Conjugacy for Free Groups under Split Extensions

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Abstract. At the present paper we show that conjugacy is *preserved* and *reflected* by the natural homomorphism defined from a semigroup S to a group G, where G defines split extensions of some free groups. The main idea in the proofs is based on a geometrical structure as applied in the paper [8].

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INTRODUCTION AND PRELIMINARIES

Let $\widehat{\mathscr{P}} = [\mathbf{x}\ ;\ \mathbf{r}]$ be a semigroup presentation for a semigroup $S\ (= S(\widehat{\mathscr{P}}))$. For each $R \in \mathbf{r}$, the words R_{+1} and R_{-1} are distinct, non-empty and positive on $\mathbf{x}$. We also let $\widehat{\mathbf{r}} = \{R_{+1}R_{-1}^{-1} : R \in \mathbf{r}\}$. Then we have a corresponding group presentation $\mathscr{P} = \langle \mathbf{x}\ ;\ \widehat{\mathbf{r}} \rangle$, for a group $G\ (= G(\mathscr{P}))$. Finally, let π be the natural homomorphism from S to G defined by $[X]_{\widehat{\mathscr{P}}} \mapsto [X]_{\mathscr{P}}$ (X is a word on $\mathbf{x}$). The focus of this paper is the *conjugacy problem*, and so *conjugacy*, that have received a good deal of attention (see, for instance, [7, 8, 10, 13]), and, in here, this problem will be studied in the meaning of the homomorphisim π. In particular it seems natural to ask the following questions for the conjugacy problem:

- If two elements of S are conjugate in S, are their images under π conjugate in G?
- If π is injective and two elements of S have conjugate images under π in G, are they conjugate in S?

If the first question can be answered positively, then we say that π *preserves* conjugacy, and if the second one can be answered positively, then we say that π *reflects* conjugacy. The main diffuculty associated with such questions is that there seems to be no standard definition of conjugacy in an arbitrary semigroup or monoid. But, in [8], Goldstein and Teymouri modified the definition given in the famous book of Lallement ([10]) to arrive at a definition of conjugacy in semigroup S (see Definitions 4 and 5 below). In fact, by this modification, π preserves conjugacy and, in addition, if $\mathscr{P}$ satisfies Adjan's conditions, say (AC) (in which $\mathscr{P}$ has the property that both left and right graphs are cycle free [1]), for S to be embeddable ([15]) in group G, then π also reflects conjugacy.

In a joint paper ([11, Proposition 3]), it was showed that the semidirect product (i.e. the split extension) $F_n \rtimes_{\varpi} \mathbb{Z}$, where ϖ is a morphism from $\mathbb{Z}$ to $Aut(F_n)$, is *word-hyperbolic* and *free-by-cyclic* for some sufficiently large values. In fact these two results on this special semidirect product imply that the conjugacy problem is solvable for it (see [3, 4]). Thus one can asks whether solvability of the conjugacy problem can be extended for semidirect products obtained by some other free groups. In this paper we try to find an answer for this question over some special free groups. Therefore, by considering the paper [8], we first re-prove the conjugacy problem on the group $G_1 = F_n \rtimes_{\varpi} \mathbb{Z}$ (semidirect product of free group of rank n by free group of rank 1) in a different manner. We basically show that π preserves and reflects conjugacy for this case. Additionally, again by considering [8], we prove only reflectivity of π

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on the group $G_2 = F^n \rtimes_{\varpi} F^2$ (semidirect product of free abelian group of rank n by free abelian group of rank 2) as a main result of this paper. The reason for us keeping re-prove the well known fact about conjugacy on the group G_1 is presenting the valuality of the method, used in here, on a known result. Therefore the proof on the group G_2 will be more understandable. The following theorem will be proved again by a geometric way.

Theorem 1 *For the group G_1, if the generator of the group $\mathbb{Z}$ is a unique sink (or a unique source) for the left graph (or the right graph) of the presentation for this group, then the conjugacy is both preserved and reflected by the natural homomorphism.*

In the following, the main theorem of this paper is given and the same method with Theorem 1 will be applied on the proof of it.

Theorem 2 *For the group G_2, if the S-diagram of this group has "symmetrical two squares", then the conjugacy is reflected by the natural homomorphism.*

As it seen from the statements of above theorems, the major idea in here is using *S-diagrams* that are effective tools in obtaining results on groups and semigroups. We refer the reader to [12] for more details on these diagrams. But we may only recall that "*a cellular diagram (simply connected diagram) is called an S-diagram if it is two-sided and no interior vertex is a sink or a source*". In studying boundary problems (such that word problems etc.) for semigroups, the importance of S-diagrams can be seen clearly by the following proposition which will be used later in the paper.

