 0$ and the Littlewood–Paley type inequality (6) we have

$$\begin{aligned}
J_{n,r}^{\beta_1} &:= \frac{1}{n^{2r}} \left\{ \sum_{\nu=1}^{n} \nu^{2\beta_1 r-1} E_\nu^{\beta_1}(f)_{p(\cdot),\omega} \right\}^{\frac{1}{\beta_1}} \\
&\leq \frac{1}{n^{2r}} \left\{ \sum_{\nu=1}^{m+1} \sum_{|\mu|=2^{\nu-1}}^{2^\nu-1} \mu^{2\beta_1 r-1} E_\mu^{\beta_1}(f)_{p(\cdot),\omega} \right\}^{\frac{1}{\beta_1}} \\
&\leq \frac{1}{n^{2r}} \left\{ \sum_{\nu=1}^{m+1} 2^{2\nu\beta_1 r} E_{2^{\nu-1}-1}^{\beta_1}(f)_{p(\cdot),\omega} \right\}^{\frac{1}{\beta_1}} \\
&\leq \frac{1}{n^{2r}} \left\{ \sum_{\nu=1}^{m+1} 2^{2\nu\beta_1 r} \left\| \sum_{|\mu|=2^{\nu-1}}^{\infty} c_\mu e^{i\mu x} \right\|_{p(\cdot),\omega}^{\beta_1} \right\}^{\frac{1}{\beta_1}} \\
&\overset{r,p}{\preccurlyeq} \frac{1}{n^{2r}} \left\{ \sum_{\nu=1}^{m+1} 2^{2\nu\beta_1 r} \left\| \left(\sum_{\mu=\nu}^{\infty} |\Delta_\mu|^2 \right)^{\frac{1}{2}} \right\|_{p(\cdot),\omega}^{\beta_1} \right\}^{\frac{1}{\beta_1}} \\
&= \left\{ \sum_{\nu=1}^{m+1} \left\| \left(\frac{2^{4\nu r}}{n^{4r}} \sum_{\mu=\nu}^{\infty} |\Delta_\mu|^2 \right)^{\frac{1}{2}} \right\|_{p(\cdot),\omega}^{\beta_1} \right\}^{\frac{1}{\beta_1}}.
\end{aligned}$$

We assume $\beta_1 = 2$. Then $2 > p^*$ and

$$J_{n,r}^2 \overset{r,p}{\preccurlyeq} \left\{ \sum_{\nu=1}^{m+1} \left\| \left(\frac{2^{4\nu r}}{n^{4r}} \sum_{\mu=\nu}^{\infty} |\Delta_\mu|^2 \right)^{\frac{1}{2}} \right\|_{p(\cdot),\omega}^{2} \right\}^{\frac{1}{2}}.$$

By Theorem H, $L^{p(\cdot)}_{\omega}$ is 2-concave [23] and we obtain

$$J^2_{n,r} \overset{r,p}{\preccurlyeq} \left\| \left(\sum_{\nu=1}^{m+1} \frac{2^{4\nu r}}{n^{4r}} \sum_{\mu=\nu}^{\infty} |\Delta_\mu|^2 \right)^{\frac{1}{2}} \right\|_{p(\cdot),\omega}. \tag{7}$$

Using Abel's transformation, $(a+b)^{1/2} \le a^{1/2} + b^{1/2}$ (for $a, b \in \mathbb{R}^+ \cup \{0\}$) and Minkowski's inequality, we get

$$\begin{aligned}
J^2_{n,r} &\overset{r,p}{\preccurlyeq} \left\| \left(\sum_{\nu=1}^{m} \frac{2^{4\nu r}}{n^{4r}} |\Delta_\mu|^2 + \frac{2^{4r(m+1)}}{n^{4r}} \sum_{\mu=m+1}^{\infty} |\Delta_\mu|^2 \right)^{\frac{1}{2}} \right\|_{p(\cdot),\omega} \\
&\le \left\| \left(\sum_{\nu=1}^{m} \frac{2^{4\nu r}}{n^{4r}} |\Delta_\mu|^2 \right)^{\frac{1}{2}} \right\|_{p(\cdot),\omega} + \left\| \left(\frac{2^{4r(m+1)}}{n^{4r}} \sum_{\mu=m+1}^{\infty} |\Delta_\mu|^2 \right)^{\frac{1}{2}} \right\|_{p(\cdot),\omega} \\
&\le \left\| \sum_{\nu=1}^{m} \frac{2^{2\nu r}}{n^{2r}} |\Delta_\nu| \right\|_{p(\cdot),\omega} + \left\| \sum_{\mu=m+1}^{\infty} |\Delta_\mu| \right\|_{p(\cdot),\omega} \\
&\le \left\| \sum_{\nu=1}^{m} \sum_{|\mu|=2^{\nu-1}}^{2^\nu-1} \frac{2^{2\nu r}}{n^{2r}} |c_\mu e^{i\mu x}| \right\|_{p(\cdot),\omega} + \left\| \sum_{|\mu|=2^m}^{\infty} c_\mu e^{i\mu x} \right\|_{p(\cdot),\omega}.
\end{aligned}$$

Since

$$\|f(\cdot) - S_n(\cdot, f)\|_{p(\cdot),\omega} \overset{r,p}{\preccurlyeq} E_n(f)_{p(\cdot),\omega}$$

we have

$$\begin{aligned}
J^2_{n,r} \overset{r,p}{\preccurlyeq} \left\| \sum_{\nu=1}^{m} \sum_{|\mu|=2^{\nu-1}}^{2^\nu-1} \frac{2^{2\nu r}}{|\mu|^{2r}} \frac{\left(\frac{|\mu|}{n}\right)^{2r}}{\left(1-\frac{\sin\frac{\mu}{n}}{\frac{\mu}{n}}\right)^r} \left(1-\frac{\sin\frac{\mu}{n}}{\frac{\mu}{n}}\right)^r |c_\mu e^{i\mu x}| \right\|_{p(\cdot),\omega} \\
+ E_{2^m-1}(f)_{p(\cdot),\omega}
\end{aligned}$$

and by Theorem B,

$$J^2_{n,r} \overset{r,p}{\preccurlyeq} \left\| \sum_{|\mu|=1}^{2^m-1} \frac{2^{2\nu r}}{|\mu|^{2r}} \frac{\left(\frac{|\mu|}{n}\right)^{2r}}{\left(1-\frac{\sin\frac{\mu}{n}}{\frac{\mu}{n}}\right)^r} \left(1-\frac{\sin\frac{\mu}{n}}{\frac{\mu}{n}}\right)^r |c_\mu e^{i\mu x}| \right\|_{p(\cdot),\omega} + \Omega_r\left(f, \frac{1}{n}\right)_{p(\cdot),\omega}.$$

Now we define

$$h_\mu := \begin{cases} \frac{2^{2\nu r}}{|\mu|^{2r}} & \text{for } 1 \le |\mu| \le 2^m - 1,\ \nu = 1, \dots, m, \\ \frac{2^{2mr}}{|\mu|^{2r}} & \text{for } 2^m \le |\mu| \le n, \\ 0 & \text{for } |\mu| > n \end{cases}$$

and

$$\lambda_\mu := \begin{cases} \frac{\left(\frac{|\mu|}{n}\right)^{2r}}{\left(1-\frac{\sin\frac{\mu}{n}}{\frac{\mu}{n}}\right)^r} & \text{for } 1 \le |\mu| \le n, \\ 0 & \text{for } |\mu| > n. \end{cases}$$

Hence, for $|\mu| = 1, 2, 3, \ldots$, $\{h_\mu\}$ satisfies (5) with $A = 2^{2r}$ and $\{\lambda_\mu\}$ satisfies (5) with $A = (1 - \sin 1)^{-r}$. Therefore taking

$$I := \left\| \sum_{|\mu|=1}^{2^m-1} \frac{2^{2\nu r}}{|\mu|^{2r}} \frac{\left(\frac{|\mu|}{n}\right)^{2r}}{\left(1-\frac{\sin\frac{\mu}{n}}{\frac{\mu}{n}}\right)^r} \left(1 - \frac{\sin\frac{\mu}{n}}{\frac{\mu}{n}}\right)^r |c_\mu e^{i\mu x}| \right\|_{p(\cdot),\omega}$$

we get

$$I = \left\| \sum_{|\mu|=1}^{\infty} h_\mu \lambda_\mu \left(1 - \frac{\sin\frac{\mu}{n}}{\frac{\mu}{n}}\right)^r |c_\mu e^{i\mu x}| \right\|_{p(\cdot),\omega}$$

and, using Theorem F twice, we have

$$\begin{aligned} J_{n,r}^2 &\overset{p}{\preccurlyeq} \frac{2^{2r}}{(1-\sin 1)^r} \left\| \sum_{|\mu|=1}^{\infty} \left(1 - \frac{\sin\frac{\mu}{n}}{\frac{\mu}{n}}\right)^r |c_\mu e^{i\mu x}| \right\|_{p(\cdot),\omega} \\ &\overset{p}{\preccurlyeq} \frac{2^{2r}}{(1-\sin 1)^r} \|(I - \sigma_{1/n})^r f\|_{p(\cdot),\omega} \\ &= \frac{2^{2r}}{(1-\sin 1)^r} \|(I - \sigma_{1/n})^{[r]} (I - \sigma_{1/n})^{r-[r]} f\|_{p(\cdot),\omega} \\ &\le \frac{2^{2r}}{(1-\sin 1)^r} \sup_{0<h_i,t<\frac{1}{n}} \left\| \prod_{i=1}^{[r]} (I - \sigma_{h_i})(I - \sigma_t)^{r-[r]} f \right\|_{p(\cdot),\omega} \\ &\overset{r,p}{\preccurlyeq} \Omega_r\left(f, \frac{1}{n}\right)_{p(\cdot),\omega}. \end{aligned}$$

Therefore

$$J_{n,r}^2 \overset{r,p}{\preccurlyeq} \Omega_r\left(f, \frac{1}{n}\right)_{p(\cdot),\omega}.$$

Now, we assume $\beta_1 = p^*$. Then $2 < p^*$ and

$$J_{n,r}^{\beta_1} \overset{r,p}{\preccurlyeq} \left\{ \sum_{\nu=1}^{m+1} \left\| \left(\frac{2^{4\nu r}}{n^{4r}} \sum_{\mu=\nu}^{\infty} |\Delta_\mu|^2 \right)^{\frac{1}{2}} \right\|_{p(\cdot),\omega}^{p^*} \right\}^{\frac{1}{p^*}}.$$

The p^*-concavity of $L^{p(\cdot)}_\omega$ (see Theorem H) and $(a+b)^{p^*/2} \geq a^{(p^*/2)} + b^{(p^*/2)}$ (for $a, b \in \mathbb{R}^+ \cup \{0\}$) imply that

$$J^{\beta_1}_{n,r} \overset{r,p}{\preccurlyeq} \left\| \left(\sum_{\nu=1}^{m+1} \left(\frac{2^{4\nu r}}{n^{4r}} \sum_{\mu=\nu}^{\infty} |\Delta_\mu|^2 \right)^{\frac{p^*}{2}} \right)^{\frac{1}{p^*}} \right\|_{p(\cdot),\omega}$$
$$\leq \left\| \left(\sum_{\nu=1}^{m+1} \frac{2^{4\nu r}}{n^{4r}} \sum_{\mu=\nu}^{\infty} |\Delta_\mu|^2 \right)^{\frac{1}{2}} \right\|_{p(\cdot),\omega}.$$

Proceeding as above (see (7)) we conclude

$$J^{\beta_1}_{n,r} \overset{r,p}{\preccurlyeq} \Omega_r\left(f, \frac{1}{n}\right)_{p(\cdot),\omega}$$

as desired. □

We have also an improvement of the inverse Theorem C.

Theorem 2. *If $p \in \mathcal{P}^{\log}_{\pm}$, $\omega^{-p_0} \in A_{(\frac{p(\cdot)}{p_0})'}$ for some $p_0 \in (1, p_*)$, $n \in \mathbb{N}$, $r \in \mathbb{R}^+$, $\gamma_1 := \min\{2, p_*\}$ and $f \in L^{p(\cdot)}_\omega$, then there is a positive constant $c_{19}(r, p)$ such that*

$$\Omega_r\left(f, \frac{1}{n}\right)_{p(\cdot),\omega} \leq \frac{c_{19}(r,p)}{n^{2r}} \left\{ \sum_{\nu=1}^{n} \nu^{2\gamma_1 r - 1} E^{\gamma_1}_{\nu-1}(f)_{p(\cdot),\omega} \right\}^{\frac{1}{\gamma_1}}$$

holds.