Proposition 3 ([14]) *Let u and v be two words in the alphabet X. Then they represent the same element in S if and only if there is an S-diagram whose boundary label is the pair (u,v).*

We note that notations $\sim_G$, $\sim_S$, $\sim_S^{(i)}$, $\sim_S^{(ec)}$ and $\sim_S^{(eic)}$ will denote conjugate in G, conjugate in S, inversely conjugate in S, elementary conjugate in S and elementary inversely conjugate in S, respectively, at the rest of this paper. Now we can give the following conjugacy definitions which build up main body of the *preserving* parts of proofs.

Definition 4 ([8]) *Let u and v be two elements in S. If there exists an element a in S (possibly empty word) such that either $ua = av$ or $au = va$, then $u \sim_S^{(ec)} v$. Moreover if there exists an element a in S such that either $uav = a$ or $vau = a$, then $u \sim_S^{(eic)} v$.*

It is seen that if $u \sim_S^{(eic)} v$, then $u \sim_G v^{-1}$. Hence, by keeping this in mind, the following definition gives a proper statement for any two elements in S to be conjugate to each other.

Definition 5 ([8]) *Let u and v be two elements in S. We say that $u \sim_S v$ (or $u \sim_S^{(i)} v$) if there exists a finite sequence of elements $u_1, u_2, \cdots, u_n$ such that "$u = u_1$ and $v = u_n$", "either $u_{k+1} \sim_S^{(ec)} u_k$ or $u_{k+1} \sim_S^{(eic)} u_k$" and "the number of elementary inverse conjugations is even for the case $u \sim_S^{(i)} v$ this number is odd".*

We note that, by Definition 5, if $u \sim_S^{(i)} v$ then $u \sim_G v^{-1}$.

Remark 6 *Let G be a group and let u, v be both non identity elements in G. If $u \sim_G v$, then there exists a reduced annular diagram that boundary cycles are u and v ([12]). This fact will be directly used in the paper (without mentioning again) to obtain the required preservation and reflectivity of π related to $F_n \rtimes_{\varpi} \mathbb{Z}$ and $F^n \rtimes_{\varpi} F^2$.*

CONJUGACY ON THE GROUP $G_1 = F_n \rtimes_{\varpi} \mathbb{Z}$

Let F_n and $\mathbb{Z}$ be generated by $\langle x_1, x_2, \cdots, x_n \rangle$ and $\langle t \rangle$, respectively. By assuming each morphism satisfying the compatibility conditions $(x_i)\varpi_t = \prod_{i,j=1}^{n} x_j^{\alpha_{ij}}$, Cohen and Suciu showed in a joint paper [6, Section 1.1] that

$$\mathscr{P} = \left\langle x_1, x_2, \cdots, x_n, t \;;\; t^{-1} x_i t = x_1^{\alpha_{i1}} x_2^{\alpha_{i2}} \cdots x_n^{\alpha_{in}} \right\rangle \qquad (1)$$

is a presentation for the semidirect product $F_n \rtimes_{\varpi} \mathbb{Z}$, where α_{ij} $(1 \leq i, j \leq n)$'s are some integers.

Remark 7 *It is a well known fact that, in general, $\mathscr{P}$ in* (1) *cannot be a presentation for the semidirect product of F_n by $\mathbb{Z}$ since the morphism given by $x_i \longmapsto x_1^{\alpha_{i1}} x_2^{\alpha_{i2}} \cdots x_n^{\alpha_{in}}$ not need to define an automorphism of F_n for every elections of the integers α_{ij}'s. So, for each i, we should have assumed that conditions $(x_i)\varpi_t = t^{-1} x_i t = \prod_{i,j=1}^{n} x_j^{\alpha_{ij}}$ must hold to $\mathscr{P}$ in* (1) *be a semidirect product presentation, as done in* [6].