Remark 3. Since x^γ is convex for $\gamma_1 = \min\{2, p_*\}$, we have

$$\left(\nu\nu^{2r-1} E_\nu(f)_{p(\cdot),\omega}\right)^{\gamma_1} - \left((\nu-1)\nu^{2r-1} E_\nu(f)_{p(\cdot),\omega}\right)^{\gamma_1}$$
$$\leq \left(\sum_{\mu=1}^{\nu} \mu^{2r-1} E_\mu(f)_{p(\cdot),\omega} \right)^{\gamma_1} - \left(\sum_{\mu=1}^{\nu-1} \mu^{2r-1} E_\mu(f)_{p(\cdot),\omega} \right)^{\gamma_1}.$$

Summing the last inequality with $\nu = 1, 2, 3, \ldots$ we find

$$\sum_{\nu=1}^{n} \left\{ \left(\nu\nu^{2r-1} E_\nu(f)_{p(\cdot),\omega}\right)^{\gamma_1} - \left((\nu-1)\nu^{2r-1} E_\nu(f)_{p(\cdot),\omega}\right)^{\gamma_1} \right\}$$
$$\leq \sum_{\nu=1}^{n} \left\{ \left(\sum_{\mu=1}^{\nu} \mu^{2r-1} E_\mu(f)_{p(\cdot),\omega} \right)^{\gamma_1} - \left(\sum_{\mu=1}^{\nu-1} \mu^{2r-1} E_\mu(f)_{p(\cdot),\omega} \right)^{\gamma_1} \right\}$$

and hence

$$\left\{\sum_{\nu=1}^{n}\nu^{2\gamma_1 r-1}E_{\nu-1}^{\gamma_1}(f)_{p(\cdot),\omega}\right\}^{1/\gamma_1}\leq 2\sum_{\nu=1}^{n}\nu^{2r-1}E_{\nu-1}(f)_{p(\cdot),\omega}.$$

The last inequality implies that the inequality in Theorem 2 is better than the inequality in Theorem C. Furthermore, in some cases, the inequalities in Theorems 1 and 2 give more precise results: If

$$E_n(f)_{p(\cdot),\omega}\asymp\frac{1}{n^{2r}},\quad n\in\mathbb{N},$$

then from Theorems B and C we have

$$\Omega_r\left(f,\frac{1}{n}\right)_{p(\cdot),\omega}\asymp\frac{1}{n^{2r}}\left|\log\frac{1}{n}\right|$$

and from Theorems 1 and 2

$$\frac{c}{n^{2r}}\left|\log\frac{1}{n}\right|^{\frac{1}{\beta}}\leq\Omega_r\left(f,\frac{1}{n}\right)_{p(\cdot),\omega}\leq\frac{C}{n^{2r}}\left|\log\frac{1}{n}\right|^{\frac{1}{\gamma}}.$$

Proof of Theorem 2. As is well known,

$$\sigma^r_{t,h_1,h_2,\ldots,h_{[r]}}f:=\prod_{i=1}^{[r]}(I-\sigma_{h_i})(I-\sigma_t)^{r-[r]}f$$

has Fourier series

$$\sigma^r_{t,h_1,h_2,\ldots,h_{[r]}}f(\cdot)\sim\sum_{\nu=-\infty}^{\infty}\left(1-\frac{\sin\nu t}{\nu t}\right)^{r-[r]}\left(1-\frac{\sin\nu h_1}{\nu h_1}\right)\cdots\left(1-\frac{\sin\nu h_{[r]}}{\nu h_{[r]}}\right)c_\nu e^{i\nu\cdot}$$

and

$$\sigma^r_{t,h_1,h_2,\ldots,h_{[r]}}f(\cdot)=\sigma^r_{t,h_1,h_2,\ldots,h_{[r]}}(f(\cdot)-S_{2^{m-1}}(\cdot,f))+\sigma^r_{t,h_1,h_2,\ldots,h_{[r]}}S_{2^{m-1}}(\cdot,f).$$

From $E_n(f)_{p(\cdot),\omega}\downarrow 0$ we have

$$\begin{aligned}\|\sigma^r_{t,h_1,h_2,\ldots,h_{[r]}}(f(\cdot)-S_{2^{m-1}}(\cdot,f))\|_{p(\cdot),\omega}&\overset{r,p}{\preccurlyeq}\|f(\cdot)-S_{2^{m-1}}(\cdot,f)\|_{p(\cdot),\omega}\overset{r,p}{\preccurlyeq}E_{2^{m-1}}(f)_{p(\cdot),\omega}\\&\overset{r,p}{\preccurlyeq}\frac{1}{n^{2r}}\left\{\sum_{\nu=1}^{n}\nu^{2\gamma_1 r-1}E_{\nu-1}^{\gamma_1}(f)_{p(\cdot),\omega}\right\}^{\frac{1}{\gamma_1}}.\end{aligned}$$

On the other hand, from (6) we get

$$\|\sigma^r_{t,h_1,h_2,\dots,h_{[r]}} S_{2^{m-1}}(\cdot,f)\|_{p(\cdot),\omega} \overset{r,p}{\preccurlyeq} \left\| \left\{ \sum_{\mu=1}^{m} |\delta_\mu|^2 \right\}^{\frac{1}{2}} \right\|_{p(\cdot),\omega},$$

where

$$\delta_\mu := \sum_{|\nu|=2^{\mu-1}}^{2^\mu-1} \left(1-\frac{\sin\nu t}{\nu t}\right)^{r-[r]} \left(1-\frac{\sin\nu h_1}{\nu h_1}\right)\cdots\left(1-\frac{\sin\nu h_{[r]}}{\nu h_{[r]}}\right) c_\nu e^{i\nu x}.$$

By Lemmas 1 and 2 we have that (cf. [28])

$$\left\| \left\{ \sum_{\mu=1}^{m} |\delta_\mu|^2 \right\}^{\frac{1}{2}} \right\|_{p(\cdot),\omega} \le \left\{ \sum_{\mu=1}^{m} \|\delta_\mu\|^{\gamma_1}_{p(\cdot),\omega} \right\}^{\frac{1}{\gamma_1}}.$$