By [12], we can consider a diagram for the presentation of $F_n \rtimes_{\varpi} \mathbb{Z}$ ([9]). To obtain labels on the outer and the inner boundaries of this diagram, let us traverse these boundaries in an anticlockwise manner starting from two different points. Then we get labels where each of them is a word $U = x_1^{\alpha_{n1}} \cdots x_n^{\alpha_{nn}} x_1^{\alpha_{n-11}} \cdots x_n^{\alpha_{n-1n}} \cdots x_1^{\alpha_{21}} \cdots x_n^{\alpha_{2n}} x_1^{\alpha_{11}} \cdots x_n^{\alpha_{1n}}$ and $V = x_n x_{n-1} \cdots x_2 x_1$, respectively. So, by considering this diagram ([9]), we get the following result:

Lemma 8 *A word $W = x_i x_{i+1} \cdots x_n x_1 x_2 \cdots x_{i-1}$ $(1 \leq i \leq n)$ obtained by the generators of F_n is conjugate to*

$$x_1^{\alpha_{i1}} \cdots x_n^{\alpha_{in}} x_1^{\alpha_{i+11}} \cdots x_n^{\alpha_{i+1n}} \cdots x_1^{\alpha_{1n}} \cdots x_n^{\alpha_{nn}} x_1^{\alpha_{11}} \cdots x_n^{\alpha_{1n}} \cdots x_1^{\alpha_{i-11}} \cdots x_n^{\alpha_{i-1n}}. \tag{2}$$

Proof. Using the relations in (1), $Wt = tU$ in S; hence $W \sim_S^{(ec)} U$. □

Proof of Theorem 1:

The existence of the preservation for the natural homomorphism π is clear by Definition 5. For the reflectivity of π, let t (that was assumed the generator of $\mathbb{Z}$) be a unique sink (or a unique source) for the left graph (or the right graph) of presentation $\mathscr{P}$ in (1). Therefore $\mathscr{P}$ satisfies (AC) condition and so the natural homomorphism from S to $F_n \rtimes_{\varpi} \mathbb{Z}$ is one-to-one.

By [8], there are following two situations for diagrams of any groups:

1. There is a positive path of interior edges which either starts at the outer boundary and ends at the outer boundary or starts at the outer boundary ends at the inner boundary (which is suitable for our case), and also it has no self-intersections.
2. There is a positive cycle of interior edges which is the outer boundary of a proper annular subdiagram of a diagram.

Now if our diagram drawn for $F_n \rtimes_{\varpi} \mathbb{Z}$ ([9]) is convenient to second case in above, then

- the word as the label on the outer boundary of this diagram is $U = x_1^{\alpha_{n1}} \cdots x_n^{\alpha_{nn}} \cdots x_1^{\alpha_{11}} \cdots x_n^{\alpha_{1n}}$,
- words as the label on the outer boundary of this diagram and as the label on the outer boundary of subdiagram are same and equal to $W = x_n \cdots x_2 x_1$,
- the word as the label on the inner boundary of subdiagram is $V = x_1^{\alpha_{n1}} \cdots x_n^{\alpha_{nn}} \cdots x_1^{\alpha_{11}} \cdots x_n^{\alpha_{1n}}$.

These imply that $U \sim_S W$ and $W \sim_S V$, and so $U \sim_S V$.

After all, since this diagram is an annular, reduced and has n regions, we can separate it along some word t. Therefore we get

$$x_1^{\alpha_{n1}} \cdots x_n^{\alpha_{nn}} \cdots x_1^{\alpha_{11}} \cdots x_n^{\alpha_{1n}} \sim_S x_n \cdots x_2 x_1 \quad \text{or} \quad x_1^{\alpha_{11}} \cdots x_n^{\alpha_{1n}} x_1^{\alpha_{n1}} \cdots x_n^{\alpha_{nn}} \cdots x_1^{\alpha_{21}} \cdots x_n^{\alpha_{2n}} \sim_S x_1 x_n \cdots x_2 \quad \text{or}$$
$$\cdots \quad \cdots \quad \cdots \quad \text{or} \quad x_1^{\alpha_{i1}} \cdots x_n^{\alpha_{in}} \cdots x_1^{\alpha_{i+11}} \cdots x_n^{\alpha_{i+1n}} \sim_S x_i x_{i-1} \cdots x_1 x_n x_{n-1} \cdots x_{i+1}$$

$(Ut^{-1} = t^{-1}V)$, which completes the proof. □

It is known that every word is conjugate to itself. Therefore there is no necessary to draw a diagram for this situation in the above proof.

As an application, let us consider the group $F_2 \rtimes_{\varpi} \mathbb{Z}$. Although distinct automorphisms of F_2 yield distinct presentations for $F_2 \rtimes_{\varpi} \mathbb{Z}$, the diagrams related to these presentations are virtually same. Let us consider the morphism that tends x_1 to $x_1^{\varepsilon\alpha}$ and x_2 to $x_2^{\delta\beta}$, where $\varepsilon, \delta = \pm 1$ and $\alpha, \beta \in \mathbb{Z}^+$. (The note in Remark 7 is still holding). In fact, for any different positive integers α and β, the diagram can be drawn by taking $\varepsilon, \delta = +1$, $\varepsilon, \delta = -1$, $\varepsilon = +1$ and $\delta = -1$ or $\varepsilon = -1$ and $\delta = +1$. Now if we take $\varepsilon, \delta = +1$, then we get labels of the outer and the boundaries as $x_2^{-\beta} x_1^{-\alpha} x_2^{-\beta} \cdots x_1^{-\alpha}$ and $x_2^{-1} x_1^{-1} x_2^{-1} \cdots x_1^{-1}$, respectively. So, by Lemma 8, these are conjugate elements.

CONJUGACY ON THE GROUP $G_2 = F^n \rtimes_{\varpi} F^2$

Let F^n and F^2 be presented by $\langle x_1, x_2, \cdots, x_n ; x_i x_j = x_j x_i \rangle$ $(1 \leq i < j \leq n)$ and $\langle y_1, y_2 ; y_1 y_2 = y_2 y_1 \rangle$, respectively. Hence, by considering the morphism $\varpi : F^2 \to Aut(F^n)$, we obtain

$$\begin{aligned} \mathscr{Q} =< x_1, x_2, \cdots, x_n, y_1, y_2 \quad ; \quad & x_i x_j = x_j x_i \, (1 \leq i < j \leq n), \, y_1 y_2 = y_2 y_1, \\ & y_l^{-1} x_i y_l = x_1^{\alpha_{i1}} x_2^{\alpha_{i2}} \cdots x_n^{\alpha_{in}} \, (1 \leq i \leq n, 1 \leq l \leq 2) > \end{aligned} \tag{3}$$

as a presentation for G_2 ([6]) such that α_{ij}'s ($1 \leq i, j \leq n$) are integers. We note that the assumption on the morphism ϖ_{x_i} is still valid as assumed for the presentation (1). We also note that one can look at the paper [5] for the monoid version of this presentation.

Due to commutativity of each groups F^n and F^2, we should consider the diagram of the presentation $\mathcal{Q}$ given above in two cases ([9]):

- Case 1 : *Separate diagram case*

In this case the diagram of $\mathcal{Q}$ consists of three separate diagrams, where the first and the second represent the commutator relators of groups F^2 and F^n, respectively, and the third one represents the relators $x_i y_1 = y_1 x_1^{\alpha_{i1}} x_2^{\alpha_{i2}} \cdots x_n^{\alpha_{in}}$ and $x_j y_2 = y_2 x_1^{\beta_{j1}} x_2^{\beta_{j2}} \cdots x_n^{\beta_{jn}}$. We should note that although the second diagram has been considered by taking $n \geq 3$ and odd, it can also be figured out similarly for the all even number n grater than or equal to 2.

Since the conjugacy problem is solvable for free abelian groups, there will be no problem to obtain conjugate elements from the first and the second diagrams. Moreover to obtain conjugate elements from the third diagram, we can traverse the outer and inner boundaries in a clockwise manner starting from two different points, and so we get labels as the words $x_n^{-\alpha_{1n}} \cdots x_1^{-\alpha_{11}} x_n^{-\alpha_{2n}} \cdots x_1^{-\alpha_{21}} \cdots x_n^{-\alpha_{nn}} \cdots x_1^{-\alpha_{n1}}$ and $x_n^{-\beta_{1n}} \cdots x_1^{-\beta_{11}} x_n^{-\beta_{2n}} \cdots x_1^{-\beta_{21}} \cdots x_n^{-\beta_{nn}} \cdots x_1^{-\beta_{n1}}$, respectively. In fact, by using the relators in presentation (3), we can rewrite these words in a shorter form to determine the conjugate elements. In other words, for the label of outer boundary, we can write $y_1^{-1} x_1^{-1} y_1$ *instead of* $x_n^{-\alpha_{1n}} \cdots x_1^{-\alpha_{11}}$ and $y_1^{-1} x_2^{-1} y_1$ *instead of* $x_n^{-\alpha_{2n}} \cdots x_1^{-\alpha_{21}}$, etc. and, by iterating this procedure and considering the deletion operation, we finally obtain the identity $y_1 y_1^{-1}$. In addition, we also have $y_1^{-1} x_1^{-1} x_2^{-1} \cdots x_n^{-1} y_1$ for a shorter form of $x_n^{-\alpha_{1n}} \cdots x_1^{-\alpha_{11}} x_n^{-\alpha_{2n}} \cdots x_1^{-\alpha_{21}} \cdots x_n^{-\alpha_{nn}} \cdots x_1^{-\alpha_{n1}}$. By the same way, we can rewrite the word $y_2^{-1} x_1^{-1} x_2^{-1} \cdots x_n^{-1} y_2$ for the label on inner bondary. At this stage, let us label the subword $x_1^{-1} x_2^{-1} \cdots x_n^{-1}$ by W. Also let us cut the diagram along $y_2^{-1} y_1$ (from one point to another) to obtain an S-diagram, and so to get conjugate elements (see [12]). In fact, by Proposition 3 and Definition 5, we obtain $y_1^{-1} W y_1 \sim_S y_2^{-1} W y_2$, and then $y_2 y_1^{-1} W y_1 y_2^{-1} \sim_S W$. This shows that every word is conjugate to itself by considering the third diagram ([9]).