We estimate $\|\delta_\mu\|_{p(\cdot),\omega}$. Since

$$\|\delta_\mu\|_{p(\cdot),\omega} = \left\| \sum_{|\nu|=2^{\mu-1}}^{2^\mu-1} \left[|\nu|^r \left(1-\frac{\sin\nu t}{\nu t}\right)^{r-[r]} \left(1-\frac{\sin\nu h_1}{\nu h_1}\right) \right.\right.$$
$$\left.\left. \cdots\left(1-\frac{\sin\nu h_{[r]}}{\nu h_{[r]}}\right)\right] \left[\frac{1}{|\nu|^r} c_\nu e^{i\nu x}\right] \right\|_{p(\cdot),\omega},$$

using Abel's transformation we get

$$\|\delta_\mu\|_{p(\cdot),\omega} \le \sum_{|\nu|=2^{\mu-1}}^{2^\mu-2} \left| \upsilon^r \left(1-\frac{\sin\upsilon t}{\upsilon t}\right)^{r-[r]} \left(1-\frac{\sin\upsilon h_1}{\upsilon h_1}\right)\cdots\left(1-\frac{\sin\upsilon h_{[r]}}{\upsilon h_{[r]}}\right)\right.$$
$$- (\nu+1)^r \left(1-\frac{\sin(\nu+1)t}{(\nu+1)t}\right)^{r-[r]} \left(1-\frac{\sin(\nu+1)h_1}{(\nu+1)h_1}\right)$$
$$\left. \cdots\left(1-\frac{\sin(\nu+1)h_{[r]}}{(\nu+1)h_{[r]}}\right)\right| \left\| \sum_{|l|=2^{\mu-1}}^{\nu} \frac{1}{|l|^r} |c_l e^{ilx}| \right\|_{p(\cdot),\omega}$$
$$+ \left| (2^\mu-1)^r \left(1-\frac{\sin(2^\mu-1)t}{(2^\mu-1)t}\right)^{r-[r]} \left(1-\frac{\sin(2^\mu-1)h_1}{(2^\mu-1)h_1}\right)\right.$$
$$\left. \cdots\left(1-\frac{\sin(2^\mu-1)h_{[r]}}{(2^\mu-1)h_{[r]}}\right)\right| \left\| \sum_{|l|=2^{\mu-1}}^{2^\mu-1} \frac{1}{|l|^r} |c_l e^{ilx}| \right\|_{p(\cdot),\omega}.$$

We have

$$\left\| \sum_{|l|=2^{\mu-1}}^{2^\mu-1} \frac{1}{|l|^r} |c_l e^{ilx}| \right\|_{p(\cdot),\omega} \overset{r,p}{\preccurlyeq} \frac{1}{|2^{\mu-1}|^r} \left\| \sum_{|l|=2^{\mu-1}}^{2^\mu-1} |c_l e^{ilx}| \right\|_{p(\cdot),\omega}$$

$$\leq \frac{1}{|2^{\mu-1}|^r} \left\| \sum_{|l|=2^{\mu-1}}^{2^\mu-1} c_l e^{ilx} \right\|_{p(\cdot),\omega}$$

$$\overset{r,p}{\preccurlyeq} \frac{1}{2^{\mu r}} E_{2^{\mu-1}-1}(f)_{p,\omega}$$

and, similarly,

$$\left\| \sum_{|l|=2^{\mu-1}}^{\nu} \frac{1}{|l|^r} |c_l e^{ilx}| \right\|_{p(\cdot),\omega} \overset{r,p}{\preccurlyeq} \frac{1}{2^{\mu r}} E_{2^{\mu-1}-1}(f)_{p,\omega}.$$

Since $x^r(1-\frac{\sin x}{x})^r$ is non-decreasing and $(1-\frac{\sin x}{x}) \leq x^2$ for $x>0$, we obtain

$$\begin{aligned}
\|\delta_\mu\|_{p(\cdot),\omega} \overset{r,p}{\preccurlyeq} \frac{2^{-\mu r}}{t^{r-[r]} h_1 \cdots h_{[r]}} \Bigg[\sum_{|\nu|=2^{\mu-1}}^{2^\mu-2} \Bigg| &(\upsilon t)^{r-[r]} \left(1 - \frac{\sin \upsilon t}{\upsilon t}\right)^{r-[r]} \\
&\times (\upsilon h_1)\left(1 - \frac{\sin \upsilon h_1}{\upsilon h_1}\right) \cdots (\upsilon h_{[r]})\left(1 - \frac{\sin \upsilon h_{[r]}}{\upsilon h_{[r]}}\right) \\
&- ((\nu+1)t)^{r-[r]} \left(1 - \frac{\sin(\nu+1)t}{(\nu+1)t}\right)^{r-[r]} \\
&\times ((\upsilon+1)h_1)\left(1 - \frac{\sin(\nu+1)h_1}{(\nu+1)h_1}\right) \\
&\cdots ((\nu+1)h_{[r]})\left(1 - \frac{\sin(\nu+1)h_{[r]}}{(\nu+1)h_{[r]}}\right) \Bigg| \Bigg] \times E_{2^{\mu-1}-1}(f)_{p(\cdot),\omega} \\
&+ 2^{-\mu r} \Bigg| ((2^\mu-1)t)^{r-[r]} \left(1 - \frac{\sin(2^\mu-1)t}{(2^\mu-1)t}\right)^{r-[r]} \\
&\times (2^\mu-1)h_1 \left(1 - \frac{\sin(2^\mu-1)h_1}{(2^\mu-1)h_1}\right) \\
&\cdots (2^\mu-1)h_{[r]} \left(1 - \frac{\sin(2^\mu-1)h_{[r]}}{(2^\mu-1)h_{[r]}}\right) \Bigg| \times E_{2^{\mu-1}-1}(f)_{p(\cdot),\omega}
\end{aligned}$$

$$\leq 2\Big(1-\frac{\sin(2^{\mu}-1)t}{(2^{\mu}-1)t}\Big)^{r-[r]}\Big(1-\frac{\sin(2^{\mu}-1)h_1}{(2^{\mu}-1)h_1}\Big)$$
$$\cdots\Big(1-\frac{\sin(2^{\mu}-1)h_{[r]}}{(2^{\mu}-1)h_{[r]}}\Big)\times E_{2^{\mu-1}-1}(f)_{p(\cdot),\omega}$$
$$\leq 2.2^{2\mu r}t^{2r-2[r]}h_1^2\cdots h_{[r]}^2 E_{2^{\mu-1}-1}(f)_{p(\cdot),\omega}$$

and therefore

$$\|\delta_\mu\|_{p(\cdot),\omega} \overset{r,p}{\preccurlyeq} 2^{2\mu r}t^{2(r-[r])}h_1^2\cdots h_{[r]}^2 E_{2^{\mu-1}-1}(f)_{p(\cdot),\omega}.$$

Then

$$\begin{aligned}
&\left\|\sigma^r_{t,h_1,h_2,\ldots,h_{[r]}}S_{2^{m-1}}(\cdot,f)\right\|_{p(\cdot),\omega}\\
&\quad\overset{r,p}{\preccurlyeq} t^{2(r-[r])}h_1^2\cdots h_{[r]}^2\left\{\sum_{\mu=1}^{m}2^{2\mu r\gamma_1}E^{\gamma_1}_{2^{\mu-1}-1}(f)_{p(\cdot),\omega}\right\}^{\frac{1}{\gamma_1}}\\
&\quad\overset{r,p}{\preccurlyeq} t^{2(r-[r])}h_1^2\cdots h_{[r]}^2\{2^{2\gamma_1 r}E_0^{\gamma_1}(f)_{p(\cdot),\omega}\}^{\frac{1}{\gamma_1}}\\
&\qquad+t^{2(r-[r])}h_1^2\cdots h_{[r]}^2\left\{\sum_{\mu=2}^{m}\sum_{\nu=2^{\mu-2}}^{2^{\mu-1}-1}\nu^{2\gamma_1 r-1}E^{\gamma_1}_{\nu-1}(f)_{p(\cdot),\omega}\right\}^{\frac{1}{\gamma_1}}\\
&\quad\overset{r,p}{\preccurlyeq} t^{2(r-[r])}h_1^2\cdots h_{[r]}^2\left\{\sum_{\nu=1}^{2^{m-1}-1}\nu^{2\gamma_1 r-1}E^{\gamma_1}_{\nu-1}(f)_{p(\cdot),\omega}\right\}^{\frac{1}{\gamma_1}}.
\end{aligned}$$

The last inequality implies that

$$\Omega_r\Big(f,\frac{1}{n}\Big)_{p(\cdot),\omega}\overset{r,p}{\preccurlyeq}\frac{1}{n^{2r}}\left\{\sum_{\nu=1}^{n}\nu^{2\gamma_1 r-1}E^{\gamma_1}_{\nu-1}(f)_{p(\cdot),\omega}\right\}^{\frac{1}{\gamma_1}}. \qquad \square$$

As a corollary of Theorems 1 and 2 we have the following improvements of the Marchaud inequality and its converse inequality.

Corollary 1. *Under the conditions of Theorem* B *if* $r,l\in\mathbb{R}^+$, $r<l$, *and* $0<t\leq 1/2$, *then there exist positive constants* $c_{20}(l,r,p)$, $c_{21}(l,r,p)$ *such that*

$$\begin{aligned}
&c_{20}(l,r,p)t^{2r}\left\{\int_t^1\Big[\frac{\Omega_l(f,u)_{p(\cdot),\omega}}{u^{2r}}\Big]^{\beta_1}\frac{du}{u}\right\}^{\frac{1}{\beta_1}}\\
&\qquad\leq\Omega_r(f,t)_{p(\cdot),\omega}\leq c_{21}(l,r,p)t^{2r}\left\{\int_t^1\Big[\frac{\Omega_l(f,u)_{p(\cdot),\omega}}{u^{2r}}\Big]^{\gamma_1}\frac{du}{u}\right\}^{\frac{1}{\gamma_1}}
\end{aligned}$$

hold.

The following Theorem 3 and Corollary 2 are improved versions of Theorem D and Theorem E, respectively.

Theorem 3. *Under the conditions of Theorem* B *if*

$$\sum_{k=1}^{\infty} k^{\gamma_1 \alpha - 1} E_k^{\gamma_1}(f)_{p(\cdot),\omega} < \infty \tag{8}$$

for some $\alpha \in \mathbb{R}^+$, *then* $f \in W^{\alpha}_{p(\cdot),\omega}$. *Furthermore, for* $n \in \mathbb{N}$ *there exists a constant* $c_{22}(\alpha, p) > 0$ *such that*

$$E_n(f^{(\alpha)})_{p(\cdot),\omega} \leq c_{22}(\alpha, p)\left(n^{\alpha} E_n(f)_{p(\cdot),\omega} + \left\{\sum_{\nu=n+1}^{\infty} \nu^{\alpha\gamma_1 - 1} E_\nu^{\gamma_1}(f)_{p(\cdot),\omega}\right\}^{\frac{1}{\gamma_1}}\right)$$

holds.

As a corollary of Theorem 3 we have

Corollary 2. *Under the conditions of Theorem* B *there exists a constant* $c_{23}(\alpha, r, p) > 0$ *such that*

$$\Omega_r\left(f^{(\alpha)}, \frac{1}{n}\right)_{p(\cdot),\omega} \leq c_{23}(\alpha, r, p)\left(\frac{1}{n^{2r}}\left(\sum_{\nu=1}^{n} \nu^{\gamma_1(2r+\alpha)-1} E_\nu^{\gamma_1}(f)_{p(\cdot),\omega}\right)^{\frac{1}{\gamma_1}} + \left(\sum_{\nu=n+1}^{\infty} \nu^{\alpha\gamma_1-1} E_\nu^{\gamma_1}(f)_{p(\cdot),\omega}\right)^{\frac{1}{\gamma_1}}\right)$$

holds for $n \in \mathbb{N}$ *and* $\alpha, r \in \mathbb{R}^+$.

Proof of Theorem 3. Let T_n be a polynomial of the class $\mathcal{T}_n$ such that we have $E_n(f)_{p(\cdot),\omega} = \|f - T_n\|_{p(\cdot),\omega}$ and set

$$\mathcal{U}_0(x) := T_1(x) - T_0(x); \quad \mathcal{U}_\nu(x) := T_{2^\nu}(x) - T_{2^{\nu-1}}(x),\ \nu = 1, 2, 3, \ldots$$

Hence

$$T_{2^N}(x) = T_0(x) + \sum_{\nu=0}^{N} \mathcal{U}_\nu(x), \quad N = 0, 1, 2, \ldots$$

For given $\varepsilon > 0$, by (8) there exists $\eta \in \mathbb{N}$ such that

$$\sum_{\nu=2^{\eta}}^{\infty} \nu^{\gamma_1\alpha-1} E_{\nu}^{\gamma_1}(f)_{p(\cdot),\omega} < \varepsilon. \tag{9}$$

From the fractional Bernstein inequality (Lemma A) we have

$$\|\mathcal{U}_{\nu}^{(\alpha)}\|_{p(\cdot),\omega} \overset{\alpha,p}{\preccurlyeq} 2^{\nu\alpha}\|\mathcal{U}_{\nu}\|_{p(\cdot),\omega} \overset{\alpha,p}{\preccurlyeq} 2^{\nu\alpha} E_{2^{\nu-1}}(f)_{p(\cdot),\omega}, \quad \nu \in \mathbb{N}.$$