- Case 2 : *Joint diagram case*

In the following paragraph, we use some technical terms such as *seed, leaves* etc. and we refer again [12] for the details.

This case has been basically constructed over a new diagram which obtained by putting three diagrams considered above in one diagram. So to obtain this new one, we first take the second diagram (which shows commutator relators in F^n) and call it "*seed*". Then by placing the thirs diagram to the side of each related generator on the boundary of seed, we get $2n$ regions. Let us also call these new regions as "*leaves*". Since the labels join outer boundary of the third diagram to outer boundary of the second diagram are y_1 and y_2, we must take the first diagram (which represents commutator relators in F^2) between two leaves symmetrically. Hence we obtain a new diagram by putting three diagrams into one diagram ([9]).

Remark 9 *On account of the number of each generators of F^n on the boundary of seed is 2, the rank of the other free abelian group must be 2. Otherwise there would be some unneeded loops (circles) on the diagram.*

Now to make sure the existence of the diagram for presentation $\mathcal{Q}$ in (3), we need to check the word, as a label, on the boundary of this new diagram. To do that we let travel around the boundary of this new diagram starting from one point in a clockwise manner. So the required label will be $W = y_1 y_2^{-1} x_n^{-\alpha_{1n}} \cdots x_1^{-\alpha_{11}} \cdots x_n^{-\alpha_{3n}} \cdots x_1^{-\alpha_{31}} y_1^{-1} y_2 x_1^{\beta_{11}} \cdots x_n^{\beta_{1n}} \cdots x_1^{\beta_{31}} \cdots x_n^{\beta_{3n}}$. Then, by applying the same iterating procedure W as done in Case 1, we obtain $W = y_1 y_2^{-1} y_1^{-1} x_1^{-1} y_1 y_1^{-1} x_2^{-1} y_1 \cdots y_1^{-1} x_3^{-1} y_1 y_1^{-1} y_2 y_2^{-1} x_1 y_2 y_2^{-1} x_2 y_2 \cdots y_2^{-1} x_3 y_2$. After some rearrangements and reductions, we get $W = y_2^{-1} x_1^{-1} x_2^{-1} \cdots x_3^{-1} x_1 x_2 \cdots x_3 y_2$, i.e. $W = 1$.

These above procedure give the following lemma.

Lemma 10 *For $G_2 = F^n \rtimes_{\varpi} F^2$, the corresponding diagram exists.*

Proof of Theorem 2:

As previously we must construct an annular diagram to determine conjugate elements. So let us take the each word, which is the label of the outer and inner boundaries of the new diagram constructed above, as $y_2 y_1^{-1} W_1 y_1 y_2 U_1$ and $W_2 V_2$, respectively, where W_1, U_1, W_2 and V_2 are words. Then the required annular diagram has "symmetrical two squares" ([9]). But in this diagram if we cut the path y_2 we get two S-diagrams. So we have two regions, namely $\mathbb{A}$

and $\mathbb{B}$, and two outer and inner boundaries. Then we have

. the word as the label on the outer boundary of region $\mathbb{A}$ is $W_1 = x_1^{\beta_{11}} \cdots x_n^{\beta_{1n}} x_1^{\beta_{21}} \cdots x_n^{\beta_{2n}} x_1^{\beta_{41}} \cdots x_n^{\beta_{4n}} \cdots x_1^{\beta_{51}} \cdots x_n^{\beta_{5n}} x_1^{\beta_{31}} \cdots x_n^{\beta_{3n}}$.