On the other hand, it is easily seen that

$$2^{\nu\alpha} E_{2^{\nu-1}}(f)_{p(\cdot),\omega} \overset{\alpha,p}{\preccurlyeq} \left\{ \sum_{\mu=2^{\nu-2}+1}^{2^{\nu-1}} \mu^{\gamma_1\alpha-1} E_{\mu}^{\gamma_1}(f)_{p(\cdot),\omega} \right\}^{\frac{1}{\gamma_1}}, \quad \nu = 2, 3, 4, \ldots$$

For the positive integers satisfying $K < N$, we have

$$T_{2^N}^{(\alpha)}(x) - T_{2^K}^{(\alpha)}(x) = \sum_{\nu=K+1}^{N} U_{\nu}^{(\alpha)}(x), \quad x \in \boldsymbol{T},$$

and hence if K, N are large enough we obtain from (9)

$$\begin{aligned}
&\|T_{2^N}^{(\alpha)}(x) - T_{2^K}^{(\alpha)}(x)\|_{p(\cdot),\omega} \\
&\quad \leq \sum_{\nu=K+1}^{N} \|\mathcal{U}_{\nu}^{(\alpha)}(x)\|_{p(\cdot),\omega} \\
&\quad \overset{\alpha,p}{\preccurlyeq} \sum_{\nu=K+1}^{N} 2^{\nu\alpha} E_{2^{\nu-1}}(f)_{p(\cdot),\omega} \\
&\quad \overset{\alpha,p}{\preccurlyeq} \sum_{\nu=K+1}^{N} \left\{ \sum_{\mu=2^{\nu-2}}^{2^{\nu-1}} \mu^{\gamma_1\alpha-1} E_{\mu}^{\gamma_1}(f)_{p(\cdot),\omega} \right\}^{\frac{1}{\gamma_1}} \\
&\quad \overset{\alpha,p}{\preccurlyeq} \left\{ \sum_{\mu=2^{K-1}+1}^{2^{N-1}} \mu^{\gamma_1\alpha-1} E_{\mu}^{\gamma_1}(f)_{p(\cdot),\omega} \right\}^{\frac{1}{\gamma_1}} \overset{\alpha,p}{\preccurlyeq} \varepsilon^{\frac{1}{\gamma_1}}.
\end{aligned}$$

Therefore $\{T_{2^N}^{(\alpha)}\}$ is a Cauchy sequence in $L_{\omega}^{p(\cdot)}$. Then there exists $\varphi \in L_{\omega}^{p(\cdot)}$ satisfying

$$\|T_{2^N}^{(\alpha)} - \varphi\|_{p(\cdot),\omega} \to 0 \quad \text{as } N \to \infty.$$

On the other hand, we have (cf. [2, Theorem 5])

$$\|T_{2^N}^{(\alpha)} - f^{(\alpha)}\|_{p(\cdot),\omega} \to 0 \quad \text{as } N \to \infty.$$

Then $f^{(\alpha)} = \varphi$ a.e. Therefore $f \in W^{\alpha}_{p(\cdot),\omega}$.

We note that

$$\begin{aligned} E_n(f^{(\alpha)})_{p(\cdot),\omega} &\le \|f^{(\alpha)} - S_n f^{(\alpha)}\|_{p(\cdot),\omega} \\ &\le \|S_{2^{m+2}} f^{(\alpha)} - S_n f^{(\alpha)}\|_{p(\cdot),\omega} \\ &\quad + \Big\| \sum_{k=m+2}^{\infty} [S_{2^{k+1}} f^{(\alpha)} - S_{2^k} f^{(\alpha)}] \Big\|_{p(\cdot),\omega}. \end{aligned} \tag{10}$$

By Lemma A we get for $2^m < n < 2^{m+1}$

$$\|S_{2^{m+2}} f^{(\alpha)} - S_n f^{(\alpha)}\|_{p(\cdot),\omega} \overset{\alpha,p}{\preccurlyeq} 2^{(m+2)\alpha} E_n(f)_{p(\cdot),\omega} \overset{\alpha,p}{\preccurlyeq} n^{\alpha} E_n(f)_{p(\cdot),\omega}. \tag{11}$$

By (6) we find

$$\begin{aligned} &\Big\| \sum_{k=m+2}^{\infty} [S_{2^{k+1}} f^{(\alpha)} - S_{2^k} f^{(\alpha)}] \Big\|_{p(\cdot),\omega} \\ &\qquad \overset{\alpha,p}{\preccurlyeq} \Big\| \Big\{ \sum_{k=m+2}^{\infty} \Big| \sum_{|\nu|=2^k+1}^{2^{k+1}} (i\nu)^{\alpha} c_{\nu} e^{i\nu x} \Big|^2 \Big\}^{\frac{1}{2}} \Big\|_{p(\cdot),\omega} \end{aligned}$$

and therefore

$$\begin{aligned} &\Big\| \sum_{k=m+2}^{\infty} [S_{2^{k+1}} f^{(\alpha)} - S_{2^k} f^{(\alpha)}] \Big\|_{p(\cdot),\omega} \\ &\qquad \overset{\alpha,p}{\preccurlyeq} \Big(\sum_{k=m+2}^{\infty} \Big\| \sum_{|\nu|=2^k+1}^{2^{k+1}} (i\nu)^{\alpha} c_{\nu} e^{i\nu x} \Big\|_{p(\cdot),\omega}^{\gamma_1} \Big)^{\frac{1}{\gamma_1}}. \end{aligned}$$

Putting

$$|\delta_{\nu}^{*}| := \sum_{|\nu|=2^k+1}^{2^{k+1}} (i\nu)^{\alpha} c_{\nu} e^{i\nu x} = \sum_{\nu=2^k+1}^{2^{k+1}} \nu^{\alpha} 2\,\mathrm{Re}(c_{\nu} e^{i(\nu x + \alpha\pi/2)}),$$

we have

$$\|\delta_\nu^*\|_{p(\cdot),\omega} = \left\| \sum_{\nu=2^k+1}^{2^{k+1}} \nu^\alpha U_\nu(x) \right\|_{p(\cdot),\omega},$$

where $U_\nu(x) = 2\,\mathrm{Re}(c_\nu e^{i(\nu x + \alpha\pi/2)})$. Using Abel's transformation we get

$$\|\delta_\nu^*\|_{p(\cdot),\omega} \le \sum_{\nu=2^k+1}^{2^{k+1}-1} |\nu^\alpha - (\nu+1)^\alpha| \left\| \sum_{l=2^k+1}^{\nu} U_l(x) \right\|_{p(\cdot),\omega} + |(2^{k+1})^\alpha| \left\| \sum_{l=2^k+1}^{2^{k+1}-1} U_l(x) \right\|_{p(\cdot),\omega}.$$