. the word as the label on the inner boundary of the region $\mathbb{A}$ is $W_2 = x_1x_2x_4 \cdots x_{n-1}x_n \cdots x_5x_3$.

. the word as the label on the outer boundary of the region $\mathbb{B}$ is $V_1 = y_1y_2^{-1} \underbrace{x_n^{-\alpha_{1n}} \cdots x_1^{-\alpha_{11}} \cdots x_n^{-\alpha_{3n}} \cdots x_1^{-\alpha_{31}}}_{U_1} y_2y_1^{-1}$.

. the word as the label on the inner boundary of the region $\mathbb{B}$ is $V_2 = x_1^{-1}x_2^{-1}x_4^{-1} \cdots x_{n-1}^{-1}x_n^{-1} \cdots x_5^{-1}x_3^{-1}$.

It is easy to see that labels on the outer boundaries of regions $\mathbb{A}$ and $\mathbb{B}$ are equivalent to $y_2^{-1}x_1x_2x_4 \cdots x_{n-1}x_n \cdots x_5x_3y_2$ and $y_2^{-1}x_1^{-1}x_2^{-1}x_4^{-1} \cdots x_{n-1}^{-1}x_n^{-1} \cdots x_5^{-1}x_3^{-1}y_2$, respectively. Now, by considering regions $\mathbb{A}$ and $\mathbb{B}$ separately, it is seen that $W_1 \sim_S W_2$ and $V_1 \sim_S V_2$. But since W_1V_1 is equivalent to the identity, so is W_2V_2. Therefore $W_1V_1 \sim_S W_2V_2$. These yield us the proof of Theorem 2, as required. □

SOME REMARKS

This final section is based on the relationships between diagrams and *conjugacy search problem*, and between diagrams and *left (right) divisible problem* on groups $F_n \rtimes_{\varpi} \mathbb{Z}$ and $F^n \rtimes_{\varpi} F^2$.

For a group G and for the conjugate elements a and b in G, the *conjugacy search* (CS) *problem* is to find an element $c \in G$ such that $a^c = b$ (or, equivalently, $cac^{-1} = b$). For a semigroup S, we can convert this problem to find an element $c \in S$ such that $ca = bc$ or $ac = cb$ (see [7]). By Theorem 1, we stated that the conjugacy is both preserved and reflected by the natural homomorphism. To indicate that, we have just found the element t^{-1} (or t) which satisfies conjugacy. This also yields us the solvability of conjugacy search problem for the group $G_1 = F_n \rtimes_{\varpi} \mathbb{Z}$. Hence the following result is clear by Theorems 1 and 2.

Corollary 11 *The (CS) problem is solvable for the groups G_1 and G_2.*

In addition to above search problem, for a semigroup S, a word A is said to be left (right) divisible by a word B if there is a word X such that the relation $A = BX$ $(A = XB)$ holds in S. Therefore, by [2], *The left divisible (LD)* (*or, right divisible (RD)*) *problem* for a given semigroup is to find an algorithm to determine if two arbitrary words A and B whether or not A is left (right) divisible by B and, if yes, to describe a quotient X. By the proof of Theorem 1, for the words U and V, we have $\underbrace{t^{-1}V}_{A} = \underbrace{U}_{B}\underbrace{t^{-1}}_{X}$. This yields that $t^{-1}V$ is left divisible by U and Ut^{-1} is right divisible by V. Similar progress can be applied on the group $F^n \rtimes_{\varpi} F^2$ as well. Therefore we have the following result.

Corollary 12 *The (LD)* (*or (RD)*) *problem is solvable for $F_n \rtimes_{\varpi} \mathbb{Z}$ and $F^n \rtimes_{\varpi} F^2$.*

After that we can directly say that the diagram structure used for giving these above results gave a fast probabilistic algorithm for solving the conjugacy search and left (right) divisible problems.

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Record 1 of 1

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Abstract: At the present paper we show that conjugacy is preserved and reflected by the natural homomorphism defined from a semigroup S to a group G, where G defines split extensions of some free groups. The main idea in the proofs is based on a geometrical structure as applied in the paper [8].

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[1]
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