For $2^k + 1 \le \nu \le 2^{k+1}$, $k \in \mathbb{N}$ we have

$$\left\| \sum_{l=2^k+1}^{\nu} U_l(x) \right\|_{p(\cdot),\omega} \overset{\alpha,p}{\preccurlyeq} E_{2^k}(f)_{p(\cdot),\omega}$$

and since

$$(\nu+1)^\alpha - \nu^\alpha \le \begin{cases} \alpha(\nu+1)^{\alpha-1}, & \alpha \ge 1, \\ \alpha\nu^{\alpha-1}, & 0 \le \alpha < 1, \end{cases}$$

we obtain

$$\|\delta_\nu^*\|_{p(\cdot),\omega} \overset{\alpha,p}{\preccurlyeq} 2^{k\alpha} E_{2^k-1}(f)_{p(\cdot),\omega}.$$

Therefore

$$\begin{aligned} &\left\| \sum_{k=m+2}^{\infty} [S_{2^{k+1}} f^{(\alpha)} - S_{2^k} f^{(\alpha)}] \right\|_{p(\cdot),\omega} \\ &\quad \overset{\alpha,p}{\preccurlyeq} \left\{ \sum_{k=m+2}^{\infty} 2^{k\alpha\gamma_1} E_{2^k-1}^{\gamma_1}(f)_{p(\cdot),\omega} \right\}^{\frac{1}{\gamma_1}} \\ &\quad \overset{\alpha,p}{\preccurlyeq} \left\{ \sum_{\nu=n+1}^{\infty} \nu^{\gamma_1\alpha-1} E_\nu^{\gamma_1}(f)_{p(\cdot),\omega} \right\}^{\frac{1}{\gamma_1}} \end{aligned} \tag{12}$$

and using (10), (11) and (12) Theorem 3 is proved. □

Bibliography

[1] R. Akgün, Sharp Jackson and converse theorems of trigonometric approximation in weighted Lebesgue spaces, *Proc. A. Razmadze Math. Inst.* **152** (2010), 1–18.

[2] R. Akgün, Polynomial approximation of functions in weighted Lebesgue and Smirnov spaces with nonstandard growth, *Georgian Math. J.* **11** (2011), no. 2, 203-235.

[3] R. Akgün, Approximating polynomials for functions of weighted Smirnov–Orlicz spaces, *J. Funct. Spaces Appl.*, to appear.

[4] R. Akgün and D. M. Israfilov, Polynomial approximation in weighted Smirnov–Orlicz space, *Proc. A. Razmadze Math. Inst.* **139** (2005), 89–92.

[5] R. Akgün and D. M. Israfilov, Approximation by interpolating polynomials in Smirnov–Orlicz class, *J. Korean Math. Soc.* **43** (2006), no. 2, 413–424.

[6] R. Akgün and D. M. Israfilov, Approximation and moduli of fractional orders in Smirnov–Orlicz classes, *Glas. Mat. Ser. III* **43**(63) (2008), no. 1, 121–136.

[7] R. Akgün and D. M. Israfilov, Simultaneous and inverse approximation theorems in weighted Orlicz spaces, *Bull. Belg. Math. Soc. Simon Stevin* **17** (2010), no. 1, 13–28.

[8] N. K. Bari and S. B. Stečkin, Best approximations and differential properties of two conjugate functions, *Trudy Moskov. Mat. Obšč.* **5** (1956), 483–522 (in Russian).

[9] P. L. Butzer and R. J. Nessel, *Fourier Analysis and Approximation, Volume 1: One-Dimensional Theory*, Pure and Applied Mathematics 40, Academic Press, New York, London, 1971.

[10] P. L. Butzer, R. L. Stens and M. Wehrens, Approximation by algebraic convolution integrals, in: *Approximation theory and functional analysis* (Proc. Internat. Sympos. Approximation Theory, Univ. Estadual de Campinas, Campinas, 1977), pp. 71–120, North-Holland Math. Stud. 35, North-Holland, Amsterdam, New York, 1979.

[11] R. A. De Vore and G. G. Lorentz, *Constructive Approximation*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences] 303, Springer-Verlag, Berlin, 1993.

[12] L. Diening, Maximal function on generalized Lebesgue spaces $L^{p(\cdot)}$, *Math. Inequal. Appl.* **7** (2004), no. 2, 245–253.

[13] L. Diening and M. Růžička, *Calderon–Zymund operators on generalized Lebesgue spaces $L^{p(x)}$ and problems related to fluid dynamics*, J. Reine Angew. Math. **563** (2003), 197–220.

[14] L. Ephremidze, V. Kokilashvili and Y. E. Yildirir, On the inverse inequalities for trigonometric polynomial approximations in weighted Lorentz spaces, *Proc. A. Razmadze Math. Inst.* **144** (2007), 132–136.

[15] X. Fan and D. Zhao, On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$, *J. Math. Anal. Appl.* **263** (2001), no. 2, 424–446.

[16] E. A. Gadjieva, *Investigation of the properties of functions with quasimonotone Fourier coefficients in generalized Nikolskii–Besov spaces*, Author's summary of dissertation, Tbilisi, 1986 (in Russian).

[17] P. Hästö and L. Diening, *Muckenhoupt weights in variable exponent spaces*, preprint, `www.helsinki.fi/~pharjule/varsob/publications.shtml`.

[18] H. Hudzik, On generalized Orlicz–Sobolev space, *Funct. Approximatio Comment. Math.* **4** (1976), 37–51.

[19] D. M. Israfilov and R. Akgün, Approximation in weighted Smirnov–Orlicz classes, *J. Math. Kyoto Univ.* **46** (2006), no. 4, 755–770.

[20] D. M. Israfilov and R. Akgün, Approximation by polynomials and rational functions in weighted rearrangement invariant spaces, *J. Math. Anal. Appl.* **346** (2008), no. 2, 489–500.

[21] D. M. Israfilov, V. Kokilashvili and S. Samko, Approximation in weighted Lebesgue and Smirnov spaces with variable exponents, *Proc. A. Razmadze Math. Inst.* **143** (2007), 25–35.

[22] D. M. Israfilov, B. Oktay and R. Akgün, Approximation in Smirnov–Orlicz classes, *Glas. Mat. Ser. III* **40**(60) (2005), no. 1, 87–102.

[23] A. Kamińska, Indices, convexity and concavity in Musielak–Orlicz spaces, Funct. Approx. Comment. Math. **26** (1998), 67–84.

[24] V. M. Kokilashvili, Approximation of analytic functions of class E_p, *Soobshch. Akad. Nauk. Gruzin. SSR* **34** (1968), 82–102 (in Russian).

[25] V. M. Kokilashvili, The approximation of periodic functions, *Soobshch. Akad. Nauk. Gruzin. SSR* **34** (1968), 51–81 (in Russian).

[26] V. Kokilashvili and S. Samko, Singular integrals and potentials in some Banach function spaces with variable exponent, *J. Funct. Spaces Appl.* **1** (2003), no. 1, 45–59.

[27] V. Kokilashvili and S. Samko, Singular integrals in weighted Lebesgue spaces with variable exponent, *Georgian Math. J.* **10** (2003), no. 1, 145–156.

[28] V. M. Kokilashvili and S. G. Samko, A refined inverse inequality of approximation in weighted variable exponent Lebesgue spaces, *Proc. A. Razmadze Math. Inst.* **151** (2009), 134–138.

[29] V. M. Kokilashvili and S. G. Samko, Operators of harmonic analysis in weighted spaces with non-standard growth, *J. Math. Anal. Appl.* **352** (2009), no. 1, 15–34.

[30] V. Kokilashvili and Y. E. Yildirir, On the approximation in weighted Lebesgue spaces, *Proc. A. Razmadze Math. Inst.* **143** (2007), 103–113.

[31] A. A. Konyuškov, Best approximations by trigonometric polynomials and Fourier coefficients, *Mat. Sb. N.S.* **44**(86) (1958), 53–84 (in Russian).

[32] N. Korneĭchuk, *Exact Constants in Approximation Theory*, translated from the Russian by K. Ivanov, Encyclopedia of Mathematics and its Applications 38, Cambridge University Press, Cambridge, 1991.

[33] Z. O. Kováčik and J. Rákosnik, On spaces $L^{p(x)}$ and $W^{k,p(x)}$, *Czechoslovak Math. J.* **41**(116) (1991), no. 4, 592–618.

[34] P. Marcellini, Regularity and existence of solutions of elliptic equations with p,q-growth conditions, *J. Differential Equations* **90** (1991), no. 1, 1–30.

[35] J. Musielak, *Orlicz Spaces and Modular Spaces*, Lecture Notes in Mathematics 1034, Springer-Verlag, Berlin, 1983.

[36] H. Nakano, *Topology of Linear Topological Spaces*, Maruzen Co., Tokyo, 1951.

[37] W. Orlicz, Über konjugierte Exponentenfolgen, *Studia Math.* **3** (1931), 200–212.

[38] P. P. Petrushev and V. A. Popov, *Rational Approximation of Real Functions*, Encyclopedia of Mathematics and its Applications 28, Cambridge University Press, Cambridge, 1987.

[39] M. K. Potapov and B. V. Simonov, On the interrelation of the generalized Besov–Nikol'skii and Weyl–Nikol'skii classes of functions, *Anal. Math.* **22** (1996), no. 4, 299–316.

[40] K. R. Rajagopal and M. Růžička, On the modeling elektroreological materials, *Mech. Res. Commun.* **23**(1996), no. 4, 401–407.

[41] M. Růžička, *Electrorheological Fluids: Modeling and Mathematical Theory*, Lecture Notes in Mathematics 1748, Springer-Verlag, Berlin, 2000.

[42] S. Samko, Differentiation and integration of variable order and the spaces $L^{p(x)}$, in: *Operator theory for complex and hypercomplex analysis* (Mexico City, 1994), pp. 203–219, Contemp. Math. 212, Amer. Math. Soc., Providence, RI, 1998.

[43] S. Samko, On a progress in the theory of Lebesgue spaces with variable exponent: maximal and singular operators, *Integral Transforms Spec. Funct.* **16** (2005), no. 5-6, 461–482.

[44] B. Sendov and V. A. Popov, *The Averaged Moduli of Smoothness, Applications in Numerical Methods and Approximation*, Pure and Applied Mathematics, John Wiley & Sons, Chichester, 1988.

[45] I. I. Sharapudinov, The topology of the space $\mathcal{L}^{p(t)}([0,1])$, *Mat. Zametki* **26** (1979), no. 4, 613–632 (in Russian).

[46] I. I. Sharapudinov, On the uniform boundedness in $L^p(p = p(x))$ of some families of convolution operators, *Mat. Zametki* **59** (1996), no. 2, 291–302 (in Russian); translation in *Math. Notes* **59** (1996), no. 1–2, 205–212.

[47] B. V. Simonov and S. Yu. Tikhonov, Embedding theorems in the constructive theory of approximations, *Mat. Sb.* **199** (2008), no. 9, 107–148 (in Russian); translation in *Sb. Math.* **199** (2008), no. 9–10, 1367–1407.

[48] S. B. Stečkin, On the theorem of Kolmogorov–Seliverstov, *Izvestiya Akad. Nauk SSSR. Ser. Mat.* **17** (1953), 499–512 (in Russian).

[49] A. F. Timan, Theory of approximation of functions of a real variable, translated from the Russian by J. Berry, International Series of Monographs in Pure and Applied Mathematics 34, Pergamnon Press, Oxford, 1963; Russian original: Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1960.

[50] M. F. Timan, Inverse theorems of the constructive theory of functions in L_p spaces $(1 \leq p \leq \infty)$, *Mat. Sb. N.S.* **46**(88) (1958), 125–132 (in Russian).

[51] M. F. Timan, On Jackson's theorem in L_p-spaces, *Ukrain. Mat. Z.* **18** (1966), no. 1, 134–137 (in Russian).

[52] V. V. Zhikov, Averaging of functionals of the calculus of variations and elasticity theory, *Izv. Akad. Nauk SSSR Ser. Mat.* **50** (1986), no. 4, 675–710 (in Russian).

Received February 17, 2010.

Author information

Ramazan Akgün, Department of Mathematics, Faculty of Arts and Sciences,
Balikesir University, 10145 Balikesir, Turkey.
E-mail: `rakgun@balikesir.edu.tr`

Vakhtang Kokilashvili, A. Razmadze Mathematical Institute,
I. Javakhisvili State University, Tbilisi 0186, Georgia.
E-mail: `kokil@rmi.ge